Chapter 2

Preliminaries

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters. Although details are included in some case, many of the fundamental principle analysis are merely stated without proof.

Throughout this thesis, we let $\mathbb{R}$ stand for the set of all real number and $\mathbb{N}$ the set of all natural number. Let T be function (mapping) from a set X into itself. If $x \in X$, then Tx is the image of x under function T.

2.1 Metric spaces

Definition 2.1.1. Let X be a nonempty set. ***A metric on*** X ***(or distance function on*** X***)*** is a function $d : X \times X \to \mathbb{R}$ with the properties, for all $x, y, z \in X$

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$. (Symmetry).
4. $d(x, y) \leq d(x, z) + d(z, y)$ (The triangle inequality).

$d(x, y)$ is called the distance between x and y.

Example 2.1.2. Let X be a set of real numbers $\mathbb{R}$ and define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, y) = |x - y| \qquad \forall x, y \in \mathbb{R}.$$

Then d is a metric on $\mathbb{R}$ and d is called a ***usual metric***.

Example 2.1.3. Let $X = \mathbb{R}^2$ and define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \qquad \forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

Then d is a metric on $\mathbb{R}^2$.

Example 2.1.4. Let X be an arbitrary set and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$ $\forall x, y \in X$. Then d is a metric on X and we called that ***discrete metric***.

Example 2.1.5. Let $X = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous on } [a, b]\}$ and define $d : X \times X \to \mathbb{R}$ by

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)| \qquad \forall f, g \in X.$$

Then d is a metric on X.

Definition 2.1.6. A set X equipped with a metric d, denoted by (X, d), is called a ***metric space***.

Definition 2.1.7. A function of positive integer variable, designated by $f(n)$ or x_n, where $n = 1, 2, 3, ...$, is called a ***sequence***. The sequence $x_1, x_2, ...$ is also designated briefly by $\{x_n\}$.

Definition 2.1.8. A sequence $\{x_n\}$ in a metric space (X, d) is said to ***converge to*** $x \in X$ iff, for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for $n \geq N$.

In such case we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$ and x is called the limit of a sequence $\{x_n\}$. If $\lim\limits_{n\to\infty} x_n = x$ for some $x \in X$, the sequence $\{x_n\}$ is called ***convergent*** ; otherwise it is called ***divergent***.

Definition 2.1.9. Let (M, d) and (N, ρ) be metric spaces. A map $T : M \to N$ is called an ***isometry*** if $\rho(Tx, Ty) = d(x, y)$ for all $x, y \in M$.
If T is surjective (onto), then we say that M and N are isometric. A surjective isometric $T : M \to N$ is called a ***motion*** of M.

Definition 2.1.10. A sequence $\{x_n\}$ in a metric space (X, d) is called a ***Cauchy sequence*** if, for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for $n, m \geq N$.

Proposition 2.1.11. Every convergent sequence in a metric space is a Cauchy sequence.

Definition 2.1.12. A metric space (X, d) is ***complete*** if every Cauchy sequence in X converges.

Definition 2.1.13. A subset M of metric space (X, d) is ***closed*** if every sequence $\{x_n\}$ in M such that $x_n \to x$ implies $x \in M$.

Definition 2.1.14. Let (X, d) be a metric space, $x \in X$ and let r be a positive real number. The ***d-open ball of center*** x ***and radius*** r is the set

$$B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

Definition 2.1.15. Let M be a subset of a metric space (X, d). The ***closure*** of M, denoted by $\overline{M}$, is the set

$$\overline{M} = \{x \in X : B_d(x, r) \cap M \neq \varnothing \text{ for all } r \in \mathbb{R}\}.$$

2.2 Vector spaces

Definition 2.2.1. Let V be a nonempty set and K be a field. Let $+ : X \times X \to X$ and $\cdot : K \times X \to X$. ***A vector space*** $(V, +, \cdot)$ ***over a field*** K is one that satisfies these eight properties, for all $x, y, z \in V$ and for all $k, l \in K$,

1. $(x + y) + z = x + (y + z)$.
2. There exists $0 \in V$ such that $x + 0 = x = 0 + x$.
3. For all $x \in V$, there exist$-x \in V$ such that $x + (-x) = -x + x = 0$.
4. $x + y = y + x$.
5. $k(lx) = (kl)x$.
6. $(k + l)x = kx + lx$.
7. $k(x + y) = kx + ky$.
8. $1x = x$.

Definition 2.2.2. Let $(V, +, \cdot)$ be a vector space over a field K. ***A norm on*** V is a map $\| \| : V \to K$ with the properties: for all $x, y \in V$ and $k \in K$

1. $\|x\| \geq 0$.
2. $\|x\| = 0$ iff $x = 0$.
3. $\|kx\| = |k| \, \|x\|$.
4. $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2.2.3. A vector space V together with a norm is called a ***normed space***, and is denoted by $(V, \| \|)$.

The real number $\|x\|$ is called the norm of the vector x.

Proposition 2.2.4. If $(V, \| \|)$ is a normed space and $d : V \times V \to \mathbb{R}$ such that $d(x, y) = \|x - y\|$, then (V, d) is a metric space.

Proof. For each $x, y, z \in V$ we have

1. $d(x, y) = \|x - y\| \geq 0$;

2. $d(x, y) = \|x - y\| = |-1|\|y - x\| = \|y - x\| = d(y, x)$;

3. $d(x, y) = 0$ iff $\|x - y\| = 0$ iff $x - y = 0$ iff $x = y$;

4. $d(x, y) = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$.

Thus d is metric on V. Therefore V equipped with a metric d is a metric space $\square$

Definition 2.2.5. A complete normed space is called a ***Banach space***.

Definition 2.2.6. Let $(V, \| \|)$ be a normed space. A sequence $\{x_n\}$ in X is said to ***converge to*** $x \in X$ iff, for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$ for $n \geq N$. In such case we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$ and x is called the limit of a sequence $\{x_n\}$. If $\lim\limits_{n\to\infty} x_n = x$ for some $x \in X$, the sequence $\{x_n\}$ is called ***convergent*** ; otherwise it is called ***divergent***.

Definition 2.2.7. Let $(V, +, \cdot)$ be a vector space over a field K. A mapping $f : V \to K$ is called a ***linear functional*** on V if $f(x + y) = fx + fy$ and $f(kx) = kf(x)$, for all $x, y \in V$, and for all $k \in K$.

Definition 2.2.8. Let $(V, \| \|)$ be a normed space. A sequence $\{x_n\}$ in X is said to ***converge weakly to*** $x \in X$ iff, $\lim\limits_{n\to\infty} f(x_n) = f(x)$ holds for every continuous linear functional f. In such case we write $x_n \rightharpoonup x$.

2.3 Contraction mappings

A complete survey of all that has been written about contraction mappings would appear to be nearly impossible, and perhaps not really useful. In particular the wealth of applications of Banach's contraction mapping principle is astonishingly diverse. We only attempt to touch on some of the high points of this profound and seminal development in metric fixed point theory.

The origins of metric contraction principles and, ergo, metric fixed point theory itself, rest in the method of successive approximations for proving existence and uniqueness of solutions of differential equations. This method is associated with the names of such celebrated nineteenth century mathematicians such as Cauchy, Liouville, Lipschitz, Peano, and, especially, Picard. In fact the precursors of the fixed point theoretic approach are explicit in the work of Picard. However it is the Polish mathematician Stefan Banach who is credited with placing the ideas underlying the method into an abstract framework suitable for broad applications well beyond the scope of elementary differential and integral equation. Accordingly we take Banach's formulation as our point of departure in Section 2.3.2. It is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contraction condition on the mapping is easy to test and it requires only the structure of a complete metric space for its setting.

The key ingredients of the Contraction Mapping Principle as it first appeared in Banach's 1922 thesis [3] are these. (X, d) is a complete metric space and $T : X \to X$ is a *contraction mapping*. Thus there exists a constant $k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for each $x, y \in X$. From this one draws three conclusions:

(i) T has a unique fixed point, say x_0.

(ii) For each $x \in X$ the Picard sequence $\{T^n(x)\}$ converges to x_0.

(iii) The convergence is uniform if X is bounded

In fact condition (iii) can be put in much more explicit form in terms of error estimates.

(iii)$_1$ $d(T^n x, x_0) \leq \frac{k^n}{1-k} d(x, Tx)$ for each $x \in X$ and $n \geq 1$.

(iii)$_2$ $d(T^{n+1} x, x_0) \leq \frac{k^n}{1-k} d(T^{n+1} x, T^n x)$ for each $x \in X$ and $n \geq 1$.

In particular, there is an explicit rate of convergence:

(iv) $d(T^{n+1} x, x_0) \leq k d(T^n x, x_0)$.

A primary early example of an extension of Banach's principle is a theorem of Caccioppoli [19] which asserts that the Picard iterates of a mapping T converge in a complete metric space X provided for each $n \geq 1$, there exists a constant c_n such that

$$d(T^n x, T^n y) \leq c_n d(x, y)$$

for all $x, y \in X$, where $\sum_{n=1}^{\infty} c_n < \infty$.

The Contraction Mapping Principle has seen many other extensions particularly to mappings for which conclusions (i) and (ii) hold. In many of these instances (especially ones which reduce to Banach's principle under an appropriate renorming) it is possible to obtain (iii) as well. We give an overview of these facts below. We begin with an explicit proof of Banach's theorem (one of many) along with one of its canonical applications. We then take up many of the extensions. We conclude with a brief discussion of converses of Banach's theorem. Many other part are devoted to the limiting case $k = 1$, where in general it is possible to conclude at most that T has a (not necessarily unique) fixed point.

Before proceeding we turn to a simple example to illustrate the usefulness of the contraction mapping principle. Consider Volterra integral equation

$$u(x) = f(x) + \int_0^x F(x,y)u(y)dy, \tag{2.3.1}$$

where f and the kernel F are defined and continuous on, respectively, $[0,a]$ and $[0,a] \times [0,a]$. By employing the standard method of successive approximations it is possible to show that 2.3.1 has a unique continuous solution for any F. On the other hand, if the operator $T : C[0,a] \to C[0,a]$ is defined by setting

$$T(u(x)) = f(x) + \textstyle\int_0^x F(x,y)u(y)dy$$

then it is easy to see that for $u, v \in C[0,a]$

$$\|Tu - Tv\| \leq aK\|u - v\|$$

where $K = \sup\limits_{0\leq x,y\leq a} |F(x,y)|$ and $\|\cdot\|$ is the usual supremum norm on $C[0,a]$. Banach's contraction principle thus immediately yields a unique solution on any interval for which $aK < 1$. The problem is that in order to obtain a solution one must either restrict the size of the interval $[0,a]$ or the magnitude of the kernel F. This is not serious since in the first instance standard continuation arguments can then be applied to extend the solution.

On the other hand, A. Bielecki [14] discovered another way to remedy this problem. By assigning a new norm $\|\cdot\|_\lambda$, $\lambda > 0$, to $C[0,a]$ as follows:

$$\|u\|_\lambda = \mathop{sup}_{0\leq x\leq a} [exp(-\lambda x)|u(x)|],$$

it is possible to show that for all $u, v \in C[0,a]$,

$$\|Tu - Tv\|_\lambda \leq \tfrac{K}{\lambda}\|u - v\|_\lambda,$$

where K is defined as above. (For details, see e.g., [26].) It is then clear that for λ sufficiently large T is indeed a contraction mapping on the Banach space $(C[0, a], \|\cdot\|_\lambda)$. A direct application of the contraction mapping principle now yields the desired solution.

2.3.1 The contraction mapping principle

Definition 2.3.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ***lipschitzian*** if there is a constant $k \geq 0$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$.

The smallest number k is called the ***Lipschitz constant*** of T.

Definition 2.3.2. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be ***contraction*** if there is a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$.

Definition 2.3.3. Let $T : X \to X$ be a map of a set X into itself. A point $x \in X$ is called a ***fixed point of single mapping*** T if $Tx = x$.

Example 2.3.4. Let $T : \mathbb{R} \to \mathbb{R}$ be such that $Tx = 2x - 2$ for all $x \in \mathbb{R}$.

Since $T2 = 2 \cdot 2 - 2 = 2$, 2 is a fixed point of T.

Example 2.3.5. Let $T : \mathbb{R} \to \mathbb{R}$ be such that $Tx = x^2 + x - 1$ for all $x \in \mathbb{R}$.

Since $T1 = 1^2 + 1 - 1 = 1$ and $T(-1) = (-1)^2 - 1 - 1 = -1$, 1,-1 are fixed points of T.

Example 2.3.6. Let $T : \mathbb{R} \to \mathbb{R}$ be such that $Tx = x$ for all $x \in \mathbb{R}$.

Since $Tx = x$ for all $x \in \mathbb{R}$, x is a fixed point of T for all $x \in \mathbb{R}$.

Example 2.3.7. The map $T : \mathbb{R} \to \mathbb{R}$ given by $Tx = x + 1$ for all $x \in \mathbb{R}$ has no fixed point.

Definition 2.3.8. Let $T : X \to X$ be a map of a set X into itself. The set of all fixed points of T is denoted by $F(T)$:

$$F(T) = \{x \in X : Tx = x\}.$$

Example 2.3.9.

1. From Example 2.3.4, $F(T) = \{2\}$.

2. From Example 2.3.5, $F(T) = \{-1, 1\}$.

3. From Example 2.3.6, $F(T) = \mathbb{R}$.

4. From Example 2.3.7, $F(T) = \varnothing$.

Theorem 2.3.10 (Banach's Contraction Mapping Principle). *Let (X,d) be a complete metric space and let $T : X \to X$ be a contraction mapping. Then T has a unique fixed point z. Moreover, for each $x \in X$,*

$$\lim_{n\to\infty} T^n x = z$$

and in fact for each $x \in X$,

$$d(T^n x, z) \leq \tfrac{k^n}{1-k} d(z, Tz), \qquad n=1, 2, 3, \ldots$$

Proof. See section 2 of chapter 1 in [38]. □

Observes that in theorem 2.3.10, the existance of fixed point of T needs the assumption that $T : X \to X$ is a contraction mapping. The following theorem shows that if T is not a contraction mapping but T^N is a contraction mapping for some positive integer N, then T still has a unique fixed point.

Theorem 2.3.11 (Theorem 2.4 in Chapter 1[38]). *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping for which T^N is a contraction mapping for some positive integer N. Then T has a unique fixed point.*

Proof. By Banach's Theorem T^N has a unique fixed point z. So $T^N z = z$. However,

$$T^N T z = T^{N+1} z = T T^N z = T z,$$

so Tz is also a fixed point of T^N. Since the fixed point of T^N is unique, it follows that $Tz = z$. Also, if $Ty = y$ then $T^N y = y$ proving (again by uniqueness) that $y = z$. Thus T has a unique fixed point. □

2.3.2 Set-valued (Multivalued) contraction mappings

Banach's Contraction Mapping Principle extends nicely to set-valued mappings, a fact first noticed by S.Nadler [47] (also see [45]).

Definition 2.3.12. For any set X, we shall let $CL(X)$ (resp. $CB(X)$, $K(X)$) denote the class of all nonempty closed (resp. nonempty closed bounded, nonempty compact) subsets of X.

Definition 2.3.13. Let (X, d) be a metric space. The distance between a point $x \in X$ and a non-empty subset A of X is denoted and defined by

$$d(x, A) := \inf \{d(x, a) : a \in A\}.$$

Definition 2.3.14. Let (X, d) be a metric space. The ***Hausdorff metric*** is a function $H : X \times X \to \mathbb{R}$ such that

$$H(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

for all $A, B \subseteq X$.

Example 2.3.15. Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $d(x, y) = |x - y|$. Then $(\mathbb{R}, d)$ is metric space.

1. $H(\{1\}, [2, 3]) = 2$.
2. $H(\{0\}, [0, 1]) = 1$.
3. $H([1, 2], [5, 10]) = 8$.
4. $H([0, 1], [1, 3]) = 2$.
5. $H([0, 1] \cup [3, 4], [2, 3]) = 2$ etc.

Proposition 2.3.16. Let A, B be nonempty closed subsets of a non-empty set X and $x \in X$. If d is a metric on X and H is a Hausdorff metric on X, then:

1. If $d(x, A) = 0$, then $x \in A$.
2. If $H(A, B) = 0$, then $A = B$.

Proof.

1. Assume that $d(x, A) = 0$.

   Then $\inf\{d(x, a) : a \in A\} = 0$.

   Since A is a closed set, thus $d(x, a_0) = 0$ for some $a_0 \in A$.

   Hence $x = a_0 \in A$.

2. Assume that $H(A,B) = 0$.

   Therefore $H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} = 0$.

   Thus $\sup_{a\in A} d(a,B) = 0$ and $\sup_{b\in B} d(b,A) = 0$.

   It follows that $d(a,B) = 0$ for all $a \in A$ and $d(b,A) = 0$ for all $b \in B$.

   By 1., $a \in B$ for all $a \in A$ and $b \in A$ for all $b \in B$.

   That is $A \subseteq B$ and $B \subseteq A$. Therefore $A = B$.

$\square$

Definition 2.3.17. Let (X,d) be a metric space. A map $T : X \to CL(X)$ is called a ***contraction*** mapping if there is a positive constant $k < 1$ such that

$$H(Tx,Ty) \leq kd(x,y)$$

for all $x, y \in X$.

Definition 2.3.18. Let $T : X \to CL(X)$ be a map from X into $CL(X)$. A point $x \in X$ is called a ***fixed point of multivalued mapping*** T if $x \in Tx$.

Example 2.3.19. Let $T : \mathbb{R} \to CL(\mathbb{R})$ be such that $Tx = \{3, x-1\}$ for all $x \in \mathbb{R}$. Since $T3 = \{2,3\}$ and $3 \in \{2,3\}$, 3 is a fixed point of T.

Example 2.3.20. Let $T : \mathbb{R} \to CL(\mathbb{R})$ be such that $Tx = \{3, 2x-1, 3x-4\}$ for all $x \in \mathbb{R}$.

1. Since $T1 = \{-1,1,3\}$ and $1 \in \{-1,1,3\}$, 1 is a fixed point of T.

2. Since $T2 = \{1,2,3\}$ and $2 \in \{1,2,3\}$, 2 is a fixed point of T.

3. Since $T3 = \{3,5\}$ and $3 \in \{3,5\}$, 3 is a fixed point of T.

Example 2.3.21. Let $T : \mathbb{R} \to CL(\mathbb{R})$ be such that $Tx = [x,\infty)$ for all $x \in \mathbb{R}$. Since $x \in [x,\infty) = Tx$ for all $x \in \mathbb{R}$, x is a fixed point of T for all $x \in \mathbb{R}$.

Example 2.3.22.

1. From Example 2.3.19, $F(T) = \{3\}$.
2. From Example 2.3.20, $F(T) = \{1, 2, 3\}$.
3. From Example 2.3.21, $F(T) = \mathbb{R}$.

Theorem 2.3.23 (Nadler Contraction Principle [47]). *Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be a contraction mapping. Then T has a fixed point.*

Proof. See [34] □

We now let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping. Then the map $T_1 : X \to CB(X)$ given by $T_1x = \{Tx\}$ is also a contraction mapping. It follows from Nadler Contraction Principle that T_1 has a fixed point. That is there exists $x \in X$ such that $x \in T_1x = \{Tx\}$. Thus there exists $x \in X$ which $x = Tx$. Therefore T has a fixed point. In fact Nadler Contraction Principle generalizes the Banach's contraction principle.

In contrast to Banach's theorem, the preceding theorem does not assert that the fixed point is unique. Indeed, it need not be. In [52] an example in $\mathbb{R}^2$ is given of a multivalued contraction mapping whose values are compact and connected yet the mapping has a disconnected fixed point set. It is also show in [52] that if x is a fixed point of a multivalued contraction mapping T defined on a closed convex subset of a Banach space and if Tx is not a singleton, then T always has at least one additional fixed point distinct from x. On the other hand, Ricceri [54] has shown that the fixed point set is an absolute retract if X is a closed and convex subset of a Banach space and T has closed convex values.

There is an interesting stability result (due to T. C. Lim [43]) that holds for set-valued contractions (hence ordinary contractions as well). Such results find applications, for example, in the study of iterated function systems ([20], [25]). We begin with a technical lemma as follows.

Lemma 2.3.24 (Lemma 5.2 in [38]). *Let (X,d) be a complete metric space and let $T_1, T_2 : X \to CB(X)$ be two contraction mappings each having Lipschitz constant $k < 1$. Then*

$$H(F(T_1), F(T_2)) \leq \tfrac{1}{1-k} \sup_{x\in X} H(T_1x, T_2x).$$

Theorem 2.3.25 (Theorem 5.3 in [38]). *Let (X,d) be a complete metric space and let $T_i : X \to CB(X)$, i=$1, 2, 3$,..., be sequence of contraction mappings each having Lipschitz constant $k < 1$. If $\lim_{n\to\infty} H(T_nx, T_0x) = 0$ uniformly for $x \in X$, then*

$$\lim_{n\to\infty} H(F(T_n), F(T_0)) = 0$$

Proof. Let $\epsilon > 0$. Since $\lim_{n\to\infty} H(T_nx, T_0x) = 0$ uniformly it is possible to choose $N \in \mathbb{N}$ so that for $n \geq N$, $\sup_{x\in X} H(T_nx, T_0x) < (1-k)\epsilon$. By lemma 2.3.24, $H(F(T_n), F(T_0)) < \epsilon$ for all such n. $\square$

2.3.3 Generalized contraction mappings

There is a vast amount of literature dealing with technical extensions and generalizations of Banach's theorem. Most of these results involve a common underlying strategy. One assumes that a self-mapping T of a complete metric space X satisfies some general (and frequently quite complex) contractive type condition (C) which implies that

1. The sequence of Picard iterates of the mapping, or some related sequence is Cauchy.

2. The limit of such a sequence is always a fixed point of the mapping.

The condition (C) usually involves a relationship between the six distance

$$\{d(x,y), d(Tx,Ty), d(x,Ty), d(Tx,y), d(x,Tx), d(y,Ty)\}$$

for each pair $x, y \in X$, and continuity of the mapping may or may not be assumed. People who want to fully acquaint themselves with this literature are directed to the survey of Rhoades [53] which covers the period up through the mid-seventies, a paper by Hegedüs [28], a subsequent survey by Park and Rhoades [51], an analysis of [53] by Collaco and Silva [21], as well as references found in these sources. Further escalations in the level of complexity can be found in a paper by Park [50] and in Liu's recent observations [44] involving Park's conditions.

2.3.4 I-contraction mappings

The idea of *I-contraction* is extended from one mapping to two mappings such that the first mapping is single valued while the second mapping is multivalued.

Definition 2.3.26. Let $I : X \to X$ be a map of a metric space into itself. A mapping $T : X \to CL(X)$ is said to be ***I-contraction mapping*** if there is a positive constant $k < 1$ such that

$$H(Tx,Ty) \le k d(Ix,Iy)$$

for all $x, y \in X$.

Definition 2.3.27. Let (X,d) be a metric space, $x \in X$, $I : X \to X$ and $T : X \to CL(X)$.

1. x is a ***coincidence point of*** I ***and*** T iff $Ix \in Tx$.

2. x is a ***common fixed point of*** I ***and*** T iff $x = Ix \in Tx$.

Example 2.3.28. Let $I : \mathbb{R} \to \mathbb{R}$, $T : \mathbb{R} \to CL(\mathbb{R})$ be such that $Ix = 1 - x^2$ and $Tx = [0, x^2]$ for all $x \in \mathbb{R}$. Since $I\frac{1}{\sqrt{2}} = \frac{1}{2} \in [0, \frac{1}{2}) = T\frac{1}{2}$, $\frac{1}{2}$ is a coincidence point of I and T.

Example 2.3.29. Let $I : \mathbb{R} \to \mathbb{R}$, $T : \mathbb{R} \to CL(\mathbb{R})$ be such that $Ix = x + 1$ and $Tx = [x, \infty)$ for all $x \in \mathbb{R}$. Since $Ix = x + 1 \in [x, \infty) = Tx$ for all $x \in \mathbb{R}$, x is a coincidence points of I and T for all $x \in \mathbb{R}$.

Example 2.3.30. Let $I : \mathbb{R} \to \mathbb{R}$, $T : \mathbb{R} \to CL(\mathbb{R})$ be such that $Ix = x$ and $Tx = \{3, 2x - 1, 3x - 4\}$ for all $x \in \mathbb{R}$.

1. Since $T1 = \{-1, 1, 3\}$ and $I1 = 1 \in \{-1, 1, 3\}$, 1 is a common fixed point of I and T.

2. Since $T2 = \{1, 2, 3\}$ and $I2 = 2 \in \{1, 2, 3\}$, 2 is a common fixed point of I and T.

3. Since $T3 = \{3, 5\}$ and $I3 = 3 \in \{3, 5\}$, 3 is a common fixed point of I and T.

Example 2.3.31. Let $X = [1, \infty)$, $I : X \to X$, $T : X \to CL(X)$ be such that $Ix = x^2$ and $Tx = [1, x+1]$ for all $x \in \mathbb{R}$. Since $I1 = 1 \in [1, 2] = T1$, 1 is a common fixed point of I and T.

Definition 2.3.32. Let (X, d) be a metric space and let $I : X \to X$, $T : X \to CL(X)$.

1. The set of all coincidence points of I and T is denoted by $C(I, T)$:

$$C(I, T) = \{x \in D : Ix \in Tx\}.$$

2. The set of all common fixed points of I and T is denoted by $F(I, T)$:

$$F(I, T) = \{x \in D : x = Ix \in Tx\}.$$

Example 2.3.33.

1. From Example 2.3.28, $C(I,T) = \{\frac{1}{2}\}$.

2. From Example 2.3.29, $C(I,T) = \mathbb{R}$.

3. From Example 2.3.30, $F(I,T) = \{1,2,3\}$.

4. From Example 2.3.31, $F(I,T) = \{1\}$.

Definition 2.3.34 ([32]). Let (X,d) be a metric space and let $I : X \to X$, $T : X \to CL(X)$. A mapping I is said to be ***T-weakly commuting*** at $x \in X$ iff $IIx \in TIx$.

Example 2.3.35. Let $I : \mathbb{R} \to \mathbb{R}$ and $T : \mathbb{R} \to CL(\mathbb{R})$ such that $Ix = x^2$, $Tx = [0,x]$.

1. We have that $II0 = I0 = 0$ and $TI0 = T0 = \{0\}$. It follows that $II0 \in TI0$. Thus I is T-weakly commuting at 0.

2. We have that $II1 = I1 = 1$ and $TI1 = T1 = [0,1]$. It follows that $II1 \in TI1$. Thus I is T-weakly commuting at 1.

Example 2.3.36. Let $I : \mathbb{R} \to \mathbb{R}$ and $T : \mathbb{R} \to CL(\mathbb{R})$ be such that $Ix = x+1$ and $Tx = [x, x+1]$. Then we have that $IIx = I(x+1) = (x+1)+1 = x+2$ and $TIx = T(x+1) = [x+1, (x+1)+1] = [x+1, x+2]$. So $IIx \in TIx$ for all $x \in \mathbb{R}$. Thus I is T-weakly commuting at x for all $x \in \mathbb{R}$.

Example 2.3.37. Let $X = [1,\infty)$, $I : X \to X$ and $T : X \to CL(X)$ be such that $Ix = 2x$ and $Tx = [1, 2x+1]$. Then we have that $IIx = I(2x) = 2(2x) = 4x$ and $TIx = T(2x) = [1, 2(2x)+1] = [1, 4x+1]$. So $IIx \in TIx$ for all $x \in D$. Thus I is T-weakly commuting at x for all $x \in X$.

We extend the condition in Nadler Contraction Principle from $T(X) \subseteq X$ to $T(X) \subseteq I(X)$ and from T is contraction to T is I-contraction. We expect to get the preliminary result that I and T have a common fixed point. But we get that I and T have a coincidence point.

Theorem 2.3.38. *Let (X,d) be a complete metric space and let* $I : X \to X$, $T : X \to CL(X)$. *If* T *is I-contraction mapping and* $T(X) \subseteq I(X)$, *then* I *and* T *have a coincidence point.*

Afterward we try to add the following conditions in the above theorem.

1. $IIv = Iv$ at v for some $v \in C(I,T)$.

2. I is T-weakly commuting at v for some $v \in C(I,T)$.

Consequently we obtain that I and T have a common fixed point.

Theorem 2.3.39. *Let (X,d) be a complete metric space and let* $I : X \to X$, $T : X \to CB(X)$. *If* T *is I-contraction mapping,* $T(X) \subseteq I(X)$, $IIv = Iv$ *at* v *for some* $v \in C(I,T)$, *and* I *is* T*-weakly commuting at* v *for some* $v \in C(I,T)$, *then* I *and* T *have a common fixed point.*

Remark 2.3.40. If we assume that $I : X \to X$ in Theorem 2.3.39 is the identity mapping, then Theorem 2.3.39 is Nadler Contraction Principle. So Theorem 2.3.39 generalizes Nadler Contraction Principle.

2.3.5 Generalized I-contraction mappings

In 2007 Al-Thagafi and Shahzad establish new mapping which used the relation between five distance

$$\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), d(Ix, Ty), d(Iy, Tx)\}$$

which generalizes I-contraction.

Definition 2.3.41 ([1]). Let (X, d) be a metric space and let $I : X \to X$. A mapping $T : X \to CL(X)$ is said to be ***generalized I-contraction mapping*** if there is a positive constant $k < 1$ such that for all $x, y \in X$

$$H(Tx, Ty) \leq k \max\{d(Ix, Iy), d(Ix, Tx), d(Iy, Ty), \tfrac{1}{2}(d(Ix, Ty) + d(Iy, Tx))\}.$$

Afterward, Al-Thagafi and Shahzad extend the conditions in many theorems which generalizes Banach's contraction principle , Nadler Contraction Principle, and many theorems.

Theorem 2.3.42 (Theorem 2.1 in [1]). *Let (X,d) be a metric space and let* $I : X \to X$, $T : X \to CL(X)$. *If* T *is generalized I-contraction mapping,* $\overline{T(X)} \subseteq I(X)$ *and* $\overline{T(X)}$ *is complete, then* $C(I, T) \neq \varnothing$.

Moreover, if $IIv = Iv$ *and* I *is* T*-weakly commuting at* v *for some* $v \in C(I, T)$, *then* $F(I, T) \neq \varnothing$.

Example 2.3.43 ([1]). Let $X = [0, 1)$ be the usual metric space. Define $Ix = x^2$ and $Tx = [0, \frac{2}{3}x^2]$ for all $x \in X$. Then all hypotheses of Theorem 2.3.42 are satisfied. Note that $0 \in C(I, T)$. Note also that Nadler's theorem cannot be used.

2.4 Nonexpansive mappings

Mapping which are defined on metric spaces and which do not increase distances between pairs of point and their images are called nonexpansive. Thus an abstract metric space is all that is needed to define the concept. At the same time, the more interesting results seem to require some notion of topology; more specifically a topology which assures that closed metric balls are compact. This is not a serious limitation, however, because many spaces which arise naturally in functional analysis possess such topologies; most notably the weak and weak* topologies in Banach spaces.

2.4.1 Nonexpansive Mappings

Definition 2.4.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a ***nonexpansive mapping*** if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$.

Definition 2.4.2. A nonempty subset D of a normed space $(X, \| \ \|)$ is a ***convex*** iff $kx + (1-k)y \in D$ for all $x, y \in D$ and for all $k \in [0, 1]$.

Recognition of fixed point theory for nonexpansive mapping as a noteworthy avenue of research almost surely dates from the 1965 publication of likely the most widely known result in the theory.

Theorem 2.4.3. *If K is bounded closed and convex subset of a uniformly convex Banach space X and if $T : K \to K$ is nonexpansive (that is, $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in K$), then T has a fixed point.*

The above theorem was proved independently by F. Browder [17] and D. Göhde [27] in the form stated above, and by W. Kirk [35] in a more general form. (Slightly earlier Browder obtained the Hilbert space version of this result ([16]) as a by-product of the fact that a mapping T in a Hilbert space is nonexpansive if and only if the mapping $I - T$ is monotone) Browder and Kirk used the same line of argument - indeed, one which in fact yields a more general results - while the proof of Göhde relies on properties essentially unique to uniformly convex spaces. As a result, Göhde's proof reveals that fixed points in this setting may be obtained as weak limits of sequences of 'approximate' fixed point.

2.4.2 Set-valued (Multivalued) nonexpansive mappings

Definition 2.4.4. Let (X, d) be a metric space. A mapping $T : X \to CL(X)$ is called a ***nonexpansive mapping*** if

$$H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$.

The principle result in this section is due to T. C. Lim [42].

Theorem 2.4.5. *Let X be uniformly convex Banach space, let K be a bounded closed convex subset of X, and let $T : K \to K(K)$, where K(K) denotes the collection of all nonempty compact subset of K. If T is nonexpansive relative to the Hausdorff metric on K(K), then T has a fixed point (there exists a point $x \in K$ such that $x \in Tx$).*

The proof of theorem 2.4.5 uses the so-called asymptotic center method which was discussed in the section 4 of chapter 3 in [38].

Earlier versions of Theorem 2.4.5 were obtained by Markin [46] in Hilbert spaces, by Browder [18] for spaces possessing weakly continuous duality mappings. In each of these instances, the mapping is assumed to have compact convex values. It has been shown in [37] that under this additional assumption about the valued of T Theorem 2.4.5 holds in an even wider class of spaces.

Theorem 2.4.6 (Theorem 5.3 in [38]). *Let X be a Banach space and let K be a bounded closed convex subset of X. If K has the property that the asymptotic* *center of each sequence in K (Relative to K) is nonempty and compact and* T : $K \to CC(K)$ *is nonexpansive, where CC(K) denotes the collection of all nonempty compact convex subset of K endowed with the Hausdorff metric, then T has a fixed point (there exists a point* $x \in K$ *such that* $x \in Tx$*).*

Spaces which satisfy the assumptions of the above theorem include, for example, all the k-uniformly rotund spaces of Sullivan [62], the initial step in the proof involves showing that every sequence in K has a subsequence with the property that each of its subsequences has the same asymptotic radius and asymptotic center. The proof also has a topological ingredient in that it invokes the Bohnenblust-Karlin extension ([15]) of a well known fixed point theorem of Kakutani [31] for upper semicontinuous set-valued mapping.

Definition 2.4.7 ([49]). A Banach space X satisfies ***Opial's condition*** if for each sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$, then the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| \leq \lim_{n\to\infty} \inf \|x_n - y\|$$

holds for all $y \neq x$.

Definition 2.4.8. Let M be subset of a normed space X. The mapping $T : M \to CL(X)$ is said to be ***demiclosed at 0*** if for every sequence $\{x_n\}$ in M and $\{y_n\}$ in X with $y_n \in Tx_n$ such that $x_n \rightharpoonup x$ and $y_n \to 0$, then $0 \in Tx$.

As with the demiclosedness principle, Theorem 2.4.5 also holds for spaces satisfying Opial's condition. This fact is due to E. Lami Dozo [24].

Theorem 2.4.9 ([24]). *Let K be a weakly compact convex subset of a Banach space X which satisfies Opial's condition and let* $T : K \to K(K)$ *be a nonexpansive. Then there exists* $x \in K$ *such that* $x \in Tx$.

2.4.3 I-nonexpansive mappings

Definition 2.4.10. Let (X, d) be a metric space. A mapping $T : X \to CL(X)$ is called a ***I-nonexpansive mapping*** if

$$H(Tx, Ty) \leq d(Ix, Iy)$$

for all $x, y \in X$.

Definition 2.4.11. Let D be a nonempty subset of normed space $(X, \| \|)$ and $q \in D$. If $kx + (1 - k)q \in D$ for all $x \in D$ and all $k \in [0, 1]$, then D is called ***q-starshaped***.

Proposition 2.4.12. If D is a convex, then D is q-starshaped.

Proof. Since D is a convex, so $kx+(1-k)y \in D$ for all $x, y \in D$ and for all $k \in [0, 1]$. Take $y = q \in D$. Thus $kx + (1 - k)q \in D$ for all $x \in D$ and all $k \in [0, 1]$. Therefore D is q-starshaped. $\square$

Remark 2.4.13. Since the class of q-starshaped is a subclass of a class of convex, if we represent convex by q-starshaped then the theorem is general.

Definition 2.4.14. Let D be a subset of a normed space $(X, \| \|)$. A function $I : D \to D$ is ***affine*** iff it has the following two properties:

1. D is a convex.

2. $I(kx + (1 - k)y) = kIx + (1 - k)Iy$ for all $x, y \in D$ and for all $k \in [0, 1]$.

Definition 2.4.15. Let D be a subset of a normed space $(X, \| \|)$. A function $I : D \to D$ is ***q-affine*** iff it has the following two properties:

1. D is q-starshaped.

2. $I(kx + (1-k)q) = kI(x) + (1-k)q$ for all $x \in D$ and for all $k \in [0,1]$.

Latif and Tweddle [41] establish some coincidence point theorems for I-nonexpansive mapping using the commutativity condition of mapping. Recently Shahzad and Hussain obtain some coincidence point results and prove some common fixed point theorems for a more general class of noncommuting mapping which they assume that I is T-weakly commuting at v for some $v \in C(I,T)$.

Theorem 2.4.16 (Theorem 2.7 in [61]). *Let X be a nonempty weakly compact and q-starshaped subset of a Banach space X, $I : X \to X$ such that $I(X) = X$. Let $T : X \to CL(X)$ be an I-nonxpansive mapping and such that one of the following two conditions is satisfied:*

(a) *$(I - T)$ is demiclosed at 0;*

(b) *I is weakly continuous, T is compact-valued and X satisfies Opial's conditions.*

If I is T-weakly commuting and IIv=Iv for $v \in C(I,T)$, then $F(I,T) \neq \varnothing$.

2.5 Random coincidence and common random fixed points

Definition 2.5.1. Let Ω be a set. A collection Σ of subsets of Ω is called a ***sigma algebra on*** Ω iff it satisfies the following properties

1. $\Sigma \neq \varnothing$
2. If $\omega \in \Sigma$, then $\omega^c \in \Sigma$.
3. If $\omega_n \in \Sigma$ for all $n \geq 1$, then $\bigcup\limits_{n=1}^{\infty} \omega_n \in \Sigma$.

Definition 2.5.2. A pair (Ω, Σ), where Σ is a sigma algebra on Ω, is called a ***measurable space***.

Definition 2.5.3. Let D be a nonempty subset of a normed space $(X, \| \ \|)$ and (Ω, Σ) be a measurable space. Let $\xi : \Omega \to D$ and $S : \Omega \to CL(D)$.

1. A function ξ is ***measurable*** iff $\xi^{-1}(V) \in \Sigma$ for every open subset V of D.

2. A function S is ***measurable*** iff $S^{-1}(V) \in \Sigma$ for every open subset V of D, where $S^{-1}(V) = \{\omega \in \Omega : S(\omega) \cap V \neq \varnothing\}$.

3. A function ξ is a ***measurable selector*** of S iff ξ is measurable and $\xi(\omega) \in S(\omega)$ for all $\omega \in \Omega$.

Definition 2.5.4. Let D be a nonempty subset of a normed space $(X, \| \ \|)$ and (Ω, Σ) be a measurable space. Let $I : \Omega \times D \to D$ and $T : \Omega \times D \to CL(D)$.

1. For some fixed $\omega \in \Omega$, then $I(\omega, \cdot)$ is a map of D into itself such that

$$I(\omega, \cdot)(x) = I(\omega, x) \text{ for all } x \in D.$$

2. For some fixed $\omega \in \Omega$, then $T(\omega, \cdot)$ is a map from D into $CL(D)$ such that

$$T(\omega, \cdot)(x) = T(\omega, x) \text{ for all } x \in D.$$

3. For some fixed $x \in D$, then $I(\cdot, x)$ is a map from Ω into D such that

$$I(\cdot, x)(\omega) = I(\omega, x) \text{ for all } \omega \in \Omega.$$

4. For some fixed $x \in D$, then $T(\cdot, x)$ is a map from Ω into $CL(D)$ such that

$$T(\cdot, x)(\omega) = T(\omega, x) \text{ for all } \omega \in \Omega.$$

Definition 2.5.5. Let D be a nonempty subset of a normed space $(X, \| \|)$ and (Ω, Σ) be measurable space. Let $\xi : \Omega \to D$, $I : \Omega \times D \to D$ and $T : \Omega \times D \to CL(D)$. Then

1. A mapping I is a ***random operator*** iff $I(\cdot, x)$ is measurable for all $x \in D$.

2. A mapping T is a ***random operator*** iff $T(\cdot, x)$ is measurable for all $x \in D$.

3. A function ξ is a ***deterministic fixed point*** of a random operator I iff $\xi(\omega) = I(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

4. A function ξ is a ***deterministic fixed point*** of a random operator T iff $\xi(\omega) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

5. A function ξ is a ***deterministic coincidence point*** of random operator I and T iff $I(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

6. A function ξ is a ***deterministic common fixed point*** of random operator I and T iff $\xi(\omega) = I(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

7. A function ξ is a ***random fixed point*** of a random operator T iff ξ is measurable and $\xi(\omega) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

8. A function ξ is a ***random fixed point*** of a random operator I iff ξ is measurable and $\xi(\omega) = I(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

9. A function ξ is a ***random coincidence point*** of a random operator I and T iff ξ is measurable and $I(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

10. A function ξ is a ***common random fixed point*** of a random operator I and T iff ξ is measurable and $\xi(\omega) = I(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for all $\omega \in \Omega$.

Definition 2.5.6. Let D be a nonempty subset of a normed space $(X, \| \|)$ and (Ω, Σ) be measurable space. Let $I : \Omega \times D \to D$ and $T : \Omega \times D \to CL(D)$.

1. The set of all random fixed points of a random operator I is denoted by $RF(I)$.

2. The set of all random fixed points of a random operator T is denoted by $RF(T)$.

3. The set of all random coincidence points of random operators I and T is denoted by $RC(I, T)$.

4. The set of all common random fixed points of random operators I and T is denoted by $RF(I, T)$.

Definition 2.5.7. Let D be a nonempty subset of a normed space $(X, \| \|)$ and (Ω, Σ) be measurable space. Let $I : \Omega \times D \to D$ and $T : \Omega \times D \to CL(D)$.

1. A random operator I is ***nonexpansive*** iff

   $I(\omega, \cdot)$ is nonexpansive for all $\omega \in \Omega$.

2. A random operator T is ***nonexpansive*** iff

   $T(\omega, \cdot)$ is nonexpansive for all $\omega \in \Omega$.

3. A random operator T is ***I-nonexpansive*** iff

   $T(\omega, \cdot)$ is I-nonexpansive for all $\omega \in \Omega$.