

**ON BINARY QUADRATIC DIOPHANTINE EQUATIONS WITH
CERTAIN CLASS OF COEFFICIENTS**

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**A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE
(APPLIED MATHEMATICS)
FACULTY OF GRADUATE STUDIES
MAHIDOL UNIVERSITY
2012**

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Thesis
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ACKNOWLEDGEMENTS

I would like to dedicate this page to show my gratitude to everyone who helps me complete this thesis. First of all, I would like to express my thank to my advisor, Dr. Aram Tangboonduangjit, for many good suggestions, valuable guidance and comments. He also encourages and supports me in various ways. He devotes himself in helping me finish this thesis. I am very grateful for his time and knowledge.

Besides my advisor, my deepest gratitude goes to my co-advisor, Dr. Som-sak Orankitjaroen for his suggestion and time. He always reminds me many important mistakes in the process. In addition, another member of thesis committee whom I cannot forget to thank is Dr. Keng Wiboontan. He spends his worthy time to give me comments and suggestions.

Department of Mathematics, Mahidol University, provides me the necessary facilities, so that I am able to accomplish this thesis on time. For financial support during my study at Mahidol University, I must show my gratitude to the Development and Promotion of the Gifted in Science and Technology Project scholarship (DPST) under the Institute for the Promotion of Teaching Science and Technology (IPST). I am also very thankful to my friends and all members of the Department of Mathematics for their friendship and unforgettable experience.

Finally, I extend my feeling to my family for their love and caring. They always believe in me even which gave me the strength to move forward. Moreover, they have been there for me with love, understanding and support whenever I need them.

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M.Sc. (APPLIED MATHEMATICS)

THESIS ADVISORY COMMITTEE: ARAM TANGBOONDUANGJIT, Ph.D.,
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We consider binary quadratic diophantine equations of the form $x^2 - (s^2 + s)y^2 + A_n(s)x + B_n(s)y = 0$ for an integer $s \geq 1$ and $x^2 - (t^2 - t)y^2 + C_n(t)x + D_n(t)y = 0$ for an integer $t \geq 2$ where the coefficients $A_n(s), B_n(s), C_n(t), D_n(t)$ of the first degree terms can be expressed as a product of matrices of integers. This work aims to determine the general solutions of these equations by using the method of substitution to reduce the equations into the form of Pell's equations whose general solutions, in turn, can be obtained by some standard methods.

KEY WORDS : BINARY QUADRATIC DIOPHANTINE/ PELL'S EQUATION.

31 pages

สมการไดโอแฟนไทน์สองตัวแปรดีกรีสองที่มีชุดสัมประสิทธิ์เฉพาะ
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บทคัดย่อ

พิจารณาสมการไดโอแฟนไทน์สองตัวแปรดีกรีสองที่อยู่ในรูป $x^2 - (s^2 + s)y^2 + A_n(s)x + B_n(s)y = 0$ สำหรับจำนวนเต็ม $s \geq 1$ และ $x^2 - (t^2 - t)y^2 + C_n(t)x + D_n(t)y = 0$ สำหรับจำนวนเต็ม $t \geq 2$ ซึ่งสัมประสิทธิ์ $A_n(s), B_n(s), C_n(t), D_n(t)$ ของเทอมที่มีดีกรีเท่ากับหนึ่งสามารถเขียนในรูปของผลคูณของเมทริกซ์ที่มีสมาชิกเป็นจำนวนเต็ม เราสามารถหาผลเฉลยทั่วไปของสมการข้างต้นโดยวิธีการเปลี่ยนรูปของสมการเป็นสมการของเพลล์และวิธีการพื้นฐาน

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CHAPTER I

INTRODUCTION

Number theory is a branch of mathematics with strong and important connection with other branches of mathematics for century. Application of number theory can be used to solve some challenging problems. This has made number theory attractive and important until the present time.

We often encounter the problem about an equation whose unknowns belong to the set of integers called Diophantine equation, in honor of Diophantus, the Greek mathematician who first studied the subject. The field of mathematics devoted to the study of this kind of equations is called Diophantine analysis.

Although Diophantine equation has been studied and developed, most of its problems are still not completely solved. To determine whether an individual Diophantine equation possesses a solution has been one of the most challenging problems in number theory. Moreover, the question of whether there is a systematic approach to solve any Diophantine equation was posted as the 10th problem in the famous list of Hilbert's problems in 1900. The question was resolved in 1970 by Russian mathematician named Yuri Matiyasevich [3] who proved the negative answer to the question. This means that each class of Diophantine equations or even individual one may have completely different methods of finding the solutions.

Whenever we study about the Diophantine equation, we have to answer these five typical problems :

1. Are there any solutions ?
2. Are there any solutions beyond some that are found by inspection ?
3. Are there finitely or infinitely many solutions ?
4. Can all solutions be found in theory ?
5. Can one in practice compute a full list of solutions ?

For example, considering polynomial Diophantine equations with integer coefficients, we see that only low degree equations such as linear equations of the form

$ax + by = c$ where a, b , and c are integers or the second degree equation of the form $x^2 + y^2 = z^2$ can be completely solved. One of the most famous Diophantine equation of all time is the Fermat's last theorem stated that there is no integral solution of the equation $x^n + y^n = z^n$ for $n \geq 3$. This theorem was proved by Andrew Wiles in 1995. However, when the degree of the equation or the number of unknowns is higher, the method of obtaining the solutions may be harder as well.

Objectives and Goals

The binary quadratic Diophantine equation has the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where a, b, c, d, e, f are integers with $a^2 + b^2 + c^2 \neq 0$. We can only ensure the existence of the solution or determine only a particular solution of some forms of such equation. In this paper, we concentrate on the certain forms of binary quadratic Diophantine equation, namely

$$x^2 - (s^2 + s)y^2 + A_n(s)x + B_n(s)y = 0, \quad (1.1)$$

$$x^2 - (t^2 - t)y^2 + C_n(t)x + D_n(t)y = 0 \quad (1.2)$$

for integers $s \geq 1$ and $t \geq 2$. Tekcan and Ozkoc [5] are able to give the general solutions of (1.1) and Chandoul [2] of (1.2) with a certain class of coefficients of x and y . We extend the results of their works by generalizing the coefficients of the first degree terms. In particular, we give the general solutions of equations (1.1) and (1.2) whose coefficients $A_n(s), B_n(s), C_n(t)$, and $D_n(t)$ can be generated by products of integer matrices for each nonnegative integer n as follows:

$$\begin{bmatrix} B_n(s) \\ A_n(s) \end{bmatrix} = \begin{bmatrix} 0 & 2(s^2 + s) \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2s + 1 & 2(s^2 + s) \\ 2 & 2s + 1 \end{bmatrix}^n \begin{bmatrix} 2s + 1 \\ 2 \end{bmatrix} \quad (1.3)$$

and

$$\begin{bmatrix} D_n(t) \\ C_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 2(t^2 - t) \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2t - 1 & 2(t^2 - t) \\ 2 & 2t - 1 \end{bmatrix}^n \begin{bmatrix} 2t - 1 \\ 2 \end{bmatrix}. \quad (1.4)$$

Thesis Outline

This thesis includes 5 chapters. An introduction is given in Chapter 1. The objectives of the thesis are also given in this chapter. In Chapter 2, the basic knowledge and theoretical background used and applied in the thesis will be presented. The main results are stated and proved in Chapter 3. In Chapter 4, the numerical examples of the theorems in Chapter 3 are given in the table form programmed by MATHEMATICA 7 codes which are included in the appendix. In the last chapter, the conclusion and discussion are presented.

CHAPTER II

BASICS AND PRELIMINARIES

This chapter is about basic knowledge and theoretical backgrounds that may help analyze results and find the solutions of the binary quadratic Diophantine equation (1.1) and (1.2) in this thesis.

2.1 Basic Knowledge and Theoretical Backgrounds

The section is broken into two parts. The first subsection mentions about basic definitions and notations. The second subsection gives some important theorems.

2.1.1 Definitions and Notations about Continued Fraction and Pell's equation

We recall some basic definitions and notations about continued fraction and Pell's equation.

Definition 2.1. Let a_0, a_1, \dots, a_n and b_0, b_1, \dots, b_m be positive integers. The *periodic continued fraction* is defined by

$$\langle a_0, a_1, \dots, a_n, \overline{b_0, b_1, \dots, b_{m-1}} \rangle = a_0 + \frac{1}{a_1 + \frac{1}{\dots a_n + \frac{1}{b_0 + \frac{1}{b_1 + \frac{1}{\dots b_{m-1} + \frac{1}{b_0 + \frac{1}{b_1 + \frac{1}{\dots b_{m-1} + \frac{1}{\dots}}}}}}}}}}$$

where the bar indicates that b_0, b_1, \dots, b_{m-1} repeat infinitely many times. The smallest integer m is called the length of period, or more simply, the period.

Definition 2.2. Let a_0, a_1, \dots be a sequence of integers, all positive except possibly a_0 . Define sequences $\{h_n\}$ and $\{k_n\}$ as follows:

$$\begin{aligned} h_{-2} &= 0, h_{-1} = 1 \quad \text{and} \quad h_n = a_n h_{n-1} + h_{n-2}, \\ k_{-2} &= 1, k_{-1} = 0 \quad \text{and} \quad k_n = a_n k_{n-1} + k_{n-2} \end{aligned}$$

for $n \geq 0$. In particular, $\langle a_0, a_1, \dots \rangle = \frac{h_n}{k_n}$ is called the n -th convergent of continued fraction $\langle a_0, a_1, \dots \rangle$.

Definition 2.3. The Diophantine equation of the form $x^2 - dy^2 = 1$ is known as Pell's equation where d is a positive integer that is not a perfect square. A positive solution (x_1, y_1) is said to be the least positive solution or fundamental solution if $x_1 < X$ and $y_1 < Y$ for any other positive solution (X, Y) of the equation.

Throughout the following subsection, d is assumed to be a positive integer that is not a perfect square.

2.1.2 Related Theorems in Pell's equation

A technique for treating Pell's equation is to consider the continued fraction expansion of \sqrt{d} as the following propositions have illustrated.

Proposition 2.4. *The continued fraction expansion of \sqrt{d} is a periodic continued fraction of the form $\langle a_0, \overline{a_1, a_2, \dots, a_2, a_1, 2a_0} \rangle$ where $a_0 = \lfloor \sqrt{d} \rfloor$.*

Proposition 2.5. *Let the equations $x^2 - dy^2 = \pm 1$ be given and let m be the period of \sqrt{d} . Then all positive solutions of such equations are terms in the sequences $\{h_n\}$ and $\{k_n\}$ classified into cases as follows:*

- *If m is even, then the equation $x^2 - dy^2 = -1$ has no solution and the equation $x^2 - dy^2 = 1$ has solutions given by $x = h_{nm-1}$ and $y = k_{nm-1}$ for $n = 1, 2, 3, \dots$*
- *If m is odd, then the equation $x^2 - dy^2 = -1$ has solutions given by $x = h_{nm-1}$ and $y = k_{nm-1}$ for $n = 1, 3, 5, \dots$, and the equation $x^2 - dy^2 = 1$ has solutions given by $x = h_{nm-1}$ and $y = k_{nm-1}$ for $n = 2, 4, 6, \dots$*

Proposition 2.6. *Let (x_1, y_1) be a fundamental solution of $x^2 - dy^2 = 1$. Then all positive solutions (x_n, y_n) are given by*

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

for $n \geq 1$. Equivalently, (x_n, y_n) can be obtained from the recurrence relations

$$x_{n+1} = x_1x_n + y_1y_nd,$$

$$y_{n+1} = x_1y_n + y_1x_n,$$

or, expressed in the matrix form,

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 d \\ y_1 & x_1 \end{bmatrix}^n \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Note that for an arbitrary integer $N \neq \pm 1$ the equation $x^2 - dy^2 = N$ may not be solved completely. In fact, there is no assurance as to whether all positive solutions can be obtained by the method of expanding the continued fraction alone. However, as the following proposition shows, we are able to obtain infinitely many solutions of $x^2 - dy^2 = N$ provided that one solution is known.

Proposition 2.7. *If (x_1, y_1) is a fundamental solution of $x^2 - dy^2 = 1$ and (x', y') is one of the solutions of $x^2 - dy^2 = N$, then (x_n, y_n) satisfying*

$$x_n + y_n \sqrt{d} = (x' + y' \sqrt{d})(x_1 + y_1 \sqrt{d})^n,$$

or, expressed in the matrix form,

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x' & y' d \\ y' & x' \end{bmatrix} \begin{bmatrix} x_1 & y_1 d \\ y_1 & x_1 \end{bmatrix}^n \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

is also a positive solution of $x^2 - dy^2 = N$.

The proofs of Propositions 2.4 - 2.7 can be found in [1] and [4].

CHAPTER III

DIOPHANTINE ANALYSIS

This chapter is the most important in this thesis. The Diophantine analysis of the binary quadratic Diophantine equation (1.1) and (1.2) is the main topic in the chapter. First, the literature review is including. The inspiration how the problem is solved and analyzed. Then the results and the methods to solve the problems will be discussed throughly in the last two sections.

3.1 Literature Review

We summarize the previous works that related to this thesis. Those researches inspire the method to solve binary quadratic Diophantine equations (1.1) and (1.2). We will point out some concepts that will be applied in this thesis.

3.1.1 Main Concepts in Previous Works

Tekcan and Ozkoc [5] consider integer solutions of (1.1) where $A(s) = -(4s + 2)$ and $B(s) = 4s^2 + 4s$. The form of the this equation is

$$x^2 - (s^2 + s)y^2 - (4s + 2)x + (4s^2 + 4s)y = 0. \quad (3.1)$$

Chandoul [2] is able to give the solutions of (1.2) with a certain class of coefficients $C(t) = -(16t - 4)$ and $D(t) = 16t^2 + 16$. The form of the this equation is

$$x^2 - (t^2 - t)y^2 - (16t - 4)x + (16t^2 + 16)y = 0. \quad (3.2)$$

They proceed in the similar manner of each other's works to obtain the solution of the binary quadratic Diophantine equations. First, they transform x and y by the transformation $x = u + \alpha$ and $y = v + \beta$, respectively. Then by the algebraic manipulation, the binary quadratic Diophantine equation can be written as Pell's equations

as follows

$$u^2 - (s^2 + s)v^2 = 1, \quad (3.3)$$

$$u^2 - (t^2 - t)v^2 = 1, \quad (3.4)$$

where $\sqrt{s^2 + s}$ and $\sqrt{t^2 - t}$ are not perfect square.

Instead of dealing with the complicated equations in (3.1) and (3.2), they figure out the easier ones, i.e., Pell's equations (3.3) and (3.4). Now it is obvious to see that the coefficients must have some special properties to be able to be reduced the Diophantine equations into Pell's equations after transformations and simplifications.

We extend the results of their works by generalizing the coefficients of the first degree terms. In particular, we give the general solutions of equations (1.1) and (1.2) whose coefficients $A(s)$, $B(s)$, $C(t)$, and $D(t)$ are of some specific forms.

3.2 The Resolution

In this section, we illustrate the main idea of how to obtain the general solutions of equations (1.1) and (1.2) in detail. In essence we reduce equations into the form of Pell's equations by appropriate substitutions and then apply related theorems from the previous chapter to solve the equations. Since $s^2 + s$, and $t^2 - t$ are not perfect square, we can expand their square roots as periodic continued fractions as follows:

$$\begin{aligned} \sqrt{s^2 + s} &= s + (\sqrt{s^2 + s} - s) \\ &= s + \frac{1}{\frac{1}{\sqrt{s^2 + s} - s}} \\ &= s + \frac{1}{\frac{1}{\sqrt{s^2 + s} - s} \times \frac{\sqrt{s^2 + s} + s}{\sqrt{s^2 + s} + s}} \\ &= s + \frac{1}{\frac{\sqrt{s^2 + s} + s}{s}} \\ &= s + \frac{1}{\frac{2s + \sqrt{s^2 + s} - s}{s}} \\ &= s + \frac{1}{2 + \frac{\sqrt{s^2 + s} - s}{s}} \\ &= s + \frac{1}{2 + \frac{\sqrt{s^2 + s} - s}{s} \times \frac{\sqrt{s^2 + s} + s}{\sqrt{s^2 + s} + s}} \end{aligned}$$

$$\begin{aligned}
 &= s + \frac{1}{2 + \frac{1}{\sqrt{s^2+s+s}}} \\
 &= s + \frac{1}{2 + \frac{1}{2s + \sqrt{s^2+s-s}}},
 \end{aligned}$$

$$\begin{aligned}
 \sqrt{t^2-t} &= t-1 + (\sqrt{t^2-t} - (t-1)) \\
 &= t-1 + \frac{1}{\frac{1}{\sqrt{t^2-t} - (t-1)}} \\
 &= t-1 + \frac{1}{\frac{1}{\sqrt{t^2-t} - (t-1)} \times \frac{\sqrt{t^2-t} + (t-1)}{\sqrt{t^2-t} + (t-1)}} \\
 &= t-1 + \frac{1}{\frac{\sqrt{t^2-t} + (t-1)}{t-1}} \\
 &= t-1 + \frac{1}{\frac{2(t-1) + \sqrt{t^2-t} - (t-1)}{t-1}} \\
 &= t-1 + \frac{1}{2 + \frac{\sqrt{t^2-t} - (t-1)}{t-1}} \\
 &= t-1 + \frac{1}{2 + \frac{\sqrt{t^2-t} - (t-1)}{t-1} \times \frac{\sqrt{t^2-t} + (t-1)}{\sqrt{t^2-t} + (t-1)}} \\
 &= t-1 + \frac{1}{2 + \frac{1}{\sqrt{t^2-t} + (t-1)}} \\
 &= t-1 + \frac{1}{2 + \frac{1}{2(t-1) + \sqrt{t^2-t} - (t-1)}}.
 \end{aligned}$$

Thus

$$\sqrt{s^2+s} = \langle s, \overline{2}, 2s \rangle, \tag{3.5}$$

$$\sqrt{t^2-t} = \langle t-1, \overline{2}, 2(t-1) \rangle \tag{3.6}$$

for integers $s \geq 1$ and $t \geq 2$. Next we write $x = u + \alpha$ and $y = v + \beta$ for some α and β and substitute into (1.1) and (1.2) to get

$$(u + \alpha)^2 - (s^2 + s)(v + \beta)^2 + A_n(s)(u + \alpha) + B_n(s)(v + \beta) = 0, \tag{3.7}$$

$$(u + \alpha)^2 - (t^2 - t)(v + \beta)^2 + C_n(t)(u + \alpha) + D_n(t)(v + \beta) = 0. \tag{3.8}$$

Expanding and grouping all the like terms in (3.7) and (3.8), we obtain

$$\begin{aligned} & u^2 - (s^2 + s)v^2 \\ & + (A_n(s) + 2\alpha)u + (B_n(s) - 2(s^2 + s)\beta)v \\ & + \alpha^2 - (s^2 + s)\beta^2 + A_n(s)\alpha + B_n(s)\beta = 0, \end{aligned} \quad (3.9)$$

$$\begin{aligned} & u^2 - (t^2 - t)v^2 \\ & + (C_n(t) + 2\alpha)u + (D_n(t) - 2(t^2 - t)\beta)v \\ & + \alpha^2 - (t^2 - t)\beta^2 + C_n(t)\alpha + D_n(t)\beta = 0. \end{aligned} \quad (3.10)$$

Posing the condition that all coefficients of the first degree terms in (3.9) and (3.10) be zero, we get

$$\alpha = -A_n(s)/2 \quad \text{and} \quad \beta = B_n(s)/2(s^2 + s), \quad (3.11)$$

$$\alpha = -C_n(t)/2 \quad \text{and} \quad \beta = D_n(t)/2(t^2 - t). \quad (3.12)$$

The constant terms in (3.9) and (3.10) become

$$\frac{B_n^2(s) - (s^2 + s)A_n^2(s)}{4(s^2 + s)}, \quad (3.13)$$

$$\frac{D_n^2(t) - (t^2 - t)C_n^2(t)}{4(t^2 - t)}. \quad (3.14)$$

Consequently, the choices of α and β in (3.11) and (3.12) and the coefficients $A_n(s)$, $B_n(s)$, $C_n(t)$, and $D_n(t)$ satisfying (1.3) and (1.4) enable the equations (1.1) and (1.2) to be reduced into a form of Pell's equations.

3.3 Main Theorems and Proofs

The main theorems about the Diophantine analysis of the binary quadratic Diophantine equations (1.1) and (1.2) are divided into two following theorems.

Theorem 3.1. *All integer solutions of the binary quadratic Diophantine equation*

$$x^2 - (s^2 + s)y^2 + A_n(s)x + B_n(s)y = 0$$

for each integer $s \geq 1$ with the coefficients $A_n(s)$ and $B_n(s)$ ($n \geq 0$) satisfying

$$\begin{bmatrix} B_n(s) \\ A_n(s) \end{bmatrix} = \begin{bmatrix} 0 & 2(s^2 + s) \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2s + 1 & 2(s^2 + s) \\ 2 & 2s + 1 \end{bmatrix}^n \begin{bmatrix} 2s + 1 \\ 2 \end{bmatrix}$$

are given by

$$(x_m, y_m) = \left(u_m - \frac{A_n(s)}{2}, v_m + \frac{B_n(s)}{2(s^2 + s)} \right),$$

where u_m, v_m are defined by

$$\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \begin{bmatrix} 2s + 1 & 2(s^2 + s) \\ 2 & 2s + 1 \end{bmatrix}^m \begin{bmatrix} 2s + 1 \\ 2 \end{bmatrix}.$$

Proof. By setting $\alpha = -A_n(s)/2$ and $\beta = B_n(s)/2(s^2 + s)$ as in (3.11), it follows that the equation (3.9) is in the form

$$u^2 - (s^2 + s)v^2 + \frac{B_n^2(s) - (s^2 + s)A_n^2(s)}{4(s^2 + s)} = 0. \quad (3.15)$$

Now we consider the Diophantine equation

$$B_n^2(s) - (s^2 + s)A_n^2(s) = -4(s^2 + s). \quad (3.16)$$

Obviously, one of the solutions in the above equation is $(0, 2)$ and since $\sqrt{s^2 + s} = \langle s, \overline{2}, \overline{2s} \rangle$ has an even period $m = 2$, it follows that the fundamental solution of Pell's equation $B_n^2(s) - (s^2 + s)A_n^2(s) = 1$ is $(2s + 1, 2)$ which can be found in the first convergent of the sequences $\{h_{2n-1}\}$ and $\{k_{2n-1}\}$. Thus by Proposition 2.7, $A_n(s)$ and $B_n(s)$ expressed in (1.3) are also the solution of the equation (3.16). Hence the binary quadratic Diophantine equation is reduced to Pell's equation of the following form:

$$u^2 - (s^2 + s)v^2 = 1. \quad (3.17)$$

Applying Proposition 2.6, we obtain u_m and v_m in the matrix below as solutions of (3.17).

$$\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \begin{bmatrix} 2s + 1 & 2(s^2 + s) \\ 2 & 2s + 1 \end{bmatrix}^m \begin{bmatrix} 2s + 1 \\ 2 \end{bmatrix}$$

Finally, the general solutions of the binary quadratic Diophantine equation (1.1) can be obtained by substitutions $x = u + \alpha$ and $y = v + \beta$. This completes the proof.

Theorem 3.2. *All integer solutions of the binary quadratic Diophantine equation*

$$x^2 - (t^2 - t)y^2 + C_n(t)x + D_n(t)y = 0$$

for each integer $t \geq 2$ with the coefficients $C_n(t)$ and $D_n(t)$ ($n \geq 0$) satisfying

$$\begin{bmatrix} D_n(t) \\ C_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 2(t^2 - t) \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2t - 1 & 2(t^2 - t) \\ 2 & 2t - 1 \end{bmatrix}^n \begin{bmatrix} 2t - 1 \\ 2 \end{bmatrix}$$

are given by

$$(x_m, y_m) = \left(u_m - \frac{C_n(t)}{2}, v_m + \frac{D_n(t)}{2(t^2 - t)} \right),$$

where (u_m, v_m) are defined by

$$\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \begin{bmatrix} 2t - 1 & 2(t^2 - t) \\ 2 & 2t - 1 \end{bmatrix}^m \begin{bmatrix} 2t - 1 \\ 2 \end{bmatrix}.$$

Proof. We proceed in the similar manner as in the proof of Theorem 3.1. By putting $\alpha = -C_n(t)/2$ and $\beta = D_n(t)/2(t^2 - t)$, the equation (3.10) can be written as

$$u^2 - (t^2 - t)v^2 + \frac{D_n^2(t) - (t^2 - t)C_n^2(t)}{4(t^2 - t)} = 0.$$

Now we consider the Diophantine equation

$$D_n^2(t) - (t^2 - t)C_n^2(t) = -4(t^2 - t). \quad (3.18)$$

It is not difficult to see that the solution of (3.18) is $(0, 2)$. Proposition 2.5 and the fact that $\sqrt{t^2 - t} = \langle t - 1, \overline{2, 2(t - 1)} \rangle$ which has even period $m = 2$ imply that $(2t - 1, 2)$ is the least solution of Pell's equation $D_n^2(t) - (t^2 - t)C_n^2(t) = 1$. Again by Proposition 2.7, $C_n(s)$ and $D_n(s)$ expressed in (1.4) are also the solution of (3.18). Now we have the binary Diophantine equation written in form of Pell's equation

$$u^2 - (t^2 - t)v^2 = 1. \quad (3.19)$$

Using Proposition 2.6, we see that u_m and v_m expressed in the matrix below are the solutions of (3.19).

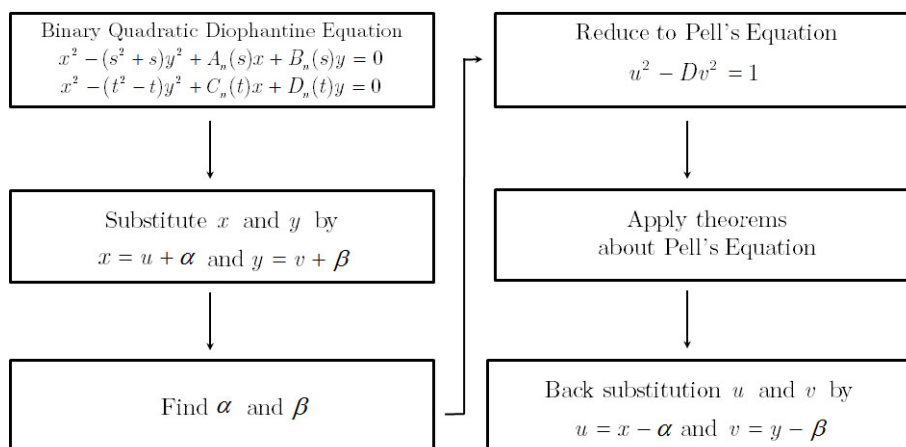
$$\begin{bmatrix} u_m \\ v_m \end{bmatrix} = \begin{bmatrix} 2t - 1 & 2(t^2 - t) \\ 2 & 2t - 1 \end{bmatrix}^m \begin{bmatrix} 2t - 1 \\ 2 \end{bmatrix}$$

We then get the general solution of the binary quadratic Diophantine equation (1.2) by substitutions $x = u + \alpha$ and $y = v + \beta$. Thus the proof is complete.

3.4 Concept Diagram

The figure shown below is a concept diagram of how the theorems are proved.

Figure 3.1: Concept diagram of proof of main theorems.



CHAPTER IV

NUMERICAL EXAMPLES

In this chapter, some numerical examples are posed in form of tables in order to verify and assure the theorems in Chapter 3. All of the table are from the program coded by MATHEMATICA 7 included in the appendix.

4.1 Tables of Solutions

The examples of coefficients $A_n(s)$, $B_n(s)$, $C_n(t)$, and $D_n(t)$ are shown in the following tables.

Table 4.1: The coefficients of $A_n(s)$ and $B_n(s)$ when $s = 3$.

A_n	B_n
146	504
2030	7032
28274	97944
393806	1364184
5485010	19000632
76396334	264644664
1064063666	3686024664
14820494990	51339700632
206422866194	715069784184

Table 4.2: The coefficients of $C_n(t)$ and $D_n(t)$ when $t = 3$.

C_n	D_n
74	180
730	1788
7226	17700

Table 4.3: The coefficients of $C_n(t)$ and $D_n(t)$ when $t = 3$ (cont.)

C_n	D_n
71530	175212
708074	1734420
7009210	17168988
69384026	169955460
686831050	1682385612
6798926474	16653900660
67302433690	164856620988

The solutions of the binary quadratic Diophantine equations (1.1) and (1.2) when specified coefficients are shown in Table 4.4 , 4.5 , 4.6, and 4.7. We see that the solutions follows the theorems.

Table 4.4: The solutions of the binary quadratic Diophantine equation (1.1) when $s = 3$ and $A_3(3) = 28274, B_3(3) = 97944$.

x_m	y_m	$x^2 - 12y^2 + 28274x + 97944y$
-14064	4102	0
-13122	4374	0
0	8162	0
182766	60922	0
2728368	795774	0
38184030	1030942	0
532017696	153588442	0
7410233358	2139158274	0
103211418960	29794578422	0
1437549801726	414984890662	0
20022485974848	5779993841874	0
278877254015790	80504928846602	0
3884259070415856	1121289009961582	0
54100749731975838	15617541210566574	0

Table 4.5: The solutions of the binary quadratic Diophantine equation (1.1) when $s = 3$ and $A_3(3) = 28274$, $B_3(3) = 97944$ (cont.)

x_m	y_m	$x^2 - 12y^2 + 28274x + 97944y$
753526237177415520	217524287937921482	0
10495266570752011086	3029722489920285202	0
146180205753350909328	42198590570946022374	0

Table 4.6: The solutions of the binary quadratic Diophantine equation (1.2) when $t = 3$ and $C_3(3) = 7226$, $D_3(3) = 17700$.

x_m	y_m	$x^2 - 6y^2 + 7226x + 17700y$
-3576	1490	0
-3248	1624	0
0	2950	0
32152	16076	0
350424	146010	0
3500992	1432224	0
34688400	14164430	0
343411912	140200276	0
3399459624	1387826530	0
33651213232	13738053224	0
333112701600	135992693910	0
3297475831672	1346188874076	0
32641645644024	13325896035050	0
323118980637472	131912771464624	0
3198548160759600	1305801818599390	0
31662362626987432	12926105414517476	0
313425078109143624	127955252326563570	0

Table 4.7: The solutions of the binary quadratic Diophantine equation (1.2) when $t = 3$ and $C_3(3) = 7226$, $D_3(3) = 17700$ (cont.)

x_m	y_m	$x^2 - 6y^2 + 7226x + 17700y$
3102588418464477712	1266626417851106424	0
30712459106535662400	12538308926184488870	0
304022002646892175192	124116462843993770476	0

CHAPTER V

CONCLUSION

In this thesis, we are successful in solving specific forms of the binary quadratic Diophantine equation. Our method of finding the solutions is based on the concepts of continued fractions and Pell's equation. We wish to explore further the possibility of n being a negative integer for the coefficients $A_n(s)$, $B_n(s)$, $C_n(t)$, and $D_n(t)$ in equations (1.1) and (1.2). Another direction for further research is to consider other forms of coefficients that have similar properties with the ones studied in this thesis.

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- [5] Tekcan, A., & Ozkoc, A. (2010). The Diophantine Equation $x^2 - (t^2 + t)y^2 - (4t + 2)x + (4t^2 + 4t)y = 0$. *Revista Matematica Complutense*, 23, 251–260.

APPENDIX

PROGRAMS BY MATHEMATICA 7

Program for Generating Solutions (x_m, y_m) **of** $x^2 - (s^2 + s)y^2 + A_n(s)x + B_n(s)y = 0$

```
(*Input s*)
s = 3;
(*Input n to specify Subscript[A, n](s), Subscript[B, n](s)*)
n = 3;
Subscript[A, n] = Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]];
Subscript[B, n] = First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]];
Print[Subscript["A", n] "=" Subscript[A, n]]
```

```

Print[Subscript["B", n] "=" Subscript[B, n]]
(*Input k the number of solutions
(Subscript[x, m], Subscript[y, m]*)
k = 20;
pp = Table[{
  First[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
  }), m].({
    {2 s + 1},
    {1}
  })]] + Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
  }).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
  }), n].({
    {2 s + 1},
    {1}
  })]] {1/(2 (s^2 + s)), -(1/2)}],
  Last[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
  }), m].({
    {2 s + 1},
    {1}
  })]] + First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
  }).MatrixPower[({

```

```

        {2 s + 1, 2 (s^2 + s)},
        {2, 2 s + 1}
    }), n].({
    {2 s + 1},
    {1}
    })] {1/(2 (s^2 + s)), -(1/2)}],
(First[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
    }), m].({
    {2 s + 1},
    {1}
    })]] + Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
    }).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
    }), n].({
    {2 s + 1},
    {1}
    })]] {1/(2 (s^2 + s)), -(1/2)}]]^2
- (s^2 + s) (Last[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
    }), m].({
    {2 s + 1},
    {1}
    })]] + First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}

```

```

    }).MatrixPower[({
      {2 s + 1, 2 (s^2 + s)},
      {2, 2 s + 1}
    }), n].({
      {2 s + 1},
      {1}
    })] {1/(2 (s^2 + s)), -(1/2)}}]^2
+ (Last[Expand[({
  {0, 2 (s^2 + s)},
  {2, 0}
}).MatrixPower[({
  {2 s + 1, 2 (s^2 + s)},
  {2, 2 s + 1}
}), n].({
  {2 s + 1},
  {1}
})]]]) (First[Expand[MatrixPower[({
  {2 s + 1, 2 (s^2 + s)},
  {2, 2 s + 1}
}), m].({
  {2 s + 1},
  {1}
})]]] + Last[Expand[({
  {0, 2 (s^2 + s)},
  {2, 0}
}).MatrixPower[({
  {2 s + 1, 2 (s^2 + s)},
  {2, 2 s + 1}
}), n].({
  {2 s + 1},
  {1}

```

```

    )]] {1/(2 (s^2 + s)), -(1/2)}})
+ (First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]] (Last[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), m].({
    {2 s + 1},
    {1}
})]] + First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]] {1/(2 (s^2 + s)), -(1/2)}})], {m, k}];
Table3 = Prepend[pp, {Subscript[x, m], Subscript[y, m],
    x^2 - (s^2 + s) y^2
    + Subscript[A, n] x + Subscript[B, n] y}];
Grid[Table3, Frame -> All]

```

Program for Generating Solutions (x_m, y_m) **of** $x^2 - (t^2 - t)y^2 + C_n(t)x + D_n(t)y = 0$

```
(*Input s*)
s = 3;
(*Input n to specify Subscript[A, n](s), Subscript[B, n](s)*)
n = 3;
Subscript[A, n] = Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]];
Subscript[B, n] = First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]];
Print[Subscript["A", n] "=" Subscript[A, n]]
Print[Subscript["B", n] "=" Subscript[B, n]]
(*Input k the number of solutions
(Subscript[x, m], Subscript[y, m])*)
k = 20;
```

```

pp = Table[{
  First[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
  }), m].({
    {2 s + 1},
    {1}
  }))] + Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
  }).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
  }), n].({
    {2 s + 1},
    {1}
  }))] {1/(2 (s^2 + s)), -(1/2)}],
  Last[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
  }), m].({
    {2 s + 1},
    {1}
  }))] + First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
  }).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
  }), n].({
    {2 s + 1},

```

```

{1}
    ]]] {1/(2 (s^2 + s)), -(1/2)}],
(First[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), m].({
{2 s + 1},
{1}
})]] + Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
{2 s + 1},
{1}
})]] {1/(2 (s^2 + s)), -(1/2)}]]^2
- (s^2 + s) (Last[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), m].({
{2 s + 1},
{1}
})]] + First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
}).MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({

```

```

        {2 s + 1},
        {1}
    ]] {1/(2 (s^2 + s)), -(1/2)}}]^2
+ (Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
})].MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]]] (First[Expand[MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), m].({
    {2 s + 1},
    {1}
})]]] + Last[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}
})].MatrixPower[({
    {2 s + 1, 2 (s^2 + s)},
    {2, 2 s + 1}
}), n].({
    {2 s + 1},
    {1}
})]]] {1/(2 (s^2 + s)), -(1/2)}}]
+ (First[Expand[({
    {0, 2 (s^2 + s)},
    {2, 0}

```

```

    }).MatrixPower[({
      {2 s + 1, 2 (s^2 + s)},
      {2, 2 s + 1}
    }), n].({
      {2 s + 1},
      {1}
    })] (Last[Expand[MatrixPower[({
      {2 s + 1, 2 (s^2 + s)},
      {2, 2 s + 1}
    }), m].({
      {2 s + 1},
      {1}
    })]] + First[Expand[({
      {0, 2 (s^2 + s)},
      {2, 0}
    }).MatrixPower[({
      {2 s + 1, 2 (s^2 + s)},
      {2, 2 s + 1}
    }), n].({
      {2 s + 1},
      {1}
    })]] {1/(2 (s^2 + s)), -(1/2)}})], {m, k}];
Table3 = Prepend[pp, {Subscript[x, m], Subscript[y, m],
  x^2 - (s^2 + s) y^2
  + Subscript[A, n] x + Subscript[B, n] y}];
Grid[Table3, Frame -> All]

```

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