CHAPTER III

MAIN RESULTS

In this chapter, we study the existence and uniqueness of semilinear impulsive periodic systems, and semilinear impulsive periodic systems with parameter perturbation. The first section we introduce basic notation and basic assumptions,. In the second section, we study properties of impulsive evolution operators, existence and uniqueness of mild solution. Finally in the third section, we study the existence and uniqueness of T_0-periodic solutions for semilinear impulsive periodic systems with parameter perturbation.

3.1 Notations

Let $\mathcal{L}(X)$ be the space of bounded linear operators in the Banach space X, and let $I := [0, T_0]$ be a closed bounded interval of the real line

Definition 3.1.1 A sequence, (τ_k) is said to be an *impulsive moment* if $0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_k < \ldots$, and $\tau_k \to \infty$ as $k \to \infty$.

We now introduce the piecewise continuous function spaces. Let X be a Banach space and $0 < T_0 < \infty$.

(1) $PC([0,\infty), X) \equiv \{\ x : [0,\infty) \to X |\ \ x$ is continuous at $t \in [0, T_0],\ t \neq \tau_k,\ x$ is continuous from left and the right limit $x(\tau_k^+)$ exists at $t = \tau_k,\ \forall k \in \mathbb{N}\}$.

(2) $PC^1([0,\infty), X) \equiv \{x \in PC([0,\infty), X) |\ \dot{x} \in PC([0,\infty), X)\}$

(3) $PC_{T_0}([0,\infty), X) \equiv \{x \in PC([0,\infty), X) |\ x(t) = x(t + T_0),\ \forall t \geq 0\}$.

3.2 Semilinear Impulsive Periodic Systems

We consider the following semilinear impulsive periodic systems

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t)), & t \neq \tau_k, \\ \Delta x(t) = B_k x(t), & t = \tau_k, \end{cases} \tag{3.2.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. Futhermore we suppose that $A(t), t \in [0, T_0]$ is a closed densely defined linear unbounded operator on X, satisfying the following assumptions (A1), (A2) and (A3) :

Assumption (A1) ;

(A1.1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots, \quad \tau_k \to \infty$ as $k \to \infty$ and there exists a positive integer σ such that $\tau_{k+\sigma} = \tau_k + T_0$ for all $k \in \mathbb{N}$.

(A1.2) $B_k \in \mathcal{L}(X)$ such that $B_{k+\sigma} = B_k$ for all $k \in \mathbb{N}$ and there exists constant $h_k(\rho) > 0$ such that

$$\|B_k(x) - B_k(y)\|_X \le h_k(\rho)\|x - y\|_X,$$

for all $k \in \mathbb{N}$ and all $x, y \in X$ such that $\|x\|_X, \|y\|_X \le \rho$.

(A1.3)$f : [0, \infty) \times X \to X$ is an operator such that $f(t+T_0, x) = f(t, x)$ and $t \mapsto f(t, x)$ is strongly measurable. For every $\rho > 0$, there exist constants $K_1(\rho),\ K_2(\rho) > 0$ such that

$$\|f(t, x)\|_X \le K_1(\rho)$$

and
$$\|f(t, x) - f(t, y)\|_X \le K_2(\rho)\|x - y\|_X,$$
for all $t \ge 0$ and all $x, y \in X$ such that $\|x\|_X, \|y\|_X \le \rho$.

Assumption (A2) ;

(A2.1) The domain $\mathcal{D}(A(t)) = \mathcal{D}$ is independent of t and dense in X for $t \in [0, T_0]$.

(A2.2) For $t \ge 0$ the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all $\lambda \in \mathbb{C}$ with $Re(\lambda) \le 0$, and there is a constant M of λ and t such that

$$\|R(\lambda, A(t))\|_{\mathcal{L}(X)} \le M(1 + |\lambda|)^{-1}$$

for all $Re(\lambda) \le 0$.

(A2.3) There exists constants $L > 0$ and $0 < \alpha \le 1$ such that

$$\|(A(t) - A(s))A^{-1}(\tau)\|_{\mathcal{L}(X)} \le L|t - s|^{\alpha}$$

for $t, s, \tau \in [0, T_0]$.

Assumption (A3) ;

(A3.1) There exists $T_0 > 0$ such that $A(t + T_0) = A(t)$ for all $t \in [0, T_0]$.

(A3.2) For all $t \geq 0$, the resolvent $R(\lambda, A(t))$ is compact.

3.2.1 Impulsive Evolution Operator

Lemma 3.2.1 *Let assumption (A2) hold. The Cauchy problem*

$$\dot{x}(t) + A(t)x(t) = 0, \quad t \in (0, T_0] \qquad \textit{with} \qquad x(0) = x_0 \tag{3.2.2}$$

has a unique evolution system $\{U(t,s)|0 \leq s \leq t \leq T_0\}$ *in* X *satisfying the following properties :*

(1) $U(t,s) \in \mathcal{L}(X)$, *for* $0 \leq s \leq t \leq T_0$ *;*

(2) $U(t,r)U(r,s) = U(t,s)$, *for* $0 \leq s < r < t \leq T_0$, $r \neq \tau_k$ *;*

(3) $U(.,.)x \in C(\triangle, X)$, *for* $x \in X, \triangle \equiv \{(t,s) \in [0,T_0] \times [0,T_0]|0 \leq s \leq t \leq T_0\}$ *;*

(4) for $0 \leq s \leq t \leq T_0, U(t,s) : X \to \mathcal{D}$ *and* $t \to U(t,s)$ *is strongly differentiable in* X. *The derivative* $(\frac{\partial}{\partial t})U(t,s) \in \mathcal{L}(X)$ *and it is strongly continuous on* $0 \leq s < t \leq T_0$ *; moreover ,*

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s) \qquad \textit{for} \quad 0 \leq s < t \leq T_0,$$

$$\|\frac{\partial}{\partial t}U(t,s)\|_{\mathcal{L}(X)} = \|A(t)U(t,s)\|_{\mathcal{L}(X)} \leq \frac{c}{t-s}$$

$$\|A(t)U(t,s)A(s)^{-1}\|_{\mathcal{L}(X)} \leq c \qquad \textit{for} \quad 0 \leq s \leq t \leq T_0;$$

(5) for every $v \in \mathcal{D}$ *and* $t \in (0, T_0], U(t,s)v$ *is differentiable with respect to* s *on* $0 \leq s \leq t \leq T_0$

$$\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v.$$

And , for each $x_0 \in X$, *the Cauchy problem (3.2.2) has a unique classical solution* $x \in C^1([0,T_0], X)$ *given by*

$$x(t) = U(t,0)x_0, \qquad t \in [0, T_0].$$

Lemma 3.2.2 *Let assumption (A2) and (A3) hold. Then evolution system and* $\{U(t,s)|0 \le s \le t \le T_0\}$ *in* X *also satisfying the following two properties :*

(1) $U(t+T_0, s+T_0) = U(t,s)$, *for* $0 \le s \le t \le T_0$ *;*

(2) $U(t,s)$ *is compact operator for* $0 \le s < t \le T_0$.

In order to construct an impulsive evolution operator and investigate its properties.

First consider the following Cauchy problem :

$$\begin{cases} \dot{x}(t) &= A(t)x(t), \quad t \in [0,T_0], \quad t \neq \tau_k, \\ \Delta x(\tau_k) &= B_k x(\tau_k), \quad t = \tau_k, \quad k = 1,2,\cdots,\sigma \\ x(0) &= x_0. \end{cases} \tag{3.2.3}$$

For every $x_0 \in X$, $\mathcal{D}$ is an invariant subspace of B_k, using Lemma 3.2.1, step by step one can verify that the Cauchy problem (3.2.3) has a unique classical solution $x \in PC^1([0,T_0],X)$ represented by $x(t) = \mathcal{S}(t,0)x_0$, where $\mathcal{S}(.,.) : \triangle \to X$ given by

$$\mathcal{S}(t,s) = \begin{cases} U(t,s), & \tau_{k-1} \le s \le t \le \tau_k, \\ U(t,\tau_k^+)(I+B_k)U(\tau_k,s), & \tau_{k-1} \le s < \tau_k < t \le \tau_{k+1}, \\ U(t,\tau_k^+)\left[\prod_{s<\tau_j<t}(I+B_j)U(\tau_j,\tau_{j-1}^+)\right](I+B_i)U(\tau_i,s), & \\ & \tau_{i-1} \le s < \tau_i \le \ldots \le \tau_k < t \le \tau_{k+1}. \end{cases} \tag{3.2.4}$$

The operator $\mathcal{S}(t,s)$ $\big((t,s) \in \triangle\big)$ is called *impulsive evolution operator.*

Lemma 3.2.3 *Let assumption (A1.1) , (A1.2) , (A2) and (A3) hold. The impulsive evolution operator* $\mathcal{S}(t,s)$ *has the following properties :*

(1) $\mathcal{S}(t,s) \in \mathcal{L}(X)$, *for* $0 \le s \le t \le T_0$ *;*

(2) for $0 \le s \le t \le T_0$,$\mathcal{S}(t+T_0, s+T_0) = \mathcal{S}(t,s)$ *;*

(3) for $0 \le t \le T_0$, $\mathcal{S}(t+T_0, 0) = \mathcal{S}(t,0)\mathcal{S}(T_0,0)$ *;*

(4) $\mathcal{S}(t,s)$ *is compact operator , for* $0 \le s < t \le T_0$.

Proof. By (1) of Lemma 3.2.1 and assumption (A1.1) , (A1.2) , $\mathcal{S}(t,s) \in \mathcal{L}(X)$, for $0 \leq s \leq t \leq T_0$. By (1) of Lemma 3.2.2 and assumption (A1.1) , (A1.2) , $\mathcal{S}(t+T_0, s+T_0) = \mathcal{S}(t,s)$, for $0 \leq s \leq t \leq T_0$. By (2) of Lemma 3.2.1 , (1) of Lemma 3.2.2 and assumption (A1.1) , (A1.2) , $\mathcal{S}(t+T_0, 0) = \mathcal{S}(t,0)\mathcal{S}(T_0,0)$. By (2) of Lemma 3.2.2 and assumption (A1.1) , (A1.2) , one can obtain that $\mathcal{S}(t,s)$ is compact operator , for $0 \leq s \leq t \leq T_0$.

3.2.2. Definitions of Solutions

Definition 3.2.4 For every $x_0 \in X$ and $f \in L^1([0,\infty), X)$, the function $x \in PC([0,\infty), X)$ given by

$$x(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)f(s,x(s))ds \tag{3.2.5}$$

for all $t \in [0, T_0]$, is said to be a mild solution of system (3.2.1).

Definition 3.2.5 A function $x \in PC([0,\infty), X)$ is said to be a periodic mild solution of system (3.2.1) if it is a mild solution and there exists $T_0 > 0$ such that $x(t+T_0) = x(t)$ for all $t \geq 0$.

Definition 3.2.6 A function $x \in PC([0,\infty), X)$ is said to be a T_0-periodic mild solution of system (3.2.1) if it is a mild solution and $x(t+T_0) = x(t)$ for all $t \geq 0$.

3.2.3 Existence and Uniqueness of Periodic Mild Solutions

Consider the following impulsive system ,

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t,x(t)), \qquad t \in [0,T_0], \quad t \neq \tau_k, \\ \Delta x(\tau_k) &= B_k x(\tau_k), \qquad t = \tau_k, \quad k = 1,2,\cdots,\sigma \\ x(0) &= x_0, \end{cases} \tag{3.2.6}$$

where $A(t)$ is a closed densely defined linear unbounded operator on X, and $f : [0,\infty) \times X \to X$. By mild solution of (3.2.6), we shall mean that a function $x \in PC([0,T_0],X)$ satisfies the following integral equation ;

$$x(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)f(s,x(s))ds.$$

Theorem 3.2.7 *Suppose $A(t), t \in [0,T_0]$ is a closed densely defined linear unbounded operator on X. If assumption (A1) hold, then system (3.2.6) has a unique mild solution $x \in C([0,T_0],X)$.*

Proof. Firstly, we consider the following general differential equation without impulse

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t,x(t)), \qquad t > 0 \\ x(0) &= x_0, \end{cases} \tag{3.2.7}$$

Define a closed ball

$$\mathcal{B}(x_0,1) = \{x \in C([0,T_1],X) | \|x(t) - x(0)\|_X < 1, 0 \le t \le T_1\}$$

where T_1 will be chosen later. Define a map Q on $\mathcal{B}(x_0,1)$ by

$$(Qx)(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)f(s,x(s))ds$$

and let $M = \sup\limits_{t\in[0,T_0]} \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)}$.

Using assumption (A1.3), one can verify that $Q : \mathcal{B}(x_0,1) \to \mathcal{B}(x_0,1)$.

We have

$$\begin{aligned} \|(Qx)(t) - x_0\|_X &\le \|\mathcal{S}(t,0)x_0 - x_0\|_X + \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \|f(s,x(s))\|_X ds \\ &\le \|\mathcal{S}(t,0)x_0 - x_0\|_X + MK_1(\rho)t. \end{aligned}$$

Since $\mathcal{S}(t,s)$ is the strongly continuous, there exists $\tau' > 0$ such that

$$\|\mathcal{S}(t,0)x_0 - x_0\|_X \le \frac{1}{2}, \qquad \text{for all } t \in [0,\tau'].$$

Now, let $0 < \tau'' < \frac{1}{2MK_1(\rho)}$. Set $T_1' = min\{\tau', \tau''\}$, we have

$$\|(Qx)(t) - x_0\|_X \leq 1, \qquad \text{for all } \; t \in [0, T_1'].$$

This mean that $(Qx)(t) \in \mathcal{B}(x_0, 1)$. Hence, $Q : \mathcal{B}(x_0, 1) \to \mathcal{B}(x_0, 1)$.

Let $x, y \in \mathcal{B}(x_0, 1)$. Using assumption (A1.3), we have

$$\begin{aligned} \|(Qx)(t) - (Qy)(t)\|_X &\leq \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \|f(s, x(s)) - f(s, y(s))\|_X ds \\ &\leq MK_2(\rho)t\|x - y\|_X. \end{aligned}$$

Now, let $0 < T_1'' < \frac{1}{2MK_2(\rho)}$, then

$$\|(Qx)(t) - (Qy)(t)\|_X \leq \frac{1}{2}\|x - y\|_X.$$

This means that the map Q is contraction map.

We shall choose $T_1 = min\{T_1', T_1''\} > 0$ (small enough) such that Q is a contraction map on $\mathcal{B}(x_0, 1)$. By contraction map principle, there exists a unique fixed point, this implies that (3.2.7) has a unique mild solution on $[0, T_1]$.

Suppose $x(.)$ is a mild solution of (3.2.7), then we have

$$\begin{aligned} \|(x)(t)\|_X &\leq \|\mathcal{S}(t,0)\|_{\mathcal{L}(X)}\|x_0\|_X + \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)}\|f(s, x(s))\|_X ds \\ &\leq M\|x_0\|_X + MK_1(\rho)t. \end{aligned}$$

By Gronwall inequality, we have obtain

$$\|(x)(t)\|_X \leq M\|x_0\|_X + MK_1(\rho)t \equiv \bar{M}.$$

That is, there exists a constant $\bar{M} = M\|x_0\|_X + MK_1(\rho)t > 0$ such that $\|x(t)\|_X \leq \bar{M}$ for all $t \in [0, T_0]$. Then we can prove the global existence of the mild solution of system (3.2.7) on $[0, T_0]$. □

Theorem 3.2.8 *Suppose $A(t), t \in [0, T_0]$ be a closed densely defined linear unbounded operator on X. If assumptions (A1) hold, then system (3.2.1) has a unique mild solution $x \in PC([0, T_0], X)$.*

Proof. For $t \in [0, \tau_1]$, Theorem 3.2.7 implies that system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \quad 0 < t \leq \tau_1, \quad x(0) = x_0,$$

has a mild solution on $I_1 = [0, \tau_1]$ which satisfies

$$x_1(t) = \mathcal{S}(t, 0)x_0 + \int_0^t \mathcal{S}(t, s)f(s, x_1(s))ds, \quad t \in [0, \tau_1].$$

Now, define

$$x_1(\tau_1) = \mathcal{S}(\tau_1, 0)x_0 + \int_0^{\tau_1} \mathcal{S}(\tau_1, s)f(s, x_1(s))ds,$$

so that $x_1(\cdot)$ is left continuous at τ_1. Next, on $I_2 = (\tau_1, \tau_2]$, consider system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \quad \tau_1 < t < \tau_2, \quad x_1(\tau_1^+) = (I + B_1)x_1(\tau_1),$$

Since $x_1 \in X$, we can use Theorem 3.2.7 again to get a mild solution on $(\tau_1, \tau_2]$ which satisfying

$$x_2(t) = \mathcal{S}(t, \tau_1)x_1(\tau_1^+) + \int_{\tau_1}^t \mathcal{S}(t, s)f(s, x_2(s))ds.$$

Now, define $x_2(\tau_2)$ accordingly so that $x_2(\cdot)$ is left continuous at τ_2. It is easy to see that Theorem 3.2.7 can be applied to interval $(\tau_1, \tau_2]$ to verify that $x_2(\tau_2) \in X$. Repeat the procedure above, use step-by-step approach on intervals $I_k = (\tau_{k-1}, \tau_k]$, $k = 3, 4, \ldots, \sigma$ $(\tau_\sigma = T_0)$ to get a mild solutions

$$x_k(t) = \mathcal{S}(t, \tau_{k-1})x_{k-1}(\tau_{k-1}^+) + \int_{\tau_{k-1}}^t \mathcal{S}(t, s)f(s, x_k(s))ds.$$

for $t \in (\tau_{k-1}, \tau_k]$ and define $x_k(\tau_k)$ accordingly with $x_k(\cdot)$ left continuous at τ_k and $x_k(\tau_k) \in X$, $k = 1, 2, \ldots, \sigma$. Thus we obtain $x \in PC([0, T_0], X)$ is a mild solution of system (3.2.1) and given by

$$x(t) = \begin{cases} x_1(t), & 0 \leq t \leq \tau_1, \\ x_k(t), & \tau_{k-1} < t \leq \tau_k, \ k = 2, 3, \ldots, \sigma. \end{cases}$$

Next, by mathematical induction we can show that (3.2.5) is satisfied on $[0, T_0]$. First, (3.2.5) is satisfied on $[0, \tau_1]$. If (3.2.5) is satisfied on $(\tau_{k-1}, \tau_k]$, then for $t \in (\tau_k, \tau_{k+1}]$,

$$\begin{aligned} x(t) = \;& x_{k+1}(t) = \mathcal{S}(t, \tau_k)x_k(\tau_k^+) + \int_{\tau_k}^t \mathcal{S}(t, s)f(s, x_{k+1}(s))ds \\ = \;& \mathcal{S}(t, \tau_k)(I + B_k)x(\tau_k) + \int_{\tau_k}^t \mathcal{S}(t, s)f(s, x_{k+1}(s))ds \\ = \;& \mathcal{S}(t, \tau_k)(I + B_k)\Big[\mathcal{S}(\tau_k, 0)x_0 + \int_0^{\tau_k} \mathcal{S}(\tau_k, s)f(s, x(s))ds\Big] \\ & + \int_{\tau_k}^t \mathcal{S}(t, s)f(s, x_{k+1}(s))ds \\ = \;& \mathcal{S}(t, 0)x_0 + \int_0^{\tau_k} \mathcal{S}(t, s)f(s, x(s))ds + \int_{\tau_k}^t \mathcal{S}(t, s)f(s, x(s))ds \\ = \;& \mathcal{S}(t, 0)x_0 + \int_0^t \mathcal{S}(t, s)f(s, x(s))ds. \end{aligned}$$

Thus (3.2.5) is also true on $(\tau_k, \tau_{k+1}]$. Therefore (3.2.5) is true on $[0, T_0]$.

Next, we want to show that a mild solution is unique on $PC([0, T_0], X)$. Suppose that x, y are mild solutions of system (3.2.1) on $PC([0, T_0], X)$. Then by Theorem 3.2.7, we have

$$\begin{aligned} \|x(t) - y(t)\|_X \le \;& \int_0^t \|\mathcal{S}(t, s)\|_{\mathcal{L}(X)} \|f(s, x(s)) - f(s, y(s))\|_X ds \\ \le \;& MK_2(\rho) \int_0^t \|x(s) - y(s)\|_X ds. \end{aligned}$$

It follows from Gronwall Lemma, we obtain $\|x(t) - y(t)\| = 0$ for all $t \in [0, T_0]$. That is, $x = y$. Therefore, system (3.2.1) has a unique mild solution. This completes the proof. □

To be able to apply the method in Pazy . we also need the following lemma.

Lemma 3.2.9 *Consider the nonhomogeneous initial value problem*

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t, x(t)), \quad t > 0 \\ x(0) &= x_0. \end{cases} \tag{3.2.8}$$

If $f \in L^1([0,\infty), X)$, then for every $x_0 \in X$ the initial value problem (3.2.8) has a unique solution which satisfies

$$x(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)f(s,x(s))ds, \qquad 0 \le t \le T_0.$$

We consider the following system,

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t,x(t)), & t \ge 0 \\ x(0) = x_0, \end{cases} \tag{3.2.9}$$

and we suppose that it has a global mild solution $x(t)$.

We also consider the following system,

$$\begin{cases} \dot{y}(t) = A(t)y(t) + f(t,x(t)), & t \ge 0 \\ y(0) = x(0). \end{cases} \tag{3.2.10}$$

By Lemma 3.2.9, system (3.2.10) has a unique mild solution $y(t)$. Let $P : C(X,X) \to X$ be the Poincare mapping, defined by

$$Px = y(T_0) = \mathcal{S}(T_0,0)x_0 + \int_0^{T_0} \mathcal{S}(t,s)f(s,x(s))ds.$$

Finally, we consider the following system,

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t,x(t)), & t \ge 0 \\ x(0) = Px, \end{cases} \tag{3.2.11}$$

which by Lemma 3.2.9 also has a unique mild solution $x(t)$.

We are now in a position to state and prove the basic tool for the proof existence of periodic mild solution.

Theorem 3.2.10 *System (3.2.9) has a T_0-periodic mild solution if and only if the mapping P has a fixed point.*

Proof. Let x be a T_0-periodic mild solution of system (3.2.9). Then x is clearly a T_0-periodic mild solution of system (3.2.10). Since x is T_0-periodic mild solution, $x(0) = x(T_0)$. Therefore $x(0) = x(T_0) = Px$, where x satisfy (3.2.11) and so $Px = x$. Conversely, let x be a fixed point of P. By definition, x satisfies

(3.2.10) and since $x(0) = y(0)$. By Lemma 3.2.9 , show that $x(t) \equiv y(t)$ and hence $x(T_0) = y(T_0)$. Since $Px = x$, it follows from (3.2.11) that $x(0) = Px = y(T_0) = x(T_0)$. That is, $x(0) = x(T_0)$. The function $\psi(t) := x(t + T_0)$ is also a mild solution of (3.2.9). Since f is T_0-periodic, $\dot{\psi}(t) = \dot{x}(t + T_0) = A(t + T_0)x(t + T_0) + f(t + T_0, x(t + T_0)) = A(t)\psi(t) + f(t, \psi(t))$. Therefore $x(t) = x(t + T_0)$ for all $t \geq 0$. i.e., system (3.2.9) has a T_0-periodic mild solution. This completes the proof. □

3.3 Semilinear Impulsive Periodic Systems with Parameter Perturbations

We consider the semilinear impulsive periodic system with parameter perturbations as the following

$$\begin{cases} \dot{x}(t) = & A(t)x(t) + f(t, x(t)) + p(t, x(t), \xi), \ t \neq \tau_k, \\ \Delta x(t) = & B_k x(t) + c_k + q_k(x(t), \xi), \qquad t = \tau_k, \end{cases} \tag{3.3.1}$$

where $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-)$ for all $k \in \mathbb{N}$. In addition to assumptions (A1), we introduce the following assumption

Assumption (A4) ;

(A4.1) $c_k \in X$ and $c_{k+\sigma} = c_k$ for all $k \in \mathbb{N}$.

(A4.2) The Fréchet derivative $\dfrac{\partial}{\partial x} f(t, x)$ exists in $[0, \infty) \times X$. For each $y \in X$, $t \mapsto \dfrac{\partial}{\partial x} f(t, x)y$ is strongly measurable, $x \mapsto \dfrac{\partial}{\partial x} f(t, x)y$ is continuous. For every $\rho > 0$, there exists a constant $K_3(\rho) > 0$ such that

$$\left\| \frac{\partial}{\partial x} f(t, x) \right\|_{\mathcal{L}(X)} \leq K_3(\rho)$$

for all $t \geq 0$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

(A4.3) $p : [0, \infty) \times S_\rho \times \Lambda \to X$ is measuable for t such that $p(t + T_0, x, \xi) = p(t, x, \xi)$ and $q_k : S_\rho \times \Lambda \to X$ such that $q_{k+\sigma}(x, \xi) = q_k(x, \xi)$, where $\Lambda \equiv (-\widetilde{\xi}, \widetilde{\xi}), (\widetilde{\xi} > 0)$ and $S_\rho = \{x \in PC([0, \infty), X) | \|x\|_{PC} < \rho\}$ and there exists a nonnegative function ω such that

$$\lim_{\xi \to 0} \omega(\xi) = \omega(0) = 0$$

and for any $t \geq 0, x, y \in S_\rho$ and $\xi \in \Lambda$ such that

$$\|p(t,x,\xi) - p(t,y,\xi)\|_X \leq \omega(\xi)\|x-y\|_X$$

and $$\|q_k(x,\xi) - q_k(y,\xi)\|_X \leq \omega(\xi)\|x-y\|_X.$$

(A4.4) The Fréchet derivative $\dfrac{\partial}{\partial x}B_k(x)$ exists in X. For every $\rho > 0$, there exists a constant $\bar{h}_k(\rho) > 0$ such that

$$\left\|\frac{\partial}{\partial x}B_k(x)\right\|_{\mathcal{L}(X)} \leq \bar{h}_k(\rho)$$

for all $t \geq 0$, $k \in \mathbb{N}$ and all $x \in X$ such that $\|x\|_X \leq \rho$.

3.3.1 Definitions of Solutions

Definition 3.3.1 A function $x \in PC([0,\infty), X)$ is said to be a mild solution of impulsive system (3.3.1) with initial condition $x(0) = x_0 \in X$ if x is given by

$$\begin{aligned} x(t) = \;& \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)[f(s,x(s)) + p(s,x(s),\xi)]ds \\ & + \sum_{0<\tau_k<t} \mathcal{S}(t,\tau_k)[c_k + q_k(x(\tau_k),\xi)]. \end{aligned} \tag{3.3.2}$$

Definition 3.3.2 A function $x \in PC([0,\infty), X)$ is said to be a periodic mild solution of system (3.3.1) if it is a mild solution and there exists $T_0 > 0$ such that $x(t+T_0) = x(t)$ for all $t \geq 0$.

Definition 3.3.3 A function $x \in PC([0,\infty), X)$ is said to be a T_0-periodic mild solution of system (3.3.1) if it is a mild solution and $x(t+T_0) = x(t)$ for all $t \geq 0$.

3.3.2 Existence and Uniqueness of Mild Solutions

At first, we consider the following reference system

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + f(t, x(t)), \quad t \neq \tau_k, \\ \Delta x(t) &= B_k x(t), \quad t = \tau_k, \end{cases} \tag{3.3.3}$$

and assume that $x_{T_0}(t)$ is a T_0- periodic mild solution of the reference system (3.3.3) which satisfies

$$x_{T_0}(t) = \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)f(s,x(s))ds. \tag{3.3.4}$$

Next, we consider the following variation system

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + \frac{\partial}{\partial x} f(t, x_{T_0}(t))x(t), \quad t \neq \tau_k, \\ \Delta x(t) &= \frac{\partial}{\partial x} B_k(x_{T_0}(t))x(t), \quad t = \tau_k, \end{cases} \tag{3.3.5}$$

and assume that the variation system (3.3.5) has only trivial solution.

Theorem 3.3.4 *Let assumption (A1) and (A4) holds. Suppose $x_{T_0}(t)$ be a T_0-periodic mild solution of the reference system (3.3.3) satisfies*

$$\rho_0 = \sup_{t \in [0,T_0]} \|x_{T_0}(t)\|_X.$$

Assume that

1. *system (3.3.5) has only trivial solution,*

2. *let $\tilde{\xi} > 0$ and $\varepsilon_o \in (0, \rho - \rho_0)$ such that $\eta < 1$ with*

$$\eta := M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0)]\sigma + [T_0 + \sigma] \sup_{\xi \in [0,\tilde{\xi}]} \omega(\xi)\Big)$$

where

$$M = \sup_{0 \leq s \leq t \leq T_0} \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)},$$

$$\bar{h}_k(\varepsilon_0) = \sup_{k \in \mathbb{N},\, \|y\| \leq \varepsilon_0} \left\| \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k) + y(\tau_k)) \right\|_X,$$

3. the following inequality is valid

$$\sup_{t\in[0,T_0],\,|\xi|\le\widetilde{\xi}} \Bigg\| \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)[p(s,x_{T_0}(s),\xi)]ds$$

$$+ \sum_{0\le\tau_k<T_0} \mathcal{S}(t,\tau_k)[c_k + q_k(x_{T_0}(\tau_k),\xi)] \Bigg\|_X \le \varepsilon_0(1-\eta).$$

Then for any constant $\rho > \rho_0 > 0$, *there exists a sufficiently small* $\widetilde{\xi} > 0$ *such that for every fixed* $\xi \in [0,\widetilde{\xi}]$ *system (3.3.1) has a unique* T_0*-periodic mild solution* $x^{\xi}_{T_0}(t)$ *satisfying*

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\| < \varepsilon_0 \quad \textit{for all} \quad t \ge 0 \tag{3.3.6}$$

and
$$\lim_{\xi\to 0} x^{\xi}_{T_0}(t) = x_{T_0}(t) \qquad \textit{uniformly on } t.$$

Proof. Let $x(t) = x_{T_0}(t) + y(t)$, then we can change system (3.3.1) into

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + \frac{\partial}{\partial x} f(t,x_{T_0}(t))y(t) + o(t,y(t)) + p(t,x_{T_0}(t)+y(t),\xi), \; t \ne \tau_k, \\ \Delta y(t) &= \frac{\partial}{\partial x} B_k(x_{T_0}(t))y(t) + o_k(y(t)) + c_k + q_k(x_{T_0}(t)+y(t),\xi), \qquad t = \tau_k, \end{aligned} \tag{3.3.7}$$

where

$$\begin{aligned} o(t,y(t)) &= f(t,x_{T_0}(t)+y(t)) - f(t,x_{T_0}(t)) - \frac{\partial}{\partial x} f(t,x_{T_0}(t))y(t) \\ o_k(y(t)) &= B_k(x_{T_0}(t)+y(t)) - B_k(x_{T_0}(t)) - \frac{\partial}{\partial x} B_k(x_{T_0}(t))y(t) \end{aligned} \tag{3.3.8}$$

Let $\quad PC_{T_0}([0,T_0];X) := \{x \in PC([0,T_0];X) \mid x(0) = x(T_0)\}$

with norm

$$\|x\|_{PC_{T_0}} = \sup_{t\in[0,T_0]} \|x(t)\|_X.$$

Let us define

$$\mathcal{B} := \mathcal{B}(\varepsilon_0) = \{y \in PC_{T_0}([0,T_0];X) \mid \|y\|_{PC_{T_0}} \le \varepsilon_0\} \tag{3.3.9}$$

and an operator $\Omega : \mathcal{B} \to PC_{T_0}([0,T_0];X)$ such that

$$\begin{aligned} \Omega(x)(t) := \mathcal{S}(t,0)x_0 &+ \int_0^t \mathcal{S}(t,s)\Big[o(s,y(s)) + p(s,x_{T_0}(s)+y(s),\xi)\Big]ds \\ &+ \sum_{0\le\tau_k<t} \mathcal{S}(t,\tau_k)[o_k(y(\tau_k)) + c_k + q_k(x_{T_0}(\tau_k)+y(\tau_k),\xi)]. \end{aligned} \tag{3.3.10}$$

If $y \in \mathcal{B}$, then

$$\begin{aligned} \|x\|_{PC_{T_0}} &= \|x_{T_0} + y\|_{PC_{T_0}} \\ &\leq \|x_{T_0}\|_{PC_{T_0}} + \|y\|_{PC_{T_0}} \\ &\leq \rho_0 + \varepsilon_0 \\ &\leq \rho_0 + (\rho - \rho_0) = \rho. \end{aligned}$$

From equation (3.3.10), we have

$$\begin{aligned} \Omega(x_{T_0})(t) = \; & \mathcal{S}(t,0)x_0 + \int_0^t \mathcal{S}(t,s)\Big[p(s, x_{T_0}(s), \xi)\Big] ds \\ & + \sum_{0 \leq \tau_k < t} \mathcal{S}(t,\tau_k)[c_k + q_k(x_{T_0}(\tau_k), \xi)]. \end{aligned} \tag{3.3.11}$$

For any $x, x_{T_0} \in \mathcal{B}$, then we have

$$\begin{aligned} & \|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} \\ & \leq \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \|o(s, y(s)) + p(s, x_{T_0}(s) + y(s), \xi) - p(s, x_{T_0}(s), \xi)\|_X ds \\ & + \sum_{0 \leq \tau_k < t} \|\mathcal{S}(t,\tau_k)\|_{\mathcal{L}(X)} \|o_k(y(\tau_k)) + q_k(x_{T_0}(\tau_k) + y(\tau_k), \xi) - q_k(x_{T_0}(\tau_k), \xi)\|_X \\ & \leq \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \left(\left\| f(t, x_{T_0}(s) + y) - f(t, x_{T_0}(s)) - \frac{\partial}{\partial x} f(t, x_{T_0}(s))y \right\|_X \right. \\ & \left. + \|p(s, x_{T_0}(s) + y(s), \xi) - p(s, x_{T_0}(s), \xi)\|_X \right) ds \\ & + \sum_{0 \leq \tau_k < t} \|\mathcal{S}(t,\tau_k)\|_{\mathcal{L}(X)} \left(\left\| B_k(x_{T_0}(\tau_k) + y) - B_k(x_{T_0}(\tau_k)) - \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k))y \right\|_X \right. \\ & \left. + \|q_k(x_{T_0}(\tau_k) + y(\tau_k), \xi) - q_k(x_{T_0}(\tau_k), \xi)\|_X \right) \\ & \leq \int_0^t \|\mathcal{S}(t,s)\|_{\mathcal{L}(X)} \left(\|f(t, x_{T_0}(s) + y) - f(t, x_{T_0}(s))\| + \left\| \frac{\partial}{\partial x} f(t, x_{T_0}(s))y \right\|_X \right. \\ & \left. + \|p(s, x_{T_0}(s) + y(s), \xi) - p(s, x_{T_0}(s), \xi)\|_X \right) ds \\ & + \sum_{0 \leq \tau_k < t} \|\mathcal{S}(t,\tau_k)\|_{\mathcal{L}(X)} \big(\|B_k(x_{T_0}(\tau_k) + y) - B_k(x_{T_0}(\tau_k))\| \\ & + \left\| \frac{\partial}{\partial x} B_k(x_{T_0}(\tau_k))y \right\|_X + \|q_k(x_{T_0}(\tau_k) + y(\tau_k), \xi) - q_k(x_{T_0}(\tau_k), \xi)\|_X \Big) \\ & \leq M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0) + \omega(\xi)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0) + \omega(\xi)]\sigma \Big) \|x - x_{T_0}\|_{PC_{T_0}}. \end{aligned}$$

Let us choose $\widetilde{\xi} > 0$ and $\varepsilon_0 \in (0, \rho - \rho_0)$ such that $\eta < 1$ with

$$\eta := M\Big([K_2(\varepsilon_0) + K_3(\varepsilon_0)]T_0 + [h_k(\varepsilon_0) + \bar{h}_k(\varepsilon_0)]\sigma + [T_0 + \sigma] \sup_{\xi \in [0, \widetilde{\xi}]} \omega(\xi)\Big) \quad (3.3.12)$$

So
$$\|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} \leq \eta \|x - x_{T_0}\|_{PC_{T_0}} \quad (3.3.13)$$

It follows from (3.3.11), (3.3.13) and assumption (3) that

$$\begin{aligned}\|\Omega(x)\|_{PC_{T_0}} &\leq \|\Omega(x) - \Omega(x_{T_0})\|_{PC_{T_0}} + \|\Omega(x_{T_0})\|_{PC_{T_0}} \\ &\leq \eta \|x - x_{T_0}\|_{PC_{T_0}} + \varepsilon_0(1 - \eta) \\ &\leq \eta\varepsilon_0 + \varepsilon_0(1 - \eta) = \varepsilon_0\end{aligned}$$

from which we know that $\Omega(x) \in \mathcal{B}$, then $\Omega : \mathcal{B} \to \mathcal{B}$ is a contraction mapping. Therefore, there exists a unique fixed point $y_1(t) \in \mathcal{B}$. From the fact that $y_1(t)$ is a solution of system (3.3.7), we know $x^{\xi}_{T_0}(t) = x_{T_0}(t) + y_1(t)$ is a T_0- periodic mild solution of (3.3.1) and satisfies

$$\|x^{\xi}_{T_0}(t) - x_{T_0}(t)\| = \|y_1(t)\| < \varepsilon_0.$$

So we have $\qquad \lim_{\xi \to 0} x^{\xi}_{T_0} = x_{T_0}(t) \qquad$ uniformly on t.

This completes the proof. □