CHAPTER II

MATHEMATICAL PRELIMINARIES

In this chapter, we review the theoretical background from functional analysis, real analysis which will be used throughout this thesis. Most theories are without prooves which can be found in the standard textbooks (see Ahmed (1991), Erwin Kreyszig (1978) and Pazy (1983) for example).

2.1 Elements of Functional Analysis

2.1.1 Normed Linear Spaces

Definition 2.1.1 Let X be a vector space over field $\mathbb{F}$, (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{R}$ is said to be a *norm* on X if it satisfies :

(N1) $\|x\| \geq 0$, for all $x \in X$

(N2) $\|x\| = 0 \;\Leftrightarrow\; x = 0$, for all $x \in X$

(N3) $\|\alpha x\| = |\alpha|\|x\|$, $\alpha \in \mathbb{F}$, for all $x \in X$

(N4) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$

Definition 2.1.2 A sequence $\{x_n\}$ in a normed space $(X, \|\cdot\|)$ is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that

$$\|x_m - x_n\| < \varepsilon, \quad \text{for all } m, n > N.$$

Definition 2.1.3 A normed linear space X is said to be *complete* if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Definition 2.1.4 A normed linear space X is said to be a *Banach space* if it is complete.

2.1.2 Linear operators

Definition 2.1.5 Let X and Y be vector spaces. A *linear operator* or a *linear map* T from X into Y is a function $T : X \to Y$ such that

(i) $T(x+y) = T(x) + T(y)$ for all $x, y \in X$,

(ii) $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and $\alpha \in \mathbb{F}$.

Definition 2.1.6 Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. Then T is said to be *bounded* if there exists $M > 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Theorem 2.1.7 *Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. Then the following statement are equivalent :*

(i) *T is continuous at 0, the zero vector in X,*

(ii) *T is continuous on X,*

(iii) *T is bounded on X.*

Let X and Y be normed spaces. Consider the set $\mathcal{L}(X, Y)$ consisting of all bounded linear operator from X to Y. $\mathcal{L}(X, Y)$ becomes a normed linear space if we define vector operations in a natural way and define the operator norm $\|T\|_{\mathcal{L}(X,Y)} = \sup\limits_{\|x\|_X \leq 1} \|Tx\|_Y$. If $X = Y$, we simply write $\mathcal{L}(X)$. Moreover, we have the following theorem.

Theorem 2.1.8 *If X is a normed linear space and Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space.*

Lemma 2.1.9 *If $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators, then $ST : X \to Z$ is also a bounded linear operator. Moreover,*

$$\|ST\| \leq \|S\|\|T\|.$$

Theorem 2.1.10 *(Uniform Boundedness Principle). Let X and Y be Banach spaces and $\mathcal{T} \subset \mathcal{L}(X, Y)$. Then,*

$$\sup_{T \in \mathcal{T}} \|Tx\|_Y < \infty, \ \forall x \in X \quad \textit{implies that} \quad \sup_{T \in \mathcal{T}} \|T\|_{\mathcal{L}(X,Y)} < \infty.$$

2.1.3 Closed Operators

Definition 2.1.11 Let X and Y be normed linear spaces and $T : X \to Y$ a function. The *graph* of T, denote by $\mathcal{G}(T)$, is defined by

$$\mathcal{G}(T) = \{(x, Tx) \mid x \in X\} \subset X \times Y.$$

If T is linear, it is easy to verify that $\mathcal{G}(T)$ is a linear subspace of $X \times Y$. We say that the map $T : X \to Y$ has a *closed graph* or T is a *closed operator* if $\mathcal{G}(T)$ is a closed subspace of $X \times Y$.

The following lemma gives a characterization of the closedness of a linear operator in terms of sequences.

Lemma 2.1.12 *Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. Then T has a closed graph if and only if for every sequence $\{x_n\}$ in X, if $x_n \to x$ and $Tx_n \to y$, then $y = Tx$.*

Theorem 2.1.13 *(Closed Graph Theorem). Suppose that X and Y are Banach spaces and $T : X \to Y$ a linear operator. Then T is bounded if and only if T has a closed graph.*

Definition 2.1.14 Let X be a Banach space, Y a subspace(not necessarily closed) of X and let $A : D(A) \subset X \to X$ be a linear operator in X. The subspace Y of X is an *invariant subspace* of A if $A : D(A) \cap Y \to Y$.

2.1.4 Compact Linear Operators

First, we recall the following facts from topology.

Definition 2.1.15 A subset M of a topological space X is *compact* if every open cover of M contains a finite subcover.

Definition 2.1.16 Let X and Y be normed spaces. An operator $A : X \to Y$ is called a *compact linear operator* (or completely continuous linear operator) if A is linear and if for every bounded subset M of X, the image $A(M)$ is *relatively compact*, that is, the closure $\overline{A(M)}$ is compact.

Definition 2.1.17 (ε-net, total boundedness). Let B be a subset of a metric space X and $\varepsilon > 0$ be given. A set $M_\varepsilon \subset X$ is called an ε-net *for* B if for every point $z \in B$ there is a point of M_ε at a distance from z less than ε. The set B is said to be *totally bounded* if for every $\varepsilon > 0$ there is a *finite* $\varepsilon-$net $M_\varepsilon \subset X$ for B, where "finite" means that M_ε is a finite set (that is, consists of finitely many points).

Lemma 2.1.18 *Let B be a subset of a metric space X.*

1. *If B is relatively compact, then B is totally bounded.*
2. *If B is totally bounded and X is complete, then B is relatively compact.*
3. *If B is totally bounded, then for every $\varepsilon > 0$ it has a finite $\varepsilon-$net $M_\varepsilon \subset B$.*

Theorem 2.1.19 *Let $T : X \to X$ be a compact linear operator and $S : X \to X$ a bounded linear operator on a normed space X. Then TS and ST are compact.*

The following fixed point theorems are the main tools in the proof of the existence of periodic mild solutions for linear and semilinear periodic systems with impulses.

Definition 2.1.20 Let X be a Banach space and let $A : X \to X$ be an operator (not necessarily linear). A *fixed point* of A is a point $x \in X$ such that

$$Ax = x.$$

In other words, a fixed point of A is solution of the equation

$$Ax = x, \ x \in X.$$

Definition 2.1.21 Let X be a Banach space and let $A : X \to X$ be an operator. The operator A is called *Lipschitz continuous* (or, briefly, A is Lipschitz) if

$$||Ax - Ay|| \leq L||x - y||$$

for some constant L and all $x, y \in X$. If $0 \leq L < 1$ is called a *contraction.*

Theorem 2.1.22 *(The Contraction Mapping Theorem). Let X be a Banach space and let $A : X \to X$ be a contraction. Then the equation*

$$Ax = x$$

has a unique solution in X, i.e., A has a unique fixed point x. Further, this fixed point may be obtained by the method of successive approximations as follow:

$$x_0 \in X \text{ arbitrary}, \quad x_n = Ax_{n-1}(n \geq 1)\,; \; x = \lim_{n \to \infty} x_n.$$

Corollary 2.1.23 *Let X_0 be a closed subset of the Banach space X and assume that A maps X_0 into itself and is a contraction on X_0. The equation $Ax = x$ has a unique solution $x \in X_0$.*

Theorem 2.1.24 *(Schauder Fixed Point Theorem). Let G be a compact convex subset in a Banach space B and let T be a continuous mapping of G into itself. Then T has a fixed point.*

Corollary 2.1.25 *Let G be a compact convex subset in a Banach space B and let T be a continuous mapping of G into itself such that the image TG is relatively compact. Then T has a fixed point.*

Corollary 2.1.26 *Let G be a compact convex subset in a Banach space B and let T be a continuous mapping of G such that $TG \subseteq G$. Then,T has at least a fixed point in G.*

Theorem 2.1.27 *(Leray-Schauder Fixed Point Theorem). Let G be a compact mapping of a Banach space B into itself and suppose there exists a constant M such that*

$$\|x\|_B < M$$

for all $x \in B$ and $\lambda \in [0,1]$ satisfying $x = \lambda Gx$. Then G has a fixed point.

The proof can be found in Gilbarg and Trudinger (1977).

Theorem 2.1.28 *(Arzela-Ascoli). Let X and Y be Banach spaces, $G \subset X$ be compact and $\mathcal{F} \subset C(G,Y)$. Suppose that*

1. *for each $x \in G$, the set $\{F(x) \mid F \in \mathcal{F}\}$ is relatively compact in Y.*
2. *$\mathcal{F}$ is uniformly bounded, i.e.,*

$$\sup_{F \in \mathcal{F},\, x \in G} \|F(x)\|_Y < \infty.$$

3. *$\mathcal{F}$ is equicontinuous, i.e., for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|F(x) - F(y)\|_Y < \varepsilon, \text{ when ever } \|x - y\|_X < \delta, \ F \in \mathcal{F}, \ x, y \in G.$$

Then there exists a sequence $\{F_k\} \subseteq \mathcal{F}$ and $F_0 \in C(G,Y)$ such that

$$\lim_{k \to \infty} \|F_k - F_0\|_{C(G,Y)} = 0$$

where $C(G,Y)$ denotes the supremum norm.

The proof can be found in Xunjing Li and Jiongmin Yong (1995).

2.1.5 Spectral Properties of Compact Linear Operaters

In this section, we consider spectral properties of a compact linear operator $T : X \to X$ on a normed space X. For this purpose we use the operator

$$T_\lambda = T - \lambda I \qquad (\lambda \in \mathbb{C}) \tag{2.1.1}$$

where I is the identity operator on X.

Definition 2.1.29 Let X be a complex Banach space and let $T : D(T) \subset X \to X$ be a linear, not necessarily bounded operator. The *resolvent set* $\rho(T)$ of T is the set of all complex numbers λ for which $T - \lambda I$ is invertible, i.e., $(T - \lambda I)^{-1}$ is a bounded linear operator in X, that is, the resolvent set $\rho(T)$ of T is given by

$$\rho(T) = \{ \lambda \in \mathbb{C} : (T - \lambda I)^{-1} \in \mathcal{L}(X)\},$$

I is the identity operator on X. When $\lambda \in \rho(T)$, $R(\lambda, T) = (T - \lambda I)^{-1}$ is called the *resolvent operator* of T at λ.

Theorem 2.1.30 *The set of eigenvalues of a compact linear operator $T : X \to X$ on a normed space X is countable (perhaps finite or even empty), and the only possible point of accumulation is $\lambda = 0$.*

Theorem 2.1.31 *Let $T : X \to X$ be a compact linear operator on a normed space X. Then for every $\lambda \neq 0$ the null space $\mathcal{N}(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional.*

Theorem 2.1.32 *Let $T : X \to X$ be a compact linear operator on a normed space X. Then for every $\lambda \neq 0$ the range of $T_\lambda = T - \lambda I$ is closed.*

2.2 Integration Theory

In this section, we review some basic concept of measurable functions and Bochner integral for Banach space valued functions. We then state some standard convergence theorems for integrals. For details and proofs we refer to

Zeidler (1990), unless we state otherwise.

2.2.1 Measurable Functions

Let $M \subset \mathbb{R}^n$ be a measurable set and X a Banach space.

Definition 2.2.1 1. A function $f : M \to X$ is called a *step function* if there exist finitely many pairwise disjoint measurable subsets M_i of M such that $|M_i| < \infty$ for all i and element a_i of X such that

$$f(x) = \begin{cases} a_i, & \text{if} \quad x \in M_i, \\ 0, & \text{otherwise.} \end{cases}$$

That is, f is constant on each set M_i.

2. The integral of a step function is defined to be

$$\int_M f dx = \sum_i |M_i| a_i.$$

3. A function $f : M \to X$ is called (strongly) *measurable* if there exists a sequence $\{f_n\}$ of step functions $f_n : M \to X$ such that

$$\lim_{n \to \infty} f_n(x) = f(x) \quad \text{for almost all} \quad x \in M.$$

4. (Measurable functions via substitution). Let X, U be real and separable Banach spaces, $M \subseteq \mathbb{R}^n$ be measurable, $f : M \times U \to X$ and $u : M \to U$. Set

$$F(x) = f(x, u(x)).$$

If the function $u : M \to U$ is measurable, then the function $F : M \to X$ is also measurable provided that f satisfies the *Caratheodory condition* :

(i) $x \mapsto f(x, u)$ is measurable on M for all $u \in U$.

(ii) $u \mapsto f(x, u)$ is continuous on U for almost all $x \in M$.

2.2.2 Bochner Integral

Definition 2.2.2 A function $f : \Omega \to X$ is called *simple* if there exist $x_1, x_2, \ldots, x_n \in X$ and $E_1, E_2, \ldots, E_n \in \mathcal{M}$ such that

$$f(x) = \sum_{i=1}^{n} x_i \chi_{E_i}(x),$$

where χ_{E_i} is the characteristic function of a measurable set E_i and the set E_i are pairwise disjoint with union Ω.

Definition 2.2.3 A function $f : \Omega \to X$ is called *Bochner integrable* if Ω is measurable and there exists a sequence $\{f_n\}$ of simple functions $f_n : \Omega \to X$ such that

1. $f(x) = \lim_{n\to\infty} f_n(x)$ for almost all $x \in \Omega$,

2. given $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\int_\Omega \|f_m(x) - f_n(x)\|_X dx < \varepsilon \quad \text{for all} \quad m, n \geq n_0(\varepsilon).$$

Theorem 2.2.4 *A strongly measurable function* $f : \Omega \to X$ *is Bochner integrable if and only if* $\int_\Omega \|f(x)\| dx < \infty$.

Theorem 2.2.5 *If* $B \in \mathcal{L}(X)$ *and the integral* f *exists, then the integral of* Bf *exists and*

$$\int_\Omega Bf(x)dx = B\int_\Omega f(x)dx.$$

Theorem 2.2.6 *(Majorant criterion). Let* $f : \Omega \to X$ *be measurable. If there exists* $g : \Omega \to \mathbb{R}$ *such that* $\|f(x)\|_X \leq g(x)$ *for almost all* $x \in \Omega$ *and* $\int_\Omega g(x)dx$ *exists, then* f *is integrable and*

$$\left\|\int_\Omega f(x)dx\right\|_X \leq \int_\Omega \|f(x)\|_X dx \leq \int_\Omega g(x)dx.$$

2.2.3 Fréchet Derivative

Definition 2.2.7 A function f defined on an open subset D of a normed space X with values in a normed space Y is *Fréchet differentiable* at $x \in D$ if there exists a bounded linear operator $\partial f(x) \in \mathcal{L}(X, Y)$ such that if

$$\rho(x, h) := f(x+h) - f(x) - \partial f(x)h, \quad (x, x+h \in D),$$

then

$$\lim_{h \to 0} \frac{\|\rho(x, h)\|_X}{\|h\|_Y} = 0.$$

The operator $\partial f(x)$ is called the *Fréchet differential* or *Fréchet derivative* of f at x. Obviously, Fréchet differentiability implies continuity. The mean value theorem holds for Fréchet differentiable maps : we need it in the form

$$\|f(x) - f(y)\| \leq \|x - y\|_X \sup_{z \in I} \|\partial f(z)\|_{(X,Y)}$$

(I the segment joining x and y) valid for D convex. The Fréchet differentiable is of course the calculus differential if $X = \mathbb{R}^m$.

2.3 Differential Equations on Banach Spaces

In this section, we introduce the concept and results on semigroups of operators via differential equations on Banach spaces which are abstract formulation of initial value problem for partial differential equations. For more details and proofs, we refer to Fattorini (1999).

2.3.1 The Homogeneous Initial Value Problem

Let X be a Banach space and let $A(t) : D(A(t)) \subset X \to X$ be a given operator. Consider the differential equation on X given by

$$\begin{cases} \dot{x}(t) = A(t)x(t), & t > 0 \\ x(0) = x_0. \end{cases} \tag{2.3.1}$$

Definition 2.3.1 The Cauchy problem (2.3.1) is said to have a *classical solution* if for each given $x_0 \in D(A(t))$ there exists a function $x \in C([0,\infty),X)$ satisfying the following properties :

(i) $x \in C([0,\infty),X) \cap C^1((0,\infty),X)$,

(ii) $x(t) \in D(A(t))$ for all $t > 0$,

(iii) (2.3.1) is satisfied, i.e., $\begin{cases} \dot{x}(t) = A(t)x(t), t > 0 \\ x(0) = x_0. \end{cases}$

Theorem 2.3.2 *Let* $\overline{D(A(t))} = X, \rho(A(t)) \neq \emptyset$. *Then (2.3.1) has a unique classical solution* $x(t)$ *which is continuously differentiable on* $[0,\infty)$, *for every initial value* $x_0 \in D(A(t))$ *if and only if* $A(t), t \in [0,T_0]$ *is a closed densely defined linear unbounded operator on* X.

2.3.2 The Inhomogeneous Initial Value Problem

Consider the inhomogeneous initial value problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t), & t > 0 \\ x(0) = x_0, \quad x_0 \in X \end{cases} \tag{2.3.2}$$

where $A(t), t \in [0,T_0]$ is a closed densely defined linear unbounded operator on X and $f \in L^1_{loc}([0,\infty),X)$.

Definition 2.3.3 A function $x : [0,T) \to X$ is a (classical) solution of (2.3.2) on $[0,T)$ if x is continuous on $[0,T)$, continuously differentiable on $(0,T)$, $x(t) \in D(A(t))$ for $0 < t < T$ and (2.3.2) is satisfied on $[0,T)$.

Theorem 2.3.4 *(Existence and Uniqueness). Let* $A(t), t \in [0,T_0]$ *is a closed densely defined linear unbounded operator on* X. *If* $f \in L^1([0,T],X)$ *then for* *every* $x_0 \in X$ *the initial value problem (2.3.2) has at most one solution. If it has a solution, this solution is given by*

$$x(t) = U(t, o)x_0 + \int_0^t U(t, s)f(s)ds, \quad 0 \le t \le T. \tag{2.3.3}$$

Definition 2.3.5 A function $x \in C([0, T], X)$ is said to be a *mild solution* of (2.3.2) corresponding to the initial state $x_0 \in X$ and the input $f \in L^1([0, T], X)$ if x is given by (2.3.3).

The definition of the mild solution of (2.3.2) coincides when $f \equiv 0$ with the definition of $U(t, o)x_0$ as the mild solution of the corresponding homogeneous equation. It is therefore clear that not every mild solution of (2.3.2) is a (classical) solution even in the case $f \equiv 0$.

Theorem 2.3.6 *Let* $A(t), t \in [0, T_0]$ *is a closed densely defined linear unbounded operator on* X, *let* $f \in L^1([0, T], X)$ *be continuous on* $(0, T)$ *and let*

$$v(t) = \int_0^t U(t, s)f(s)ds, \quad 0 \le t \le T.$$

The initial value problem (2.3.2) has a solution x *on* $[0, T)$ *for every* $x_0 \in D(A(t))$ *if one of the following conditions is satisfied;*

(i) $v(t)$ *is continuously differentiable on* $(0, T)$.

(ii) $v(t) \in D(A(t))$ *for* $0 < t < T$ *and* $A(t)v(t)$ *is continuous on* $(0, T)$.

Corollary 2.3.7 *Let* $A(t), t \in [0, T_0]$ *is a closed densely defined linear unbounded operator on* X, $f(s)$ *is continuously differentiable on* $[0, T]$ *then the initial value problem (2.3.2) has a solution* u *on* $[0, T)$ *for every* $x_0 \in D(A(t))$.

Corollary 2.3.8 *Let* $A(t), t \in [0, T_0]$ *is a closed densely defined linear unbounded operator on* X *and* $f \in L^1([0, T], X)$ *be continuous on* $(0, T)$. *If* $f(s) \in D(A(t))$, *then the initial value problem (2.3.2) has a solution on* $[0, T)$.

2.3.3 Semilinear Initial Value Problem and Perturbations Theory.

Consider the semilinear initial value problem

$$\dot{x}(t) = A(t)x(t) + f(t, x(t)), \qquad x(s) = \zeta, \tag{2.3.4}$$

where $A(t), t \in [0, T_0]$ is a closed densely defined linear unbounded operator on X and $f : [0, \infty) \times X \to X$. The assumption on $A(t)$ is that the initial value problem for the linear equation

$$\dot{x}(t) = A(t)x(t) \tag{2.3.5}$$

is well posed in $0 \leq t \leq T$, as defined in Fattorini (1999), pp 207. Below, $S(t, s)$ denotes the solution operator of (2.3.5), defined and strongly continuous in the triangle $0 \leq s \leq t \leq T$.

Define a solution of (2.3.4) as a solution of the integral equation

$$x(t) = S(t, s)\zeta(t) + \int_s^t S(t, \tau)f(\tau, x(\tau))d\tau. \tag{2.3.6}$$

We summarize in this section the necessary existence-uniqueness theory of (2.3.4). Result will be proved under two hypotheses on $f(t, x)$. The second hypothesis is stronger than the first.

Hypothesis I. $f(t, x)$ is strongly measurable in t for fixed x. For every $c > 0$ there exists $K(\cdot, c) \in L^1(0, T)$ such that

$$\|f(t, x)\| \leq K(t, c) \qquad (0 \leq t \leq T, \|x\| \leq c). \tag{2.3.7}$$

Hypothesis II. $f(t, x)$ is strongly measurable in t for fixed x. For every $c > 0$ there exists $K(\cdot, c)$, $L(\cdot, c) \in L^1(0, T)$ such that (2.17) holds and

$$\|f(t, x') - f(t, x)\| \leq L(t, c)\|x' - x\| \qquad (0 \leq t \leq T, \|x\|, \|x'\| \leq c). \tag{2.3.8}$$

Theorem 2.3.9 *Assume Hypothesis II holds in $0 \leq t \leq T$. Then the integral equation*

$$x(t) = \zeta(t) + \int_s^t S(t, \tau)f(\tau, x(\tau))d\tau \tag{2.3.9}$$

has a unique solution in some interval $s \le t \le T'$, where $s \le T' \le T$.

Theorem 2.3.10 *Let $x_1(\cdot)$ (respectively, $x_2(\cdot)$) be solution of (2.3.9) in $s \le t \le T'$ with $\zeta(t) = \zeta_1(t)$. (respectively, with $\zeta(t) = \zeta_2(t)$). Let c be a bound for $\|x_1(t)\|$, $\|x_2(t)\|$ in $s \le t \le T'$. Then*

$$\|x_1(t) - x_2(t)\| \le \sup_{s \le t \le T'} \|\zeta_1(t) - \zeta_2(t)\| \exp\left(M \int_s^t L(\tau, c) d\tau\right) \ (s \le t \le T).$$

In particular, if $x_1(\cdot)$ (respectively, $x_2(\cdot)$) is solution of (2.3.6) with $\zeta = \zeta_1$ (respectively, with $\zeta = \zeta_2$), then

$$\|x_1(t) - x_2(t)\| \le M\|\zeta_1 - \zeta_2\| \exp\left(M \int_s^t L(\tau, c) d\tau\right) \quad (s \le t \le T). \tag{2.3.10}$$

Lemma 2.3.11 *Let $x(t)$ be a solution of (2.3.9) in an interval $[s, T')$. Assume that*

$$\|x(t)\| \le c \quad (s \le t \le T'). \tag{2.3.11}$$

Then $x(\cdot)$ can be extended to an interval $[s, T'')$ with $T'' > T'$ (that is, a solution of (2.3.9) coinciding with $x(\cdot)$ in $s \le t \le T'$ exists in $[0, T'']$).

Corollary 2.3.12 *The solution $x(\cdot)$ of (2.3.9) exists in $s \le t \le T$ or in an interval $[s, T_m), T_m \le T$ and*

$$\sup_{t \to T_m^-} \|x(t)\| \le \infty. \tag{2.3.12}$$

Corollary 2.3.13 *Assume that there exists $K(\cdot) \in L^1(0, T)$ such that*

$$\|f(t, x)\| \le K(t)(1 + \|x\|) \quad (0 \le t \le T, x \in X). \tag{2.3.13}$$

Then (2.3.11) holds in every interval where the solution $x(t)$ of (2.3.9) exists accordingly, $x(t)$ exists in $s \le t \le T$.

The following theorem is one of the main tools in the proof of the existence of periodic mild solutions for the semilinear impulsive periodic systems with parameter perturbations discussed in this thesis. Its proof can be found in Fattorini (1999), pp.213.

Theorem 2.3.14 *Let the Cauchy problem for (2.3.5) be well posed in* $s \leq t \leq T$ *and let* $\{B(t), 0 \leq t \leq T\}$ *be a family of bounded linear operators in* X *such that (a) for each* $x \in X, t \to B(t)x$ *is strongly measurable, (b) there exists* $\alpha(\cdot) \in L^1(0,T)$ *such that*

$$\|B(t)\| \leq \alpha(t) \quad (0 \leq t \leq T). \tag{2.3.14}$$

Then the Cauchy problem for

$$\dot{x}(t) = (A(t) + B(t))u(t) \tag{2.3.15}$$

is well posed in $0 \leq t \leq T$*, solution of (2.3.15) with* $x(s) = \zeta$ *understood as solutions of the integral equation*

$$x(t) = S(t,s)\zeta + \int_s^t S(t,\tau)B(\tau)u(\tau)d\tau \tag{2.3.16}$$

If $U(t,s)$ *be the solution operator of (2.3.15), solutions of the inhomogeneous equation*

$$\dot{x}(t) = (A(t) + B(t))x(t) + f(t), \quad x(s) = \zeta \tag{2.3.17}$$

with $f(\cdot) \in L^1(0,T)$*, understood as solutions of the integral equation*

$$x(t) = S(t,s)\zeta + \int_s^t S(t,\tau)B(\tau)(x(\tau) + f(\tau))d\tau, \tag{2.3.18}$$

can be expressed by the variation of constants formula

$$x(t) = U(t,s)\zeta + \int_s^t U(t,\tau)f(\tau)d\tau. \tag{2.3.19}$$

2.4 Gronwall's Lemma

Theorem 2.4.1 *For* $t > t_0$ *let a nonnegative piecewise continuous function* $x(t)$ *satisfy*

$$x(t) \leq c + \int_{t_0}^t v(s)x(s)ds + \sum_{t_0 \leq \tau_n < t} b_n x(\tau_n)$$

where $c \geq 0, b_n \geq 0, v(s) > 0, x(t)$ *has discontinuous points of the first kind at* τ_n. *Then we have*

$$x(t) \leq c \prod_{t_0 \leq \tau_n < t} (1 + b_n) exp\Big(\int_{t_0}^{t} v(s) ds \Big).$$