Chapter 3

Main results

This chapter is the main goal of this thesis which contains the following sections:

3.1 Deterministic coincidence and common fixed points.

- **3.1.1** Coincidence and common fixed points for generalized I-contraction mappings.
- **3.1.2** Coincidence and common fixed points for generalized I-nonexpansive mappings.
- **3.1.2** Invariant Approximations.

3.2 Random coincidence and common random fixed points.

- **3.2.1** Random coincidence points.
- **3.2.2** Common random fixed points.

In section 3.1.1 we will introduce ***property (W.P.)*** and ***property (W.P.)***** which generalizes many conditions of existence of fixed point, coincidence point, and common fixed point in several works and theorems (see [1], [2], [22], [23], [24], [29], [30], [32], [40], [41], [47], [59], [60], [61], [63], [64]).

Also we establish some coincidence and common fixed points theorems for generalized I-contraction multivalued mappings without assumption that I is T-weakly commuting. Afterward we give some examples with support above paragraph.

In section 3.1.2 we will use ***property (W.P.)*** and ***property (W.P.)**** establish some coincidence point and common fixed point for generalized I-nonexpansive multivalued mappings without assumption that I is T-weakly commuting.

In section 3.1.3 we will derive invariant approximations on generalized I-nonexpansive multivalued mapping by apply our theorems in before section.

In section 3.2 we will finish this chapter by extending theorems in section 3.1 to random version. The existence of coincidence point and common fixed point on generalized I-nonexpansive multivalued mapping extended to random coincidence point and common random fixed point on random operator mapping.

3.1 Deterministic coincidence and common fixed points

3.1.1 Coincidence and common fixed points for generalized I-contraction mappings

In 2007 Al-Thagafi and Shahzad [1] establish coincidence point and common fixed points theorems under generalized I-contraction mapping (see Theorem 2.3.42). In this section we introduce new property and establish new coincidence points and common fix points theorem which generalizes Theorem 2.3.42 and many theorems (see [2], [22], [23], [24], [29], [30], [32], [40], [41], [47], [59], [60], [61], [63], [64]). Now we begin with the following definition.

Definition 3.1.1. Let D be a nonempty set and let $I : D \to D$, $T : D \to CL(D)$. The mappings I and T are said to satisfy the ***property (W.P.)*** on D if there exists a sequence $\{x_n\}$ in D, some $u \in D$ and $A \in CL(D)$ such that

$$\lim_{n\to\infty} Ix_n = Iu \in A = \lim_{n\to\infty} Tx_n.$$

Definition 3.1.2. Let D be a nonempty set and $T : D \to CL(D)$.
The mapping T is said to satisfy the ***property (W.P.)**** on D if there exists a sequence $\{x_n\}$ in D, some $u \in D$ and $A \in CL(D)$ such that

$$\lim_{n\to\infty} x_n = u \in A = \lim_{n\to\infty} Tx_n.$$

Lemma 3.1.3. *Let D be a metric space and $T : D \to CL(D)$. If $I : D \to D$ is an identity mapping and T satisfy the property (W.P.)* on D, then I and T satisfy the property (W.P.) on D.*

Proof. Let $I : D \to D$ be an identity mapping and T satisfy the property (W.P.)*. Then there exists a sequence $\{x_n\}$ in D, some $u \in D$ and $A \in CL(D)$ such that

$$\lim_{n\to\infty} x_n = u \in A = \lim_{n\to\infty} Tx_n.$$

Since $Iu = u$,

$$\lim_{n\to\infty} Ix_n = u = Iu \in A = \lim_{n\to\infty} Tx_n.$$

Thus I and T satisfy the property (W.P.). □

Example 3.1.4. Let $D = [0, \infty)$. Define $I : D \to D$ and $T : D \to CL(D)$ by

$$Ix = \tfrac{x}{4} \text{ and } Tx = \{\tfrac{3x}{4}\} \text{ for all } x \in D.$$

Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$.
Since $\lim_{n\to\infty} Ix_n = 0 = I0$ and $\lim_{n\to\infty} Tx_n = \{0\}$, $\lim_{n\to\infty} Ix_n = I0 \in \{0\} = \lim_{n\to\infty} Tx_n$.
Therefore I and T satisfy the property (W.P.).

Example 3.1.5. Let $D = [0, \infty)$. Define $I : D \to D$ and $T : D \to CL(D)$ by

$$Ix = x + 1 \text{ and } Tx = [0, x + 2] \text{ for all } x \in D.$$

Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$.

Since $\lim\limits_{n\to\infty} Ix_n = 1 = I0$ and $\lim\limits_{n\to\infty} Tx_n = [0, 2]$, $\lim\limits_{n\to\infty} Ix_n = I0 \in [0, 2] = \lim\limits_{n\to\infty} Tx_n$.
Therefore I and T satisfy the property (W.P.).

Example 3.1.6. Let $D = [-1, 1]$. Define $I : D \to D$ and $T : D \to CL(D)$ by

$$Ix = x^2 \text{ and } Tx = [(\tfrac{x}{4})^2, 1] \text{ for all } x \in D.$$

Consider the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$.

Since $\lim\limits_{n\to\infty} Ix_n = 1 = I1$ and $\lim\limits_{n\to\infty} Tx_n = \{1\}$, $\lim\limits_{n\to\infty} Ix_n = I1 \in \{1\} = \lim\limits_{n\to\infty} Tx_n$.
Therefore I and T satisfy the property (W.P.).

Let D be a nonempty subset of a metric space (X, d) and let $I : D \to D$, $T : D \to CL(D)$. Throughout this section we let $\varphi_{I,T}(x, y)$ stand for
$\max\left\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), \frac{1}{2}[d(Ix, Ty) + d(Iy, Tx)]\right\}$, for every $x, y \in D$.
If I is the identity mapping of D, $\varphi_{I,T}(x, y)$ will be denoted by $\varphi_T(x, y)$. Next we again give definition of generalized I-contraction and generalized contraction.

Definition 3.1.7. Let D be a nonempty set and $I : D \to D$, $T : D \to CL(D)$.

1. T is a ***generalized I-contraction*** on D if

   $H(Tx, Ty) \leq k\varphi_{I,T}(x, y)$ for all $x, y \in D$ and for some $k \in [0, 1)$.

2. T is a ***generalized contraction*** on D if

   $H(Tx, Ty) \leq k\varphi_T(x, y)$ for all $x, y \in D$ and for some $k \in [0, 1)$.

Lemma 3.1.8. *Let D be a nonempty subset of a metric space (X, d) and let $I : D \to D$, $T : D \to CL(D)$. If T is a generalized I-contraction, $\overline{T(D)} \subseteq I(D)$, and $\overline{T(D)}$ is complete, then I and T satisfy the property (W.P.).*

Proof. Let $x_0 \in D$. As $T(D) \subseteq \overline{T(D)} \subseteq I(D)$, we construct a sequence $\{x_n\}$ in D such that $Ix_n \in Tx_{n-1} \subseteq T(D)$ for all $n \geq 1$. We conclude, as in Theorem 2.1 [1], that $\{Ix_n\}$ is a Cauchy sequence in $T(D)$. It follows from the completeness of $\overline{T(D)}$ that $Ix_n \to z \in \overline{T(D)} \subseteq I(D)$ where $z = Iu$ for some $u \in D$. So there exists a sequence $\{x_n\}$ in D, such that

$$z = \lim_{n\to\infty} Ix_n = Iu \in \lim_{n\to\infty} Tx_n \in CL(D).$$

Thus I and T satisfy the property (W.P.). $\square$

Now we obtain our first theorem in this section.

Theorem 3.1.9. *Let D be a nonempty subset of a metric space (X, d) and let $I : D \to D$, $T : D \to CL(D)$, and I and T satisfy the property (W.P.) on D. If T is a generalized I-contraction on D, then $C(I,T) \neq \varnothing$. Moreover, if $IIv = Iv$ for some $v \in C(I,T)$, then $F(I,T) \neq \varnothing$.*

Proof. Since I and T satisfy the property (W.P.), there exists a sequence $\{x_n\}$ in D, some $u \in D$ and $A \in CL(D)$ such that

$$\lim_{n\to\infty} Ix_n = Iu \in A = \lim_{n\to\infty} Tx_n.$$

Note that, for every $n \geq 1$, we have

$$\begin{aligned} H(Tx_n, Tu) &\leq k\ \varphi_{I,T}(x_n, u) \\ &= k\ \max\{d(Ix_n, Iu), d(Ix_n, Tx_n), d(Iu, Tu), \\ &\qquad \frac{1}{2}[d(Ix_n, Tu) + d(Iu, Tx_n)]\}. \end{aligned}$$

By letting $n \to \infty$, we have $H(A, Tu) \leq k\ d(Iu, Tu)$.

It follows from $d(Iu, Tu) \leq H(A, Tu) \leq k\ d(Iu, Tu)$ that $Iu \in Tu$. Hence $C(I,T)$ is nonempty.

Since there exists $v \in C(I,T)$ such that $IIv = Iv$.

We let $t := Iv$. Thus $t = Iv = IIv = It \in Tv$. It follows that

$$\begin{aligned} d(t,Tt) &\leq H(Tv,Tt) \\ &\leq k\ \varphi_{I,T}(v,t) \\ &= k\ \max\{d(Iv,It), d(Iv,Tv), d(It,Tt), \frac{1}{2}[d(Iv,Tt)+d(It,Tv)]\} \\ &= k\ \max\{d(t,t), d(t,Tv), d(t,Tt), \frac{1}{2}[d(t,Tt)+d(t,Tv)]\} \\ &= k\ d(t,Tt). \end{aligned}$$

Then $t \in Tt$ and so $t = It \in Tt$. Hence $F(I,T) \neq \varnothing$. □

Remark 3.1.10. Theorem 3.1.9 generalizes and extends the Banach Contraction Principle, Nadler's Contraction Principle [47], Theorem 2.4 of Daffer and Kaneko [22], and Theorem 2.1 of Al-Thagafi and Shahzad [1].

The following example is example of the mappings which satisfy Theorem 3.1.9.

Example 3.1.11. Let $D = [0,1)$ be the usual metric space. Define $I : D \to D$ and $T : D \to CL(D)$ by $Ix = \frac{x}{2}$ and $Tx = [0, \frac{x}{4}]$ for all $x \in D$.

(1) Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$ in D. Since $\lim_{n\to\infty} Ix_n = 0$, $I0 = 0$, $\lim_{n\to\infty} Tx_n = \{0\}$, and $0 \in \{0\}$. Therefore I and T satisfy the property (W.P.) for the sequence $\{x_n\} = \{\frac{1}{n}\}$.

(2) For each $x, y \in D$,

$$\begin{aligned} H(Tx, Ty) &= H([0, \frac{x}{4}], [0, \frac{y}{4}]) \\ &= d(\frac{x}{4}, \frac{y}{4}) \\ &= \frac{1}{2} d(\frac{x}{2}, \frac{y}{2}) \\ &= \frac{1}{2} d(Ix, Iy) \\ &= \frac{1}{2} \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), \\ &\qquad \frac{1}{2}[d(Ix, Ty) + d(Iy, Tx)]\} \\ &\leq \frac{1}{2} \varphi_{I,T}(x, y). \end{aligned}$$

Thus T is a generalized I-contraction.

(3) It follow from $II0 = I0$ and $C(I, T) = \{0\}$ that $IIv = Iv$ for some $v \in C(I, T)$.

Since (1), (2), and (3) are true, all hypotheses of Theorem 3.1.9 are satisfied. Thus T and I have a common fixed point which is 0 because $0 = I0 \in T0$.

The next example is example which support that Theorem 3.1.9 generalized Theorem 2.1 of Al-Thagafi and Shahzad [1].

Example 3.1.12. Let $D = (-1, 1]$ be the usual metric space. Define $I : D \to D$ and $T : D \to CL(D)$ by $Ix = x^2$ and $Tx = [\frac{x^2}{2} - \frac{1}{2}, 1]$ for all $x \in D$.

(1) Since $T(D) = [-0.5, 1]$ and $I(D) = [0, 1]$, $\overline{T(D)} \nsubseteq I(D)$.

(2) Consider the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ in D. Since $\lim\limits_{n\to\infty} Ix_n = 1$, $I1 = 1$, $\lim\limits_{n\to\infty} Tx_n = [0, 1]$, and $1 \in [0, 1]$. Therefore I and T satisfy the property (W.P.) for the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$.

(3) For each $x, y \in D$,

$$\begin{aligned}
H(Tx,Ty) &= H([\frac{x^2}{2}-\frac{1}{2},1],[\frac{y^2}{2}-\frac{1}{2},1]) \\
&= d(\frac{x^2}{2}-\frac{1}{2},\frac{y^2}{2}-\frac{1}{2}) \\
&= d(\frac{x^2}{2},\frac{y^2}{2}) \\
&= \frac{1}{2}d(x^2,y^2) \\
&= \frac{1}{2}d(Ix,Iy) \\
&\leq \frac{1}{2}\max\{d(Ix,Iy),d(Ix,Tx),d(Iy,Ty), \\
&\qquad\qquad \frac{1}{2}[d(Ix,Ty)+d(Iy,Tx)]\} \\
&\leq \frac{1}{2}\varphi_{I,T}(x,y).
\end{aligned}$$

Thus T is a generalized I-contraction.

(4) It follows from $II0 = I0$ and $C(I,T) = \{0\}$ that $IIv = Iv$ for some $v \in C(I,T)$.

Since (1) is true, Theorem 2.1 of M.A. Al-Thagafi and Shahzad [1] cannot be used. But (2), (3), and (4) are true, so all hypotheses of Theorem 3.1.9 are satisfied. Thus T and I have a common fixed point which is 0 because $0 = I0 \in T0$.

Corollary 3.1.13. *Let D be a nonempty subset of a metric space (X,d) and let $T : D \to CL(D)$, T satisfies the property (W.P.)* on D. If T is generalized contraction on D, then $F(T) \neq \varnothing$.*

Proof. Let $I : D \to D$ be the identity mapping. Since I is the identity mapping and T satisfies the property (W.P.)*. By Lemma 3.1.3 I and T satisfy the property (W.P.), $IIv = Iv$ for all $v \in D$. Thus $IIv = Iv$ for some $v \in C(I,T)$. It follows from Theorem 3.1.9 that $F(I,T) \neq \varnothing$. Hence $F(T) \neq \varnothing$. □

Corollary 3.1.14. *Let D be a nonempty subset of a metric space (X,d). $I : D \to D$, $T : D \to CL(D)$, $\overline{T(D)} \subseteq I(D)$, and $\overline{T(D)}$ complete. Suppose that T is generalized*

I-contraction on D. Then $C(I,T) \neq \varnothing$. Moreover, if IIv=Iv for some $v \in C(I,T)$, then $F(I,T) \neq \varnothing$.

Proof. Since Lemma 3.1.8, I and T satisfy property (W.P.) on D. As in the prove of theorem 3.1. □

Remark 3.1.15. Corollary 3.1.14 generalizes Banach Contraction Principle, Nadler's Contraction Principle [47], Theorem 2.4 of Daffer and Kaneko [22], and Corollary 2.2 of Al-Thagafi and Shahzad [1].

3.1.2 Coincidence and common fixed points for generalized I-nonexpansive mappings

In 2006 Shahzad and Hussain [61] established coincidence point and common fixed points theorems under I-nonexpansive mappings (see [61]). In this section we define generalized I-nonexpansive and generalized nonexpansive mappings. Also we use property (W.P.) and (W.P.)* establish new coincidence points and common fixed points theorems which generalizes Theorems of Shahzad and Hussain [61] and Jungck [29]. Now we begin with the following definition.

Definition 3.1.16. Let D be a nonempty subset of a normed space X and let $I : D \to D$, $T : D \to CL(D)$. For every $x, y \in D$, we define

$$\psi_{I,T}(x,y) := \max\{\|Ix - Iy\|, \tfrac{1}{2}[d(Ix,Tx) + d(Iy,Ty)], \tfrac{1}{2}[d(Ix,Ty) + d(Iy,Tx)]\}.$$

If I is the identity mapping of D, $\psi_{I,T}(x,y)$ will be denoted by $\psi_T(x,y)$ respectively. Then

1. T is ***generalized I-nonexpansive*** on D if

   $H(Tx,Ty) \leq \psi_{I,T}(x,y)$ for all $x, y \in D$.

2. T is ***generalized nonexpansive*** on D if

   $H(Tx, Ty) \leq \psi_T(x, y)$ for all $x, y \in D$.

Definition 3.1.17 ([1]). Let D be a nonempty subset of a normed space X, $I : D \to D$, and $T : D \to CL(D)$.

1. The pair (I, T) satisfies the ***coincidence point condition*** (in short, ***CPC***) on $A \in CL(D)$ if whenever $\{x_n\}$ is a sequence in A such that $d(Ix_n, Tx_n) \to 0$, we have $Iz \in Tz$ for some $z \in A$.

2. The map T satisfies the ***fixed point condition*** (for short, ***FPC***) on $A \in CL(D)$ if whenever $\{x_n\}$ is a sequence in A such that $d(x_n, Tx_n) \to 0$, we have $z \in Tz$ for some $z \in A$.

Theorem 3.1.18. *Let D be a nonempty subset of a normed space X. Let $I : D \to D$, $T : D \to CL(D)$, I and T satisfy the property (W.P.) on D, and the pair (I,T) satisfies the CPC on D. If T is a generalized I-nonexpansive on D, then $C(I, T) \neq \varnothing$. Moreover, if IIv=Iv for some $v \in C(I, T)$, then $F(I, T) \neq \varnothing$.*

Proof. Since I and T satisfy the property (W.P.), there exists a sequence $\{x_n\}$ in D, some $u \in D$ and $A \in CL(D)$ such that

$$\lim_{n\to\infty} Ix_n = Iu \in A = \lim_{n\to\infty} Tx_n.$$

Thus $d(Ix_n, Tx_n) \to 0$. As the pair (I, T) satisfies the CPC on D, there exists $w \in D$ such that $Iw \in Tw$. Therefore, $C(I, T) \neq \varnothing$.

Since there exists $v \in C(I, T)$ such that $IIv = Iv$.

Let $t := Iv$. Thus $t = Iv = IIv = It \in Tv$. It follows that

$$\begin{aligned} d(t,Tt) &\leq H(Tv,Tt) \\ &\leq \psi_{I,T}(v,t) \\ &= \max\{d(Iv,It), \frac{1}{2}[d(Iv,Tv)+d(It,Tt)], \frac{1}{2}[d(Iv,Tt)+d(It,Tv)]\} \\ &= \max\{d(t,t), \frac{1}{2}[d(t,Tv)+d(t,Tt)], \frac{1}{2}[d(t,Tt)+d(t,Tv)]\} \\ &= \frac{1}{2}d(t,Tt). \end{aligned}$$

Then $t \in Tt$ and so $t = It \in Tt$. Hence $F(I,T) \neq \varnothing$. □

Remark 3.1.19. Theorem 3.1.18 generalizes and extends Theorems 2.1, 2.2, 2.4, 2.6, 2.7, 2.8, 2.9, 2.11 of Shahzad and Hussain [61], Corollaries 3.2, 3.4 of Jungck [29].

Corollary 3.1.20. *Let D be a nonempty subset of a normed space X. Let $T : D \to CL(D)$, T satisfies the property (W.P.)* on D, and T satisfies the FPC on D. If T is a generalized nonexpansive on D, then $F(T) \neq \varnothing$.*

Proof. Let $I : D \to D$ be the identity mapping. Since I is the identity the mapping and T satisfies property (W.P.)*, by Lemma 3.1.3, I and T satisfy the property (W.P.). Since T is satisfies the FPC on D, so The pair (I,T) satisfies CPC on D. For some $v \in C(I,T)$ $IIv = Iv$. It follows from Theorem 3.1.18 that $F(I,T) \neq \varnothing$. Thus $F(T) \neq \varnothing$. □

Remark 3.1.21. Corollary 3.1.20 extends and improves Corollary 2.5 of Al-Thagafi, Shahzad [1], Theorems 1,2 of Doston [23], Theorem 3.2 of Dozo [24].

Theorem 3.1.22. *Let D be a nonempty subset of a normed space X. Let $I : D \to D$, $T : D \to CL(D)$, I and T satisfy the property (W.P.) on D, and (I-T)(D) be closed. If T is a generalized I-nonexpansive on D, then $C(I,T) \neq \varnothing$. Moreover, if IIv=Iv for some $v \in C(I,T)$, then $F(I,T) \neq \varnothing$.*

Proof. Since I and T satisfy the property (W.P.), there exists a sequence $\{x_n\}$ in D, some $u \in D$ and $A \in CL(D)$ such that

$$\lim_{n\to\infty} Ix_n = Iu \in A = \lim_{n\to\infty} Tx_n.$$

Thus $\|Ix_n - Iu\| \to 0$. Because $(I-T)(D)$ is closed. So $0 \in (I-T)(D)$. Therefore there exists $v \in D$ such that $Iv \in Tv$. Hence $C(I,T) \neq \varnothing$. Now for $F(I,T) \neq \varnothing$ follows as in the proof of theorem 3.1.18. □

Remark 3.1.23. Theorem 3.1.22 extends and improves Theorem 2.1 of Shahzad and Hussain [61].

Theorem 3.1.24. *Let D be a q-starshaped subset of a normed space X. Let $I : D \to D$, $T : D \to CL(D)$, I and T satisfy the property (W.P.) on D, I(D)=D, D weakly compact, and I-T is demiclosed at 0. If T is generalized I-nonexpansive on D, then $C(I,T) \neq \varnothing$. Moreover, if IIv=Iv for some $v \in C(I,T)$, then $F(I,T) \neq \varnothing$.*

Proof. It is evident, by the Eberlein-Smulian theorem and the definition of "demiclosed" (see Theorem 2 in [23]), that the pair (I,T) satisfies the CPC on D. Now the result follows from Theorem 3.1.18. □

Remark 3.1.25. Theorem 3.1.24 extends and improves Theorems 2.2, 2.7 of Shahzad and Hussain [61].

3.1.3 Invariant approximations

Definition 3.1.26. Let M be a subset of a normed space X and $p \in X$. We define

1. The set $B_M(p) := \{x \in M : \|x - p\| = d(p, M)\}$ is called ***the set of best approximations to*** $p \in X$ out of M.

2. $M_p := \{x \in M : \|x\| \leq 2\|p\|\}$.

3. C_0 denote the class of closed convex subsets M of X containing 0.

Remark 3.1.27. $B_M(p)$ is closed, convex and contained in $M_p \in C_0$.

The problem of obtaining invariant approximations for non-commuting mapping was considered first time by Shahzad [55, 56]. In 2003 Shahzad [58] introduce the class of R-subweakly commuting multivalued mapping. It is worth mentioning that the concept of R-subweak commutativity is a useful tool for obtaining the existence of invariant approximations for a hybrid pair of mapping. Afterward Kamran [32] introduce property $(E.A.)$ for single-valued and multivalued mapping which generalized and cover than the condition in [58]. Recently Shahzad and Hussain [61] extends and improves Theorem 3.14 of Kamran [32]. They further note that Kamran's result remains true if dropped some the condition.

Now we establish theorems which generalize many theorems of Shahzad and Hussain in [61]. Our theorems apply the results of before section.

Theorem 3.1.28. *Let X be a normed space, $I : X \to X, T : X \to CL(X)$, $M \subseteq X$, $I(B_M(p)) = B_M(p)$, I and T satisfy the property (W.P.) on $B_M(p)$, the pair (I,T) satisfies the CPC on $B_M(p)$, and*

$$\sup_{y \in Tx} \|y - p\| \leq \|Ix - p\|$$

for all $x \in B_M(p)$.
If T is a a generalized I-nonexpansive on $B_M(p)$, then $C(I,T) \cap B_M(p) \neq \varnothing$.
Moreover, if IIv=Iv for some $v \in C(I,T) \cap B_M(p)$, then $F(I,T) \cap B_M(p) \neq \varnothing$.

Proof. Let $x \in B_M(p)$ and $z \in Tx$. Since $I(B_M(p)) = B_M(p)$, $Ix \in B_M(p)$ for all $x \in B_M(p)$. It follows from the definition of $B_M(p)$ that $\|Ix - p\| = d(p, M)$. Since

$$\|z - p\| \leq \sup_{y \in Tx} \|y - p\| \leq \|Ix - p\| = d(p, M),$$

$z \in B_M(p)$. Thus $Tx \subseteq B_M(p)$ for all $x \in B_M(p)$. Since Tx is closed for all $x \in X$, Tx is closed for all $x \in B_M(p)$. Therefore $I|_{B_M(p)} : B_M(p) \to B_M(p)$, $T|_{B_M(p)} : B_M(p) \to CL(B_M(p))$. Since $C(I|_{B_M(p)}, T|_{B_M(p)}) = C(I,T) \cap B_M(p)$ and $F(I|_{B_M(p)}, T|_{B_M(p)}) = F(I,T) \cap B_M(p)$. Now the result follows from Theorem 3.1.18 with $D = B_M(p)$. □

Remark 3.1.29. Theorem 3.1.28 extends and improves Theorems 2.9, 2.11, 2.12, 2.13 of Shahzad and Hussain [61], and contains, as a special case, Theorem 3 of Latif and Bano [40], and Theorem 7 of Jungck and Sessa [30].

Corollary 3.1.30. *Let X be a normed space, $T : X \to CL(X)$, $M \subseteq X$, T satisfy the property (W.P.)* on $B_M(p)$, the mapping T satisfies the FPC on $B_M(p)$, and*

$$\sup_{y \in Tx} \|y - p\| \leq \|x - p\|$$

for all $x \in B_M(p)$.

If T is a generalized nonexpansive on $B_M(p)$, then $F(T) \cap B_M(p) \neq \varnothing$.

Proof. Let $I : X \to X$ be the identity mapping. Then $I(B_M(p)) = B_M(p)$. Since T is satisfies the FPC on $B_M(p)$, the pair (I,T) satisfies the CPC on $B_M(p)$. For some $v \in C(I,T)$ $IIv = Iv$. It follows from Theorem 3.1.28 that $F(I,T) \cap B_M(p) \neq \varnothing$. Thus $F(T) \cap B_M(p) \neq \varnothing$. □

Theorem 3.1.31. *Let X be a normed space, $I : X \to X$, $T : X \to CL(X)$, $M \subseteq X$, $I(B_M(p)) = B_M(p)$, I and T satisfy the property (W.P.) on $B_M(p)$, $(I-T)(B_M(p))$ be closed, and*

$$\sup_{y \in Tx} \|y - p\| \leq \|Ix - p\|$$

for all $x \in B_M(p)$.

If T is a generalized I-nonexpansive on $B_M(p)$, then $C(I,T) \cap B_M(p) \neq \varnothing$.

Moreover, if IIv=Iv for some $v \in C(I,T) \cap B_M(p)$, then $F(I,T) \cap B_M(p) \neq \varnothing$.

Proof. Let $x \in B_M(p)$ and $z \in Tx$. Since $I(B_M(p)) = B_M(p)$, $Ix \in B_M(p)$ for all $x \in B_M(p)$. It follows from the definition of $B_M(p)$ that $\|Ix - p\| = d(p, M)$. Since

$$\|z - p\| \leq \sup_{y \in Tx} \|y - p\| \leq \|Ix - p\| = d(p, M),$$

$z \in B_M(p)$. Thus $Tx \subseteq B_M(p)$ for all $x \in B_M(p)$. Since Tx is closed for all $x \in X$, Tx is closed for all $x \in B_M(p)$. Therefore $I|_{B_M(p)} : B_M(p) \to B_M(p)$, $T|_{B_M(p)} : B_M(p) \to CL(B_M(p))$. Since $C(I|_{B_M(p)}, T|_{B_M(p)}) = C(I,T) \cap B_M(p)$ and $F(I|_{B_M(p)}, T|_{B_M(p)}) = F(I,T) \cap B_M(p)$. Now the result follows from Theorem 3.1.22 with $D = B_M(p)$. □

Theorem 3.1.32. *Let X be a normed space, $I : X \to X$, $T : X \to CL(X)$, $M \subseteq X$, $I(B_M(p)) = B_M(p)$, $B_M(p)$ be weakly compact and q-starshaped, I and T satisfy property (W.P.) on $B_M(p)$, I-T be demiclosed at 0, and*

$$\sup_{y \in Tx} \|y - p\| \leq \|Ix - p\|$$

for all $x \in B_M(p)$.

If T is a generalized I-nonexpansive on $B_M(p)$, then $C(I,T) \cap B_M(p) \neq \varnothing$.

Moreover, if IIv=Iv for some $v \in C(I,T)$, then $F(I,T) \cap B_M(p) \neq \varnothing$.

Proof. Let $x \in B_M(p)$ and $z \in Tx$. Since $I(B_M(p)) = B_M(p)$, $Ix \in B_M(p)$ for all $x \in B_M(p)$. It follows from the definition of $B_M(p)$ that $\|Ix - p\| = d(p, M)$. Since

$$\|z - p\| \leq \sup_{y \in Tx} \|y - p\| \leq \|Ix - p\| = d(p, M),$$

$z \in B_M(p)$. Thus $Tx \subseteq B_M(p)$ for all $x \in B_M(p)$. Since Tx is closed for all $x \in X$, Tx is closed for all $x \in B_M(p)$. Therefore $I|_{B_M(p)} : B_M(p) \to B_M(p)$, $T|_{B_M(p)} : B_M(p) \to CL(B_M(p))$. Since $C(I|_{B_M(p)}, T|_{B_M(p)}) = C(I,T) \cap B_M(p)$ and $F(I|_{B_M(p)}, T|_{B_M(p)}) = F(I,T) \cap B_M(p)$. Now the result follows from Theorem 3.1.24 with $D = B_M(p)$. □

3.2 Random coincidence and common random fixed points

The study of random fixed point theorems was initiated by the Prague school of probabilities in the 1950s. Random nonlinear analysis is an important mathematical discipline which is mainly concerned with the study of random nonlinear operators and its development is required for the study of various classes of random operator equations. Random techniques have been crucial in diverse areas from pure mathematics to applied sciences. In recent years, fixed point theorems in connection with the existence of random solutions of nonlinear random operator equations have been extensively studied. For a survey of random fixed point theory and its applications and related results, we refer the reader to [4, 5, 6, 9, 57, 60, 59, 61].

In [61] Shahzad and Hussain establish some random coincidence and common random fixed points Theorems which generalize several Theorems (see [59], [63], [64]). In this section we extend and generalize Theorems of Shahzad and Hussain in [61]. Our Theorems derive results in section 3.1.

3.2.1 Random coincidence points

Theorem 3.2.1. *Let* (Ω, Σ) *be a measurable space and* D *be a separable, closed, and q-starshaped subset of a normed space* X*. Let* $I : \Omega \times D \to D$ *and* $T : \Omega \times D \to CL(D)$ *be continuous random operators such that* $I(\omega, \cdot)$ *and* $T(\omega, \cdot)$ *satisfy the property (W.P.) on* D*,* $I(\omega, \cdot)(D) = D$*,* $T(\omega, \cdot)(D)$ *is bounded, and the pair* $(I(\omega, \cdot), T(\omega, \cdot))$ *satisfies the CPC on* $A \in CL(D)$ *for every* $\omega \in \Omega$*. If* $T(\omega, \cdot)$ *a is generalized I-nonexpansive on* D *for every* $\omega \in \Omega$*, then* $RC(I, T) \neq \varnothing$*.*

Proof. By Theorem 3.1.18, I and T have a deterministic coincidence point. It follows from Theorem 4.1 in [1] that I and T have a random coincidence point. Thus $RC(I,T) \neq \varnothing$. $\square$

Remark 3.2.2. Theorem 3.2.1 generalizes and extends Theorem 3.4 of Shahzad and Hussain [61].

Corollary 3.2.3. *Let (Ω, Σ) be a measurable space and D be a separable, closed, and q-starshaped subset of a normed space X. Let $T : \Omega \times D \to CL(D)$ be continuous random operators such that $T(\omega, \cdot)$ satisfies the property (W.P.)* on D, $T(\omega, \cdot)(D)$ is bounded, and the mapping $T(\omega, \cdot)$ satisfies the FPC on $A \in CL(D)$ for every $\omega \in \Omega$. If $T(\omega, \cdot)$ is a generalized nonexpansive on D for every $\omega \in \Omega$, then $RF(T) \neq \varnothing$.*

Proof. By Theorem 3.1.20, T has a deterministic fixed point. It follows from Theorem 4.1 in [1] that T has a random fixed point. Thus $RF(T) \neq \varnothing$. $\square$

Remark 3.2.4. Corollary 3.2.3 generalizes and extends Corollary 3.6 of Shahzad and Hussain [61].

3.2.2 Common random fixed points

Lemma 3.2.5. *Let (Ω, Σ) be a measurable space and D be a subset of a normed space X. Let $I : \Omega \times D \to D$ and $T : \Omega \times D \to CL(D)$ be continuous random operators such that $T(\omega, \cdot)$ is a generalized I-nonexpansive on D for every $\omega \in \Omega$. If $RC(I,T) \neq \varnothing$ and if for every $\omega \in \Omega$, $I(\omega, I(\omega, \mu(\omega))) = I(\omega, \mu(\omega))$ for some $\mu \in RC(I,T)$, then $RF(I,T) \neq \varnothing$.*

Proof. Since $\mu : \Omega \to D$ is a random coincidence point of I and T such that $I(\omega, I(\omega, \mu(\omega)) = I(\omega, \mu(\omega))$ for all $\omega \in \Omega$. Let $\xi : \Omega \to D$ be such that

$\xi(\omega) := I(\omega, \mu(\omega))$. Since $I(\omega, \mu(\omega)) \in T(\omega, \mu(\omega))$, we have $\xi(\omega) \in T(\omega, \mu(\omega))$ for all $\omega \in \Omega$. Hence ξ is a measurable map such that

$$\xi(\omega) = I(\omega, \mu(\omega)) = I(\omega, I(\omega, \mu(\omega))) = I(\omega, \xi(\omega)) \in T(\omega, \mu(\omega)).$$

We have

$$\begin{aligned} d(\xi(\omega), T(\omega, \xi(\omega))) &\leq H(T(\omega, \mu(\omega)), T(\omega, \xi(\omega))) \\ &\leq \max\{\|I(\omega, \mu(\omega)) - I(\omega, \xi(\omega))\|), \\ &\qquad \tfrac{1}{2}[d(I(\omega, \mu(\omega)), T(\omega, \mu(\omega))) + d(I(\omega, \xi(\omega)), T(\omega, \xi(\omega)))], \\ &\qquad \tfrac{1}{2}[d(I(\omega, \mu(\omega)), T(\omega, \xi(\omega))) + d(I(\omega, \xi(\omega)), T(\omega, \mu(\omega)))]\} \\ &\leq \max\{\|\xi(\omega) - \xi(\omega)\|), \\ &\qquad \tfrac{1}{2}[d(\xi(\omega), T(\omega, \mu(\omega))) + d(\xi(\omega), T(\omega, \xi(\omega)))], \\ &\qquad \tfrac{1}{2}[d(\xi(\omega), T(\omega, \xi(\omega))) + d(\xi(\omega), T(\omega, \mu(\omega)))]\} \end{aligned}$$

for all $\omega \in \Omega$. Then $d(\xi(\omega), T(\omega, \xi(\omega))) \leq \frac{1}{2}d(\xi(\omega), T(\omega, \xi(\omega)))$.
So $d(\xi(\omega), T(\omega, \xi(\omega))) = 0$. Thus $\xi(\omega) = I(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$. Hence ξ is a common random fixed point of I and T. Thus $RF(I, T) \neq \varnothing$. □

Theorem 3.2.6. *Let (Ω, Σ) be a measurable space and D be a separable, closed, and q-starshaped subset of a normed space X. Let $I : \Omega \times D \to D$ and $T : \Omega \times D \to CL(D)$ be continuous random operators such that $I(\omega, \cdot)$ and $T(\omega, \cdot)$ satisfy the property (W.P.) on D, $I(\omega, \cdot)(D) = D$, $T(\omega, \cdot)(D)$ is bounded, and the pair $(I(\omega, \cdot), T(\omega, \cdot))$ satisfies the CPC on $A \in CL(D)$ for every $\omega \in \Omega$. If $T(\omega, \cdot)$ is a generalized I-nonexpansive on D for every $\omega \in \Omega$, then $RC(I, T) \neq \varnothing$. Moreover, if for every $\omega \in \Omega$, $I(\omega, I(\omega, \mu(\omega))) = I(\omega, \mu(\omega))$ for some $\mu \in RC(I, T)$, then $RF(I, T) \neq \varnothing$.*

Proof. By Theorem 3.2.1, $RC(I, T) \neq \varnothing$. Because for every $\omega \in \Omega$,
$I(\omega, I(\omega, \mu(\omega))) = I(\omega, \mu(\omega))$ for some $\mu \in RC(I, T)$. It follows from Lemma 3.2.5 that $RF(I, T) \neq \varnothing$. □

Remark 3.2.7. Theorem 3.2.6 generalizes and extends Theorem 3.2 of Shahzad and Latif [60], Theorem 3.4 of Tan and Yaun [63], Theorem 1 of Xu [64], Theorems 3.17, 3.18 of Shahzad [59], and Theorem 3.7 of Shahzad and Hussain [61].