CHAPTER IV

CONCLUSIONS

In this chapter, we summary results of existence and uniqueness of semilinear impulsive periodic systems with parameter perturbations which have been studied in this thesis.

4.1 Problem

This thesis has considered the following problems :

1. Semilinear impulsive periodic systems :

1.1 Semilinear impulsive periodic systems.

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{4.1.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Futhermore we suppose that $A(t), t \in [0, T_0]$ is a closed densely defined linear unbounded operator on Banach space X

1.2 Existence and uniqueness of periodic mild solutions for impulsive system.

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t, x(t)), & t \in [0, T_0], \quad t \neq \tau_k, \\ \Delta x(\tau_k) &= B_k x(\tau_k), & t = \tau_k, \quad k = 1, 2, \cdots, \sigma \\ x(0) &= x_0, \end{cases} \tag{4.1.2}$$

where $A(t)$ is a closed densely defined linear unbounded operator on X, and $f : [0, \infty) \times X \to X$.

2. Semilinear impulsive periodic systems with parameter perturbations :

2.1 Semilinear impulsive periodic systems with parameter perturbations

$$\begin{cases} \dot{x}(t) = & A(t)x(t) + f(t, x(t)) + p(t, x(t), \xi), \; t \neq \tau_k, \\ \Delta x(t) = & B_k x(t) + c_k + q_k(x(t), \xi), \qquad t = \tau_k, \end{cases} \tag{4.1.3}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Futhermore we suppose that $A(t), t \in [0, T_0]$ is a closed densely defined linear unbounded operator on Banach spaces X.

2.2 Existence and uniqueness of periodic mild solutions for reference system.

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{4.1.4}$$

2.3 The variation system.

$$\begin{cases} \dot{x}(t) = A(t)x(t) + \frac{\partial}{\partial x} f(t, x_{T_0}(t))x(t), & t \neq \tau_k, \\ \Delta x(t) = \frac{\partial}{\partial x} B_k(x_{T_0}(t))x(t), & t = \tau_k, \end{cases} \tag{4.1.5}$$

4.2 Assumptions

Assumption (A1) ;

(A1.1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots, \ \tau_k \to \infty$ as $k \to \infty$ and there exists a positive integer σ such that $\tau_{k+\sigma} = \tau_k + T_0$ for all $k \in \mathbb{N}$.

(A1.2) $B_k \in \mathcal{L}(X)$ such that $B_{k+\sigma} = B_k$ for all $k \in \mathbb{N}$ and there exists constant $h_k(\rho) > 0$ such that

$$\|B_k(x) - B_k(y)\|_X \leq h_k(\rho)\|x - y\|_X,$$

for all $k \in \mathbb{N}$ and all $x, y \in X$ such that $\|x\|_X, \|y\|_X \leq \rho$.

(A1.3)$f : [0, \infty) \times X \to X$ is an operator such that $f(t+T_0, x) = f(t, x)$ and $t \mapsto f(t, x)$ is strongly measurable. For every $\rho > 0$, there exist constants $K_1(\rho),\ K_2(\rho) > 0$ such that

$$\|f(t, x)\|_X \leq K_1(\rho)$$

and
$$\|f(t, x) - f(t, y)\|_X \leq K_2(\rho)\|x - y\|_X,$$

for all $t \geq 0$ and all $x, y \in X$ such that $\|x\|_X, \|y\|_X \leq \rho$.

Assumption (A2) ;

(A2.1) The domain $\mathcal{D}(A(t)) = \mathcal{D}$ is independent of t and dense in X for $t \in [0, T_0]$.

(A2.2) For $t \geq 0$ the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant M of λ and t such that

$$\|R(\lambda, A(t))\|_{\mathcal{L}(X)} \leq M(1 + |\lambda|)^{-1}$$

for all $Re\lambda \leq 0$.

(A2.3) There exists constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(s))A^{-1}(\tau)\|_{\mathcal{L}(X)} \leq L|t - s|^{\alpha}$$

for $t, s, \tau \in [0, T_0]$.

Assumption (A3) ;

(A3.1) There exists $T_0 > 0$ such that $A(t + T_0) = A(t)$ for all$t \in [0, T_0]$.

(A3.2) For all $t \geq 0$, the resolvent $R(\lambda, A(t))$ is compact.

Assumption (A4) ;

(A4.1) $c_k \in X$ and $c_{k+\sigma} = c_k$ for all $k \in \mathbb{N}$.

(A4.2) The Fréchet derivative $\dfrac{\partial}{\partial x} f(t, x)$ exists in $[0, \infty) \times X$. For each $y \in X$, $t \mapsto \dfrac{\partial}{\partial x} f(t, x)y$ is strongly measurable, $x \mapsto \dfrac{\partial}{\partial x} f(t, x)y$ is continuous. For every $\rho > 0$, there exists a constant $K_3(\rho) > 0$ such that

$$\left\| \frac{\partial}{\partial x} f(t, x) \right\|_{\mathcal{L}(X)} \leq K_3(\rho)$$

for all $t \geq 0$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

(A4.3) $p : [0, \infty) \times S_\rho \times \Lambda \to X$ is measuable for t such that $p(t + T_0, x, \xi) = p(t, x, \xi)$ and $q_k : S_\rho \times \Lambda \to X$ such that $q_{k+\sigma}(x, \xi) = q_k(x, \xi)$, where $\Lambda \equiv (-\tilde{\xi}, \tilde{\xi}), (\tilde{\xi} > 0)$ and $S_\rho = \{x \in PC([0, \infty), X) | \|x\|_{PC} < \rho\}$ and there exists a nonnegative function ω such that

$$\lim_{\xi \to 0} \omega(\xi) = \omega(0) = 0$$

and for any $t \geq 0, x, y \in S_\rho$ and $\xi \in \Lambda$ such that

$$\|p(t, x, \xi) - p(t, y, \xi)\|_X \leq \omega(\xi)\|x - y\|_X$$

and $$\|q_k(x,\xi)-q_k(y,\xi)\|_X \le \omega(\xi)\|x-y\|_X.$$

(A4.4) The Fréchet derivative $\frac{\partial}{\partial x}B_k(x)$ exists in X. For every $\rho > 0$, there exists a constant $\bar{h}_k(\rho) > 0$ such that

$$\left\|\frac{\partial}{\partial x}B_k(x)\right\|_{\mathcal{L}(X)} \le \bar{h}_k(\rho)$$

for all $t \ge 0$, $k \in \mathbb{N}$ and all $x \in X$ such that $\|x\|_X \le \rho$.

4.3 Results

The main results of this thesis are summarized as follows :

Theorem 4.3.1 *Suppose $A(t), t \in [0,T_0]$ is a closed densely defined linear unbounded operator on X. If assumption (A1) hold, then system (4.1.2) has a unique mild solution $x \in C([0,T_0],X)$.*

Theorem 4.3.2 *Suppose $A(t), t \in [0,T_0]$ be a closed densely defined linear unbounded operator on X. If assumptions (A1) hold, then system (4.1.1) has a unique mild solution $x \in PC([0,T_0],X)$.*

Theorem 4.3.3 *Let assumption (A1) and (A4) holds. Suppose $x_{T_0}(t)$ be a T_0-periodic mild solution of the reference system (4.1.4) satisfies*

$$\rho_0 = \sup_{t\in[0,T_0]} \|x_{T_0}(t)\|_X.$$

Assume that

1. *system (4.1.5) has only trivial solution,*

2. *let $\tilde{\xi} > 0$ and $\varepsilon_o \in (0,\rho-\rho_0)$ such that $\eta < 1$ with*

$$\eta := M\Big([K_2(\varepsilon_0)+K_3(\varepsilon_0)]T_0 + [h_k(\varepsilon_0)+\bar{h}_k(\varepsilon_0)]\sigma + [T_0+\sigma]\sup_{\xi\in[0,\tilde{\xi}]}\omega(\xi)\Big)$$

where

$$M = \sup_{0\le s\le t\le T_0} \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)},$$

$$\bar{h}_k(\varepsilon_0) = \sup_{k\in\mathbb{N},\ \|y\|\le\varepsilon_0} \left\|\frac{\partial}{\partial x}B_k(x_{T_0}(\tau_k)+y(\tau_k))\right\|_X,$$

3. the following inequality is valid

$$\sup_{t\in[0,T_0],\,|\xi|\le\tilde{\xi}} \Bigg\| \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)[p(s,x_{T_0}(s),\xi)]ds$$

$$+ \sum_{0\le\tau_k<T_0} \mathcal{S}(t,\tau_k)[c_k + q_k(x_{T_0}(\tau_k),\xi)] \Bigg\|_X \le \varepsilon_0(1-\eta).$$

Then for any constant $\rho > \rho_0 > 0$, *there exists a sufficiently small* $\tilde{\xi} > 0$ *such that for every fixed* $\xi \in [0,\tilde{\xi}]$ *system (4.1.3) has a unique* T_0*-periodic mild solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\| < \varepsilon_0 \quad \textit{for all} \quad t \ge 0 \tag{4.3.1}$$

and

$$\lim_{\xi\to 0} x^{\xi}_{T_0}(t) = x_{T_0}(t) \qquad \textit{uniformly on } t.$$