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ดิวอี้เตอร์

Calculation of Quasi-normal Modes and Phase Transition of  
Topological anti de Sitter Black Holes

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### บทคัดย่อ

งานวิจัยเป็นกาศึกษาระบบหลุมดำที่มีค่าของสเกลาร์แฮร์ใกล้กับขอบเขตของเปรียเทนโลฮ์เนอร์-ฟรีด์แมน ณ อุณหภูมิต่ำ ในมิติใดๆ ระบบนี้มีสมบัติเหมือนกับตัวนำวดยิ่งโฮโลกราฟิก โดยพบว่า การควบแน่นของอนุภาคจะลู่ออกที่อุณหภูมิต่ำ  $|\ln T|^\delta$ ,  $\delta$  คือพารามิเตอร์ที่ขึ้นกับอุณหภูมิ ส่วนค่าสภาพนำไฟฟ้าได้มีการคำนวณเชิงวิเคราะห์ โดยพบโพลในผลที่ได้ โดยทุกโพลจะเคลื่อนเข้าแกนจริง และเมื่อค่าสภาพนำมีค่าเข้าสู่ศูนย์จะพบโพลจำนวนมากขึ้น จึงเป็นผลมาจากผลเฉลยที่เป็นฟังก์ชันแอร์รีใน  $2+1$  มิติ และฟังก์ชันแกมมาใน  $3+1$  มิติ ผลการคำนวณเชิงวิเคราะห์นี้มีผลสอดคล้องกับผลการคำนวณเชิงตัวเลข

งานวิจัยที่สองนี้ เป็นการคำนวณหาควอซีนอร์มอลโหมดและความถี่ ของสนามสเกลาร์ที่มีมวลและประจุ ในระบบหลุมดำทอพอโลยี 4 มิติ มีค่าความโค้งส่วนย่อยเป็นศูนย์ และมีสนามแมกเวลล์ ที่มีการเข้าคู่กับสนามสเกลาร์ โดยทำการคำนวณเชิงวิเคราะห์ สอดคล้องกับผลคำนวณเชิงตัวเลข

งานวิจัยที่สามนี้ เป็นการศึกษา หลุมดำคาลูซา-คลาน ที่ฮอริซันแบบสควอทใน 5 มิติ โดยทำการคำนวณเชิงวิเคราะห์ หาควอซีนอร์มอลโหมดและความถี่ ของสนามสเกลาร์ที่ไม่มีมวล โดยผลที่ได้มีค่าสอดคล้องกับผลการคำนวณเชิงตัวเลข

## Abstract

We discuss holographic superconductors in an arbitrary dimension whose dual black holes have a scalar hair of mass near the Breitenlohner–Freedman bound. We concentrate on low temperatures in the probe limit. We show analytically that when the bound is saturated, the condensate diverges at low temperatures as  $|\ln T|^\delta$ , where  $\delta$  depends on the dimension. This mild divergence was missed in earlier numerical studies. We calculate the conductivity analytically and show that at low temperatures, all poles move toward the real axis. We obtain an increasingly large number of poles which approach the zeroes of the Airy function in 2+1 dimensions and of the Gamma function in 3+1 dimensions. Our analytic results are in good agreement with numerical results whenever the latter are available.

The quasinormal modes(qnm) of the scalar field interacting with the Maxwell field in the 3+1 dimensional anti de Sitter(AdS) Reissener-Nordstrom(RN) space-time are analytically calculated for the zero order perturbation. To simplify the problem we study for the case of the sectional curvature  $k=0$ . Our result for zero and small perturbative charge case is consistent with the numerical result.

We study the Kaluza-Klein black holes, with squashed horizons. The system is perturbed by a massless scalar field. We analytically calculate the solution to the wave equation, which are called the quasinormal modes. We use approximation technique for the non-rotating case, where the frequencies to solution are a set of discrete complex number. Our analytical result is very well in agreement with the numerical work. Both results imply the existence of the stable ground state and the few lower excited states.

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## สารบัญ

ผลงานวิจัยลำดับที่

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# Holographic superconductors near the Breitenlohner–Freedman bound

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## Abstract

We discuss holographic superconductors in an arbitrary dimension whose dual black holes have a scalar hair of mass near the Breitenlohner–Freedman bound. We concentrate on low temperatures in the probe limit. We show analytically that when the bound is saturated, the condensate diverges at low temperatures as  $|\ln T|^\delta$ , where  $\delta$  depends on the dimension. This mild divergence was missed in earlier numerical studies. We calculate the conductivity analytically and show that at low temperatures, all poles move toward the real axis. We obtain an increasingly large number of poles which approach the zeroes of the Airy function in 2+1 dimensions and of the Gamma function in 3+1 dimensions. Our analytic results are in good agreement with numerical results whenever the latter are available.

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(Some figures may appear in colour only in the online journal)

## 1. Introduction

Using the AdS/CFT correspondence [1], it has been shown that if an Abelian symmetry has been broken outside of a black hole in AdS space, then a scalar hair can be formed creating a holographic superconductor in the dual CFT [2–5]. In the last few years, it has been observed that these systems exhibit several characteristics seen in real world's strongly coupled superconductors and seem very promising. The AdS/CFT correspondence has also been applied to other areas of condensed matter physics [6–9]; some reviews are [10–12].

Studying the conductivity of holographic superconductors, Horowitz and Roberts [4, 13] noted that at the Breitenlohner–Freedman (BF) bound of the scalar hair of the dual black hole, quasinormal modes appeared to move toward the real axis at low temperatures. Once the back reaction to the metric was included, they showed that the quasinormal modes never became

normal. Their number was determined by the height of an effective potential associated with the wave equation of an electromagnetic perturbation. The study was done numerically which limited the ability to go to very low temperatures.

Using the analytic tools developed recently in [14], we explore the low temperature regime of holographic superconductors near the BF bound. We find that in the probe limit, when the BF bound is saturated, the condensate diverges at low temperatures as  $|\ln T|^\delta$ , where  $\delta$  is a constant that we compute and depends on the dimension of spacetime. This is a very mild divergence which explains why it was not detected in earlier numerical studies [4]. It signals the breakdown of the probe limit at very low temperatures, and by including corrections to the probe limit we determine when this occurs. When back reaction to the metric is included, the effective potential associated with the electromagnetic perturbation that determines the conductivity has a finite height. This results in a finite number of quasinormal modes. As one approaches the probe limit, the height of the potential increases with an attendant increase in the number of modes. The latter approaches the real axis as the temperature is lowered at frequencies that we compute analytically. In 2+1 (3+1) dimensions they are given in terms of the zeroes of the Airy (Gamma) function.

We emphasize that, even though we probe the low temperature regime, we do not have access to the zero temperature state. This is because for a given charge of the scalar field, there is a lower bound on the temperature (for scaling dimensions  $\Delta \leq \frac{d}{2}$ ) below which the probe approximation breaks down. Thus, even though it is often possible to take a ‘zero temperature’ limit of our expressions, the limit itself is *unphysical*. It is, however, useful for computational purposes, as *physical* low-temperature systems are close to it, as we shall show.

Our paper is organized as follows. In section 2, we review the field equations. In section 3, we calculate the critical temperature. In section 4, we discuss the low temperature regime in the probe limit. We also discuss the validity of the probe limit, and find where it breaks down for a given value of the charge  $q$ . In section 5, we calculate the conductivity at the BF bound at low temperatures. Finally, section 6 contains our concluding remarks.

## 2. The field equations

We are interested in the dynamics of a scalar field of mass  $m$  and electric charge  $q$  coupled to a  $U(1)$  vector potential in the background of a  $(d + 1)$ -dimensional AdS black hole. The action is

$$S = \int d^{d+1} \sqrt{-g} \left[ \frac{R + d(d-1)/l^2}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |(\partial_\mu - iqA_\mu)\Psi|^2 - m^2 |\Psi|^2 \right], \quad (1)$$

where  $F = dA$ . We shall adopt units in which  $l = 1$ .

To find a solution of the field equations, consider the metric *ansatz*

$$ds^2 = \frac{1}{z^2} \left[ -f(z) e^{-\chi(z)} dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)} \right] \quad (2)$$

where  $\vec{x} \in \mathbb{R}^{d-1}$  represents an AdS black hole of planar horizon. The AdS boundary is at  $z = 0$ . We shall choose units so that the horizon is at  $z = 1$ , therefore we require  $f(1) = 0$ . This is possible because of scaling symmetries of the system and can be done without loss of generality as long as one is careful to only consider physical quantities which are scale invariant.

The Hawking temperature is

$$T = -\frac{f'(1)}{4\pi} e^{-\chi(1)/2}. \quad (3)$$

Assuming that the scalar field is a real function  $\Psi(z)$  and the potential is an electrostatic scalar potential,  $A = \Phi(z)dt$ , the field equations are [3]

$$\begin{aligned} \Psi'' + \left[ \frac{f'}{f} - \frac{\chi'}{2} - \frac{d-1}{z} \right] \Psi' + \left[ \frac{q^2 \Phi^2 e^\chi}{f^2} - \frac{m^2}{z^2 f} \right] \Psi &= 0 \\ \Phi'' + \left[ \frac{\chi'}{2} - \frac{d-3}{z} \right] \Phi' - \frac{2q^2 \Psi^2}{z^2 f} \Phi &= 0 \\ -\frac{d-1}{2} \chi' + z \Psi'^2 + \frac{z q^2 \Phi^2 \Psi^2}{f^2} e^\chi &= 0 \\ \frac{f}{2} \Psi'^2 + \frac{z^2}{4} \Phi'^2 e^\chi - \frac{d-1}{2} \frac{f'}{z} + \frac{d(d-1)}{2} \frac{f-1}{z^2} + \frac{m^2 \Psi^2}{2z^2} + \frac{q^2 \Psi^2 \Phi^2 e^\chi}{2f} &= 0, \end{aligned} \tag{4}$$

where the prime denotes differentiation with respect to  $z$ , to be solved in the interval  $(0, 1)$ , where  $z = 1$  is the horizon and  $z = 0$  is the boundary.

We are interested in solving the system of nonlinear equations (4) in the limit of large  $q$  (the probe limit). To this end, we shall expand the fields as a series in  $1/q$  as follows:

$$\begin{aligned} \Psi &= \frac{1}{q} \left[ \Psi_0 + \Psi_1 \frac{1}{q^2} + \dots \right] \\ \Phi &= \frac{1}{q} \left[ \Phi_0 + \Phi_1 \frac{1}{q^2} + \dots \right] \\ f &= f_0 + f_1 \frac{1}{q^2} + \dots \\ \chi &= \chi_0 + \chi_1 \frac{1}{q^2} + \dots \end{aligned} \tag{5}$$

and consider the zeroth-order system ( $q \rightarrow \infty$ ) first and then discuss the addition of first-order ( $\mathcal{O}(1/q^2)$ ) corrections in order to obtain a physically sensible system.

Near the boundary ( $z \rightarrow 0$ ), we have  $f \rightarrow 1$ ,  $\chi \rightarrow 0$  and so approximately

$$\Psi \approx \Psi^{(\pm)} z^{\Delta_\pm}, \quad \Phi \approx \mu - \rho z^{d-2} \tag{6}$$

where

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}. \tag{7}$$

While a linear combination of asymptotics is allowed by the field equations, it turns out that any such combination is unstable [17]. However, if the horizon has negative curvature, such linear combinations lead to stable configurations in certain cases [18].

Thus, the system is labeled uniquely by the dimension  $\Delta = \Delta_\pm$ . The mass of the scalar field is bounded from below by the BF bound [16],  $m^2 \geq -\frac{d^2}{4}$ , and there appears to be a quantum phase transition at  $m^2 = 0$ . There is also a unitarity bound that requires  $\Delta > \frac{d-2}{2}$ .

Demanding at the horizon

$$\Phi(1) = 0, \tag{8}$$

(gauge choice ensuring that  $A = \Phi dt$  is regular at the horizon [2]),  $\mu$  is interpreted as the chemical potential of the dual theory on the boundary.  $\rho$  is the charge density on the boundary, and the leading coefficient in the expansion of the scalar yields vacuum expectation values of operators of dimension  $\Delta_\pm$  is

$$\langle \mathcal{O}_{\Delta_\pm} \rangle = \sqrt{2} \Psi^{(\pm)}. \tag{9}$$

The field equations admit non-vanishing solutions for the scalar below a critical temperature  $T_c$  where these operators condense. In view of (6) and (9), it is convenient to define

$$\Psi(z) = \frac{1}{\sqrt{2q}} b^\Delta z^\Delta F(z), \quad b = \langle q \mathcal{O}_\Delta \rangle^{1/\Delta} \tag{10}$$

with  $F(0) = 1$ .

### 3. The critical temperature

Above the critical temperature,  $\Psi = 0$  and the field equations are solved by the AdS Reissner-Nördstrom black hole with a flat horizon,

$$f(z) = 1 - \left(1 + \frac{(d-2)\rho^2}{4}\right) z^d + \frac{(d-2)\rho^2}{4} z^{2(d-1)}, \quad \chi(z) = 0, \quad \Phi(z) = \rho(1 - z^{d-2}), \tag{11}$$

whose Hawking temperature (3) is

$$T = \frac{d}{4\pi} \left[1 - \frac{(d-2)^2 \rho^2}{4d}\right]. \tag{12}$$

Right at the critical temperature,  $\Psi$  obeys the scalar field equation (4) in the Reissner-Nördstrom background (11) with  $\rho = \rho_c$ . Thus  $F$  (equation (10)) at  $T = T_c$  obeys the field equation

$$F'' + \left[\frac{f'}{f} + \frac{2\Delta + 1 - d}{z}\right] F' + \left[\Delta \frac{(d-\Delta)(1-f) + zf'}{z^2 f} + q^2 \rho_c^2 \frac{(1-z^{d-2})^2}{f^2}\right] F = 0. \tag{13}$$

For a given  $q$ ,  $\rho_c$  is an eigenvalue which is determined by solving this equation for  $F$  subject to the boundary condition at the AdS boundary  $F(0) = 1$ . We also demand that at the horizon  $F(1)$  be finite, and that there be no contribution of the other solution (behaving as  $F(z) \sim z^{d-2\Delta}$  as  $z \rightarrow 0$ ). The latter condition implies that  $F$  has a Taylor expansion around  $z = 0$  with the properties  $F(0) = 1, F'(0) = 0$  (as can be easily deduced from (13)).

To solve equation (13) for large  $q$  (the probe limit), use expansion (5) to write

$$F = F_0 + F_1 \frac{1}{q^2} + \dots, \quad \rho = \frac{1}{q} \left[\rho_0 + \rho_1 \frac{1}{q^2} + \dots\right]. \tag{14}$$

Then at zeroth order (the  $q \rightarrow \infty$  limit), background (11) turns into an AdS Schwarzschild black hole, so

$$f_0(z) = 1 - z^d, \quad \chi_0 = 0; \tag{15}$$

and we obtain the field equation at the critical temperature:

$$-F_0'' + \frac{1}{z} \left[\frac{d}{1-z^d} - 2\Delta - 1\right] F_0' + \Delta^2 \frac{z^{d-2}}{1-z^d} F_0 = \rho_{0c}^2 \frac{(1-z^{d-2})^2}{(1-z^d)^2} F_0. \tag{16}$$

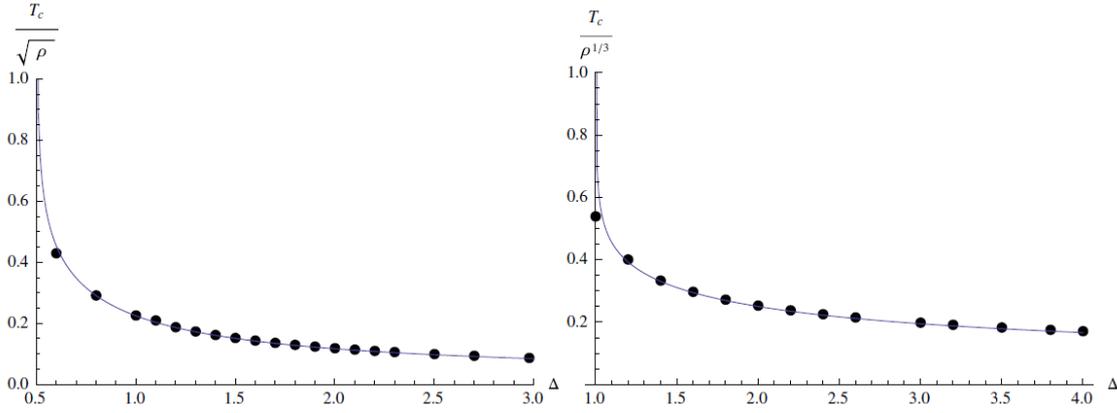
The eigenvalue  $\rho_{0c}^2$  minimizes the expression

$$\rho_{0c}^2 = \frac{\int_0^1 dz z^{2\Delta-d+1} \{(1-z^d)[F_0'(z)]^2 + \Delta^2 z^{d-2} [F_0(z)]^2\}}{\int_0^1 dz z^{2\Delta-d+1} \frac{(1-z^{d-2})^2}{1-z^d} [F_0(z)]^2}. \tag{17}$$

We can estimate the eigenvalue by substituting the trial function

$$F_0 = F_\alpha(z) \equiv 1 - \alpha z^{d-1} \tag{18}$$

which obeys the boundary conditions  $F_\alpha(0) = 1, F_\alpha'(0) = 0$  and  $F_\alpha(1)$  is finite.



**Figure 1.** The critical temperature  $T_c$  versus the scaling dimension  $\Delta$  for  $d = 3$  (left panel) and  $d = 4$  (right panel). Data points represent exact values; the solid line is obtained by minimizing (17) with the trial function (18).

For  $\Delta = \frac{d}{2}$  and  $d = 3, 4$ , we obtain, respectively,

$$\rho_{0c}^2 \approx 6.3, 4.2, \quad T_c \approx 0.15\sqrt{q\rho}, 0.2(q\rho)^{1/3} \tag{19}$$

in very good agreement with the exact  $T_c = 0.15\sqrt{q\rho}, 0.25(q\rho)^{1/3}$ . In figure 1, we extend the comparison to the entire range of the scaling dimension  $\Delta$  for  $d = 3, 4$  demonstrating the accuracy of estimate (17) with the trial function (18) for the critical temperature.

#### 4. Low temperatures

Next, we consider the low temperature regime. Because we are working in units in which the radius of the horizon is fixed ( $z = 1$ ), even at low temperatures,  $T$  is bounded. Therefore,  $\rho$  becomes large. Also, the condensate diverges in these units for the same reason, and so does  $b$  (equation (10)). We are interested in calculating scale-invariant quantities, such as the energy gap

$$E_g = \frac{\langle q\mathcal{O} \rangle^{1/\Delta}}{T_c} = \frac{b}{\rho_0^{1/(d-1)}} + \dots \tag{20}$$

We shall consider the probe limit ( $q \rightarrow \infty$ ), and then discuss how first-order corrections (in a  $1/q$  expansion) can be added to obtain a physically meaningful system.

In the probe limit, as we lower the temperature,  $F_0(z)$  (equation (14)) does not have a smooth limit as  $b \rightarrow \infty$  for  $\Delta = \Delta_+ > \frac{d}{2}$ . In this case, care should be exercised in taking this limit which corresponds to the zero temperature limit (the ground state) of the system.

We shall concentrate on the case  $\Delta = \Delta_- \leq \frac{d}{2}$  in which  $F_0(z)$  has a smooth limit as  $b \rightarrow \infty$ ; however, the limit system does not correspond to the zero temperature limit and is *not* physical. This is because the probe approximation breaks down as  $b \rightarrow \infty$ . As we shall show, the probe expansion is valid for  $b \lesssim q^{1/\Delta}$  (equation (51)). So for a given charge  $q \gg 1$ , we define the low temperature regime by

$$1 \lesssim b \lesssim q^{1/\Delta}. \tag{21}$$

We need to solve the system of zeroth-order equations,

$$\begin{aligned} -F_0'' + \frac{1}{z} \left[ \frac{d}{1-z^d} - 1 - 2\Delta \right] F_0' + \frac{\Delta^2 z^{d-2}}{1-z^d} F_0 - \frac{1}{(1-z^d)^2} \Phi_0^2 F_0 &= 0 \\ \Phi_0'' - \frac{d-3}{z} \Phi_0' - \frac{b^{2\Delta} z^{2(\Delta-1)}}{1-z^d} F_0^2 \Phi_0 &= 0. \end{aligned} \tag{22}$$

Let us first discuss the  $b \rightarrow \infty$  limit. Although it is *unphysical*, it is well defined and will be useful for our analytic calculations.

It is evident from the field equation for  $\Phi_0$  (and can be easily confirmed numerically for arbitrary regular functions  $F_0(z)$ ) that  $\Phi_0 \rightarrow 0$  for  $z \gtrsim 1/b$  (with  $b \gg 1$ ). Then for  $z \gtrsim 1/b$ , we obtain

$$F_0(z) = \mathcal{A}F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; 1; 1 - z^d\right) \tag{23}$$

which is regular at the horizon. As  $b \rightarrow \infty$ , this becomes valid in the entire interval because  $1/b \rightarrow 0$ . We deduce the  $b \rightarrow \infty$  function

$$F_0(z) = \frac{\Gamma^2(1 - \frac{\Delta}{d})}{\Gamma(1 - \frac{2\Delta}{d})} F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; 1; 1 - z^d\right), \tag{24}$$

where we used  $F_0(0) = 1$  and standard hypergeometric identities. However, this does not represent the ground state of a physical system, because as  $b \rightarrow \infty$ , the condition for a physical system (21) is violated. Nevertheless, expression (24) is useful for computational purposes.

Next, we wish to solve the equations in the low temperature regime (21). To this end, we shall use iteration as follows:

$$\begin{aligned} -F_0^{(n+1)''} + \frac{1}{z} \left[ \frac{d}{1 - z^d} - 1 - 2\Delta \right] F_0^{(n+1)'} + \frac{\Delta^2 z^{d-2}}{1 - z^d} F_0^{(n+1)} &= \frac{\mu^2}{(1 - z^d)^2} [\hat{\Phi}_0^{(n+1)}]^2 F_0^{(n)} \\ \hat{\Phi}_0^{(n+1)''} - \frac{d-3}{z} \hat{\Phi}_0^{(n+1)'} - \frac{b^{2\Delta} z^{2(\Delta-1)}}{1 - z^d} [F_0^{(n)}]^2 \hat{\Phi}_0^{(n+1)} &= 0, \end{aligned} \tag{25}$$

starting with

$$F_0^{(0)}(z) = 1, \quad \hat{\Phi}_0^{(0)}(z) = 0. \tag{26}$$

We defined

$$\Phi_0(z) = \mu \hat{\Phi}_0(z), \quad \hat{\Phi}_0(0) = 1, \tag{27}$$

where  $\mu$  is the chemical potential.

At the  $n$ th step, we obtain the electrostatic potential  $\hat{\Phi}_0^{(n+1)}$  from the second equation in (25) and subsequently the scalar field  $F_0^{(n+1)}$  from the first equation in (25). To solve the latter, note that there are three boundary conditions, two at  $z = 0$  ( $F_0^{(n+1)}(0) = 1$  and absence of the unwanted alternate behavior  $F_0^{(n+1)} \sim z^{d-2\Delta}$ ), and one at the horizon,  $z = 1$  (finiteness of  $F_0^{(n+1)}$ ). They are needed to determine the solution as well as the eigenvalue  $\mu$ . Using the two boundary conditions at  $z = 0$ , we obtain the solution in terms of the chemical potential:

$$\begin{aligned} F_0^{(n+1)}(z) = \mathcal{F}_1(z) \left[ 1 + \mu^2 \int_0^z \frac{dz'}{1 - (z')^d} (z')^{2\Delta+1-d} \mathcal{F}_2(z') [\hat{\Phi}_0^{(n+1)}(z')]^2 F_0^{(n)}(z') \right] \\ - \mathcal{F}_2(z) \mu^2 \int_0^z \frac{dz'}{1 - (z')^d} (z')^{2\Delta+1-d} \mathcal{F}_1(z') [\hat{\Phi}_0^{(n+1)}(z')]^2 F_0^{(n)}(z') \end{aligned} \tag{28}$$

where

$$\mathcal{F}_1(z) = F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; \frac{2\Delta}{d}; z^d\right), \quad \mathcal{F}_2(z) = \frac{z^{d-2\Delta}}{d - 2\Delta} F\left(1 - \frac{\Delta}{d}, 1 - \frac{\Delta}{d}; 2 - \frac{2\Delta}{d}; z^d\right). \tag{29}$$

The third boundary condition, regularity at  $z = 1$ , then fixes the chemical potential  $\mu$ .

For  $n = 0$ , we obtain the electrostatic potential

$$\hat{\Phi}_0^{(1)}(z) = \frac{2}{\Gamma(\nu)(2\Delta)^\nu} (bz)^{\frac{d-2}{2}} \left[ K_\nu\left(\frac{(bz)^\Delta}{\Delta}\right) - \frac{K_\nu\left(\frac{b^\Delta}{\Delta}\right)}{I_\nu\left(\frac{b^\Delta}{\Delta}\right)} I_\nu\left(\frac{(bz)^\Delta}{\Delta}\right) \right], \quad \nu = \frac{d-2}{2\Delta}, \tag{30}$$

where we imposed the boundary condition (8). Note that the second Bessel function has an exponentially small coefficient,  $\mathcal{O}(\sim e^{-2b^\Delta/\Delta})$ , and can be neglected at low temperatures.

The charge density is found by using (6) and (14) to be

$$\frac{\rho_0}{b^{d-2}} = -\frac{\mu}{(2\Delta)^{2\nu}}. \tag{31}$$

For the scalar field we obtain equation (28) with  $n = 0$ :

$$F_0^{(1)}(z) = \mathcal{F}_1(z) \left[ 1 + \mu^2 \int_0^z \frac{dz'}{1-(z')^d} (z')^{2\Delta+1-d} \mathcal{F}_2(z') [\hat{\Phi}_0^{(1)}(z')]^2 \right] - \mathcal{F}_2(z) \mu^2 \int_0^z \frac{dz'}{1-(z')^d} (z')^{2\Delta+1-d} \mathcal{F}_1(z') [\hat{\Phi}_0^{(1)}(z')]^2, \tag{32}$$

satisfying the correct boundary conditions at  $z = 0$ . At the horizon ( $z = 1$ ), we have a logarithmic singularity which is found using

$$\mathcal{F}_1(z) \approx -\frac{\Gamma(\frac{2\Delta}{d})}{\Gamma^2(\frac{\Delta}{d})} \ln(1-z), \quad \mathcal{F}_2(z) \approx -\frac{\Gamma(2-\frac{2\Delta}{d})}{(d-2\Delta)\Gamma^2(1-\frac{\Delta}{d})} \ln(1-z). \tag{33}$$

Near the horizon, we deduce

$$F_0^{(1)}(z) \approx -\left[ \frac{\Gamma(\frac{2\Delta}{d})}{\Gamma^2(\frac{\Delta}{d})} (1 + \mu^2 a_2) - \frac{\Gamma(2-\frac{2\Delta}{d})}{(d-2\Delta)\Gamma^2(1-\frac{\Delta}{d})} \mu^2 a_1 \right] \ln(1-z), \tag{34}$$

where

$$a_i = \int_0^1 \frac{dz}{1-z^d} z^{2\Delta+1-d} \mathcal{F}_i(z) [\hat{\Phi}_0^{(1)}(z)]^2, \quad i = 1, 2. \tag{35}$$

Demanding regularity at the horizon, we need

$$\frac{\Gamma(\frac{2\Delta}{d})}{\Gamma^2(\frac{\Delta}{d})} (1 + \mu^2 a_2) - \frac{\Gamma(2-\frac{2\Delta}{d})}{(d-2\Delta)\Gamma^2(1-\frac{\Delta}{d})} \mu^2 a_1 = 0, \tag{36}$$

which fixes the chemical potential  $\mu$ ,

$$\frac{1}{\mu^2} = \frac{\Gamma(2-\frac{2\Delta}{d})\Gamma^2(\frac{\Delta}{d})}{(d-2\Delta)\Gamma(\frac{2\Delta}{d})\Gamma^2(1-\frac{\Delta}{d})} a_1 - a_2, \tag{37}$$

as advertised.

Explicitly,

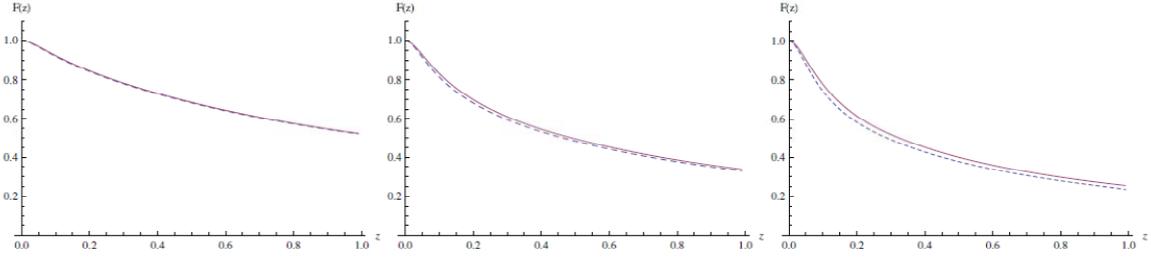
$$a_1 = \frac{1}{b^{2\Delta+2-d}} \frac{(d-2)\Gamma(1-\nu)}{(2\Delta)^{2\nu}\Gamma(\nu)} + \dots, \quad a_2 = \frac{1}{b^2} \frac{\sqrt{\pi} \Delta^{\frac{2}{d}-1} \Gamma(\frac{1}{d}) \Gamma(\frac{d-1}{d}) \Gamma(\frac{d}{2\Delta})}{(d-2\Delta)\Gamma^2(\nu) 2^{2\nu} \Gamma(\frac{d+\Delta}{2\Delta})} + \dots. \tag{38}$$

Evidently, for  $\Delta < \frac{d}{2}$ ,  $a_2 \ll a_1$  for  $b \ll 1$ ; therefore,

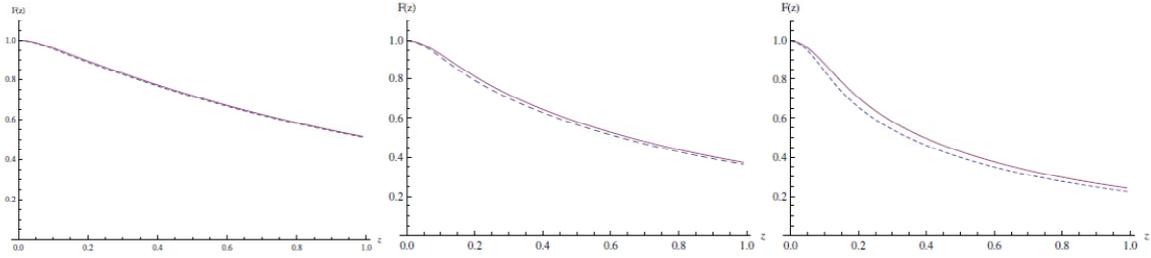
$$\mu^2 \approx \mathcal{C} b^{2\Delta+2-d}, \quad \mathcal{C} = \frac{(d-2\Delta)(2\Delta)^{2\nu}\Gamma(\nu)\Gamma(\frac{2\Delta}{d})\Gamma^2(1-\frac{\Delta}{d})}{(d-2)\Gamma(1-\nu)\Gamma(2-\frac{2\Delta}{d})\Gamma^2(\frac{\Delta}{d})}. \tag{39}$$

It is easily seen (using standard hypergeometric identities) that the low temperature expression (32) reduces to (24) as  $b \rightarrow \infty$  in the entire interval  $[0, 1]$ . However, as we have already pointed out, the latter is unphysical. By taking this limit, we do not gain access to a zero temperature system (the ground state). Nevertheless, it is useful computationally.

Before we consider the next iterative order, we note that at finite temperature, the first-order expression (32) is in excellent agreement with numerical results even at  $T/T_c \sim 0.1$ , which is the lowest temperature at which a numerical solution is available. This is shown in figures 2 and 3 in which the corresponding curves are almost indistinguishable, implying that



**Figure 2.** The field  $F$  (equation (10)) for  $\Delta = 1.2$  (left panel),  $1.4$  (middle panel),  $1.5$  (right panel) and  $d = 3$ . Solid curves are the first-order analytic expression (32) and dashed curves are exact numerical results (almost indistinguishable) at  $T/T_c \approx 0.1$ .



**Figure 3.** The field  $F$  (equation (10)) for  $\Delta = 1.6$  (left panel),  $1.8$  (middle panel),  $2$  (right panel) and  $d = 4$ . Solid curves are the first-order analytic expression (32) and dashed curves are exact numerical results (almost indistinguishable) at  $T/T_c \approx 0.2$ .

the next iterative order introduces negligible corrections to the first-order expression (32) for temperatures  $T/T_c \lesssim 0.1$ .

We can repeat the above steps for the next iterative order to calculate  $F_0^{(2)}$  and  $\hat{\Phi}_0^{(2)}$ . The resulting functions are very close to their first-order counterparts, showing that the iteration converges rather rapidly. In fact, the second-order quantities are subleading in  $1/b$ . This is the case for all values of the scaling dimension  $\Delta < \frac{d}{2}$ . In figure 4, we show the difference between first and second iterative order for  $d = 3$ ,  $\Delta = 1.4$  (all other values of  $\Delta$  are similar). The error,  $1 - \frac{F_0^{(2)}}{F_0^{(1)}}$ , is less than 5% in the entire interval  $[0, 1]$ . As the temperature decreases from  $T/T_c \sim 0.1$  to  $T/T_c \sim 0.01$ , the error decreases to less than 0.01. To demonstrate that the error is subleading in  $1/b$ , in figure 5 we plot it at the mid-point ( $z = \frac{1}{2}$ ) as well as the horizon ( $z = 1$ ) (at  $z = 0$  the error vanishes by design). Evidently, it is of order  $1/b$ , showing that the second iterative order introduces subleading corrections at low temperature.

For the charge density, we deduce from (31)

$$\rho_0 \sim b^{\frac{d}{2} + \Delta - 1}. \quad (40)$$

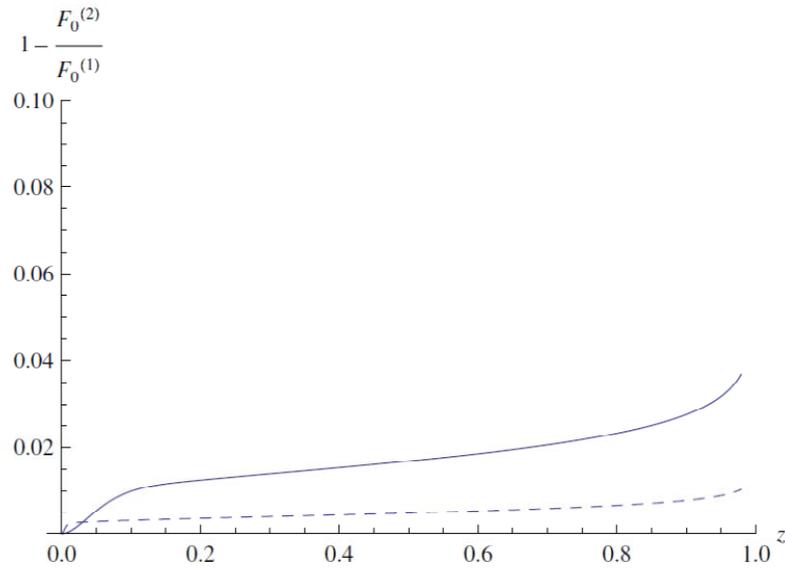
Using

$$\frac{\langle q\mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} \sim b\rho_0^{-\frac{1}{d-1}}, \quad \frac{T}{T_c} \sim \rho_0^{-\frac{1}{d-1}} \quad (41)$$

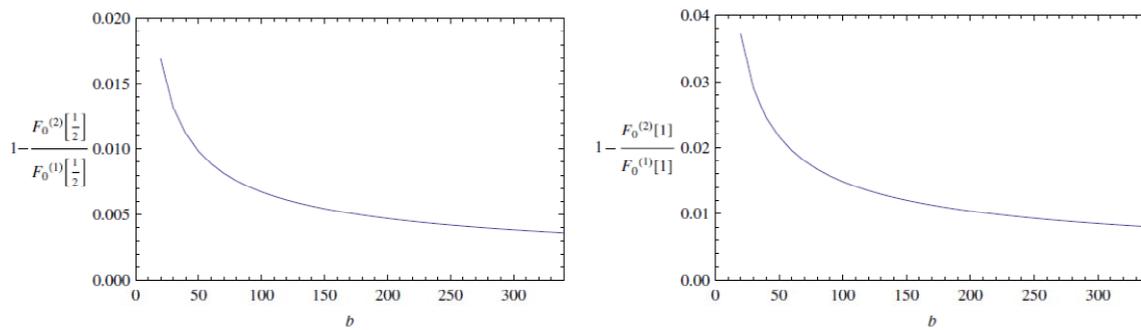
we finally obtain for the energy gap

$$E_g = \frac{\langle q\mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} = \gamma \left( \frac{T}{T_c} \right)^{-\frac{d/2 - \Delta}{d/2 + \Delta - 1}}, \quad (42)$$

showing that the condensate diverges at low temperatures (to be precise, in the regime (21)). The exponent depends on the dimensions of the operator and spacetime. The expression for



**Figure 4.** The second-order correction to the scalar field  $F$  for  $d = 3$ ,  $\Delta = 1.4$  at  $b = 20$  ( $T/T_c \sim 0.1$ ) (solid line) and  $b = 200$  ( $T/T_c \sim 0.01$ ) (dashed line).



**Figure 5.** The second-order correction to the scalar field  $F$  for  $d = 3$ ,  $\Delta = 1.4$  as a function of temperature at the mid-point,  $z = \frac{1}{2}$ , (left panel) and the horizon,  $z = 1$ , (right panel). The horizontal axis corresponds to the temperature range  $0.01 \lesssim \frac{T}{T_c} \lesssim 0.1$  with  $T$  decreasing to the right.

the exponent in (42) corrects an earlier analytic result [14]. The constant of proportionality  $\gamma$  can be found analytically. It is plotted in figure 6 versus  $\Delta$ . As  $\Delta$  approaches the BF bound,  $\gamma \rightarrow 0$ , showing that the power law behavior changes, as we discuss next.

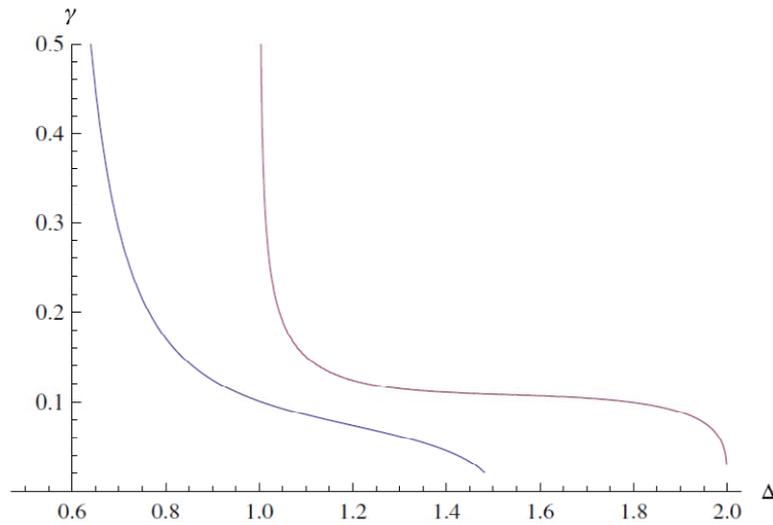
Indeed, at the end point (BF bound),  $\Delta = \frac{d}{2}$ , we need to exercise care. Letting  $\Delta = \frac{d}{2} - \epsilon$ , we obtain from (37) and (38),

$$\frac{1}{\mu^2} = \frac{(d-2)\Gamma(\frac{2}{d})}{d^{2(1-\frac{2}{d})}\Gamma(1-\frac{2}{d})} \left[ \frac{1}{2\epsilon b^{2-2\epsilon}} - \frac{1}{2\epsilon b^2} + \dots \right], \tag{43}$$

and taking the limit  $\epsilon \rightarrow 0$ , we deduce at the BF bound

$$\frac{\mu^2}{b^2} = \frac{d^{2(1-\frac{2}{d})}\Gamma(1-\frac{2}{d})}{(d-2)\Gamma(\frac{2}{d})[\ln b + \beta_d + o(b^0)]}, \tag{44}$$

where  $\beta_d$  is a constant that depends on the dimension and is easily computed (e.g., for  $d = 3$ ,  $\beta_3 \approx 1.75$ ).



**Figure 6.** The parameter  $\gamma$  in the low temperature expression (42) for the condensate versus  $\Delta$ . Curve on left (right) is for  $d = 3$  ( $d = 4$ ).

Higher order corrections are computed as before by considering the next iterative order. This introduces corrections that are subleading as  $b \rightarrow \infty$ . Already at  $T/T_c \sim 0.1$ , our first-order analytic results are almost indistinguishable from numerical results (see right panels of figures 2 and 3), the discrepancy being  $\sim 1\%$ . Below that temperature, numerical results are not available. Nevertheless the error can be estimated by calculating the correction at the second iterative order, as before. One finds that the error tends to zero as  $b \rightarrow \infty$  in the entire interval  $[0, 1]$ .

For the charge density, we have

$$\rho_0 \sim b^{d-1} (\ln b)^{-1/2}. \tag{45}$$

Therefore, the energy gap behaves as

$$E_g = \frac{(q\mathcal{O}_\Delta)^{1/\Delta}}{T_c} \sim (\ln b)^{\frac{1}{2(d-1)}} \sim \left( \ln \frac{T_c}{T} \right)^{\frac{1}{2(d-1)}}, \tag{46}$$

showing that the condensate diverges at the BF bound, albeit very mildly. This mild divergence was missed in earlier numerical studies [4].

The BF bound can also be approached from above. However, the calculation becomes considerably more involved, because for  $\Delta > d/2$ , as  $T \rightarrow 0$ , we have  $F_0 \approx 1$  near the boundary ( $z = 0$ ), but asymptotically ( $z \gtrsim 1/b$ ),  $F_0 \sim z^{d-2\Delta}$ , which does not have a smooth limit as  $T \rightarrow 0$ . Therefore, we cannot apply perturbation theory and a different approach is called for [14]. For example, one can approximate  $F_0$  by

$$F_0(z) = \begin{cases} 1, & z \leq \alpha \\ \left(\frac{z}{\alpha}\right)^{d-2\Delta}, & z > \alpha \end{cases}, \tag{47}$$

and find  $\alpha$  by a variational method. We shall not dwell on this further here.

Having understood the probe limit, we now turn to the first-order corrections in a  $1/q^2$  expansion. For  $\Delta < d/2$ , it is necessary to include these corrections in order to obtain a physical system at low temperatures, because in the  $q \rightarrow \infty$  limit the condensate diverges as  $T \rightarrow 0$  (equations (42) and (46)).

At first order, we obtain for the functions determining the metric

$$\begin{aligned}
 z f_1' - d f_1 &= \frac{(bz)^{2\Delta}}{4(d-1)} \left[ \left( m^2 + \Delta^2 f_0 + \frac{z^2 \Phi_0^2}{(bz)^{2\Delta}} \right) F_0^2 + 2\Delta z f_0 F_0 F_0' + z^2 f_0 F_0'^2 + \frac{z^4}{(bz)^{2\Delta}} \Phi_0'^2 \right] \\
 z \chi_1' &= \frac{(bz)^{2\Delta}}{d-1} \left[ \left( \Delta^2 + \frac{z^2 \Phi_0^2}{f_0^2} \right) F_0^2 + 2\Delta z F_0 F_0' + z^2 F_0'^2 \right].
 \end{aligned}
 \tag{48}$$

They can be solved at low temperature using our zeroth-order results above. We obtain

$$f_1(z) = -\frac{\Delta}{4(d-1)} (bz)^{2\Delta} [2 - z^d - z^{d-2\Delta}] + \dots, \quad \chi_1(z) = -\frac{\Delta}{2(d-1)} (bz)^{2\Delta} + \dots \tag{49}$$

For the temperature, we deduce the first-order expression

$$T = \frac{d}{4\pi} \left[ 1 + \frac{\Delta^2}{2d(d-1)} \frac{b^{2\Delta}}{q^2} + \dots \right], \tag{50}$$

showing that the temperature receives a positive correction away from the probe limit. Moreover, it is now clear when the probe limit fails. Indeed, for the expansion in  $1/q^2$  to be valid, we ought to have

$$b \lesssim q^{1/\Delta}, \tag{51}$$

justifying our earlier definition of the low energy regime (21).

For a given  $q$ , (51) places a lower bound on the temperature. While zero temperature is unattainable for finite  $q$ , the temperature can be made arbitrarily low by choosing a sufficiently large  $q$ . It follows that, even though the probe limit ( $q \rightarrow \infty$ ) is not a physical system at zero temperature, its properties are a good approximation to corresponding properties of physical systems (of finite charge  $q$  and (low) temperature). The approximation becomes better with increasing  $q$  as  $1/q^2$  corrections become smaller.

## 5. Conductivity

Next, we calculate the low temperature conductivity at the BF bound. For explicit analytic results, we concentrate on two cases  $d = 3$  and  $d = 4$ . We shall obtain the conductivity  $\sigma$  as a function of the rescaled frequency

$$\hat{\omega} = \frac{\omega}{b} = \frac{\omega}{\langle q \mathcal{O}_\Delta \rangle^{1/\Delta}}. \tag{52}$$

The function  $\sigma(\hat{\omega})$  has a well-defined limit, as  $q \rightarrow \infty$  (probe limit), down to low temperatures (in the regime (21)) at which the condensate  $\langle \mathcal{O}_\Delta \rangle$  is large. The conductivity of physical systems can be obtained as a  $1/q^2$  expansion with the conductivity in the probe limit serving as the zeroth-order term in the expansion.

### 5.1. Three dimensions

The conductivity on the AdS boundary is found by applying a sinusoidal electromagnetic perturbation in the bulk of frequency  $\omega$  obeying the wave equation

$$-\frac{d^2 A}{dr_*^2} + V A = \omega^2 A, \quad V = \frac{2q^2}{z^2} f \Psi^2, \tag{53}$$

where  $A$  is any component of the perturbing electromagnetic potential along the boundary. Equation (53) is to be solved subject to the ingoing boundary condition at the horizon

$$A \sim e^{-i\omega r_*} \sim (1-z)^{-i\omega/3}, \tag{54}$$

as  $z \rightarrow 1$  ( $r_* \rightarrow -\infty$ ), where  $r_*$  is the tortoise coordinate

$$r_* = \int \frac{dz}{f(z)} = \frac{1}{6} \left[ \ln \frac{(1-z)^3}{1-z^3} - 2\sqrt{3} \tan^{-1} \frac{\sqrt{3}z}{2+z} \right], \tag{55}$$

with the integration constant chosen so that the boundary is at  $r_* = 0$ . We wish to solve this equation at low temperatures.

Using (10) with  $d = 3$ , the wave equation reads

$$\frac{d}{dz} \left[ (1-z^3) \frac{dA}{dz} \right] - \left[ b^{2\Delta} z^{2\Delta-2} F^2(z) - \frac{\omega^2}{1-z^3} \right] A = 0. \tag{56}$$

To account for the boundary condition at the horizon, set

$$A = (1-z)^{-i\omega/3} e^{-i\omega z/3} \mathcal{A}(z), \tag{57}$$

where we included a factor  $e^{-i\omega z/3}$  for convenience, so that only  $\mathcal{A}(z)$  will contribute to the conductivity. The wave equation becomes

$$\begin{aligned} -3(1-z^3)\mathcal{A}'' + z[9z - 2(1+z+z^2)i\omega]\mathcal{A}' + \left[ 3b^{2\Delta}z^{2\Delta-2}F^2(z) \right. \\ \left. - (1+2z+3z^2)i\omega - \frac{(3+2z+z^2)(3+z+z^2+z^3)}{3(1+z+z^2)}\omega^2 \right] \mathcal{A} = 0. \end{aligned} \tag{58}$$

Regularity of the wavefunction  $\mathcal{A}$  at the horizon ( $z = 1$ ) implies the boundary condition

$$(3 - 2i\omega)\mathcal{A}'(1) + \left( b^{2\Delta}F^2(1) - 2i\omega - \frac{4\omega^2}{3} \right) \mathcal{A}(1) = 0. \tag{59}$$

At low temperatures,  $b \gg 1$ , it is convenient to rescale  $z \rightarrow z/b$ . The wave equation can be solved as a series expansion in  $1/q^2$ . The zeroth-order term is given by replacing  $F$  by  $F_0$  (equation (14)). For  $\Delta \leq \frac{3}{2}$ ,  $F_0(z)$  has a smooth limit (albeit unphysical) as  $b \rightarrow \infty$  (equation (24)). After rescaling and letting  $b \rightarrow \infty$ , we obtain the wave equation

$$-\mathcal{A}'' + [z^{2\Delta-2} - \hat{\omega}^2]\mathcal{A} = 0, \tag{60}$$

where we used  $F(z/b) \rightarrow F(0) = 1$ , as  $b \rightarrow \infty$ . For  $1 < \Delta \leq \frac{3}{2}$ , there are two linearly independent solutions,  $\mathcal{A}_\pm$ , distinguished by their asymptotic behavior:

$$\mathcal{A}_\pm \sim e^{\pm \frac{1}{2}z^\Delta}, \quad z \rightarrow \infty. \tag{61}$$

The general solution can be written as a linear combination:

$$\mathcal{A} = c^+ \mathcal{A}_+ + c^- \mathcal{A}_-. \tag{62}$$

Applying boundary condition (59), we deduce

$$\frac{c^+}{c^-} \sim e^{-\frac{2}{\Delta}b^\Delta}, \tag{63}$$

so at very low temperatures,

$$c^+ = 0, \tag{64}$$

i.e.  $\mathcal{A} \rightarrow 0$  as  $z \rightarrow \infty$ .

For  $\Delta = \frac{3}{2}$ , we obtain the exact explicit limit solution

$$\mathcal{A}(z) = \mathcal{A}_-(z) = \text{Ai}(bz - \hat{\omega}^2), \tag{65}$$

whereas  $\mathcal{A}_+(z) = \text{Bi}(bz - \hat{\omega}^2)$ , with arbitrary normalization, where we restored the scaling parameter  $b$ .

In this (unphysical) limit, the quasinormal frequencies have moved to the real axis yielding an infinite set of normal frequencies which are solutions of

$$\text{Ai}(-\hat{\omega}^2) = 0. \tag{66}$$

Thus we obtain an infinite tower of real frequencies given by the zeroes of the Airy function.

We deduce the limit conductivity (as  $q, b \rightarrow \infty$ ),

$$\sigma(\hat{\omega}) = \frac{i \text{Ai}'(-\hat{\omega}^2)}{\hat{\omega} \text{Ai}(-\hat{\omega}^2)}. \tag{67}$$

The real frequencies that solve (66) are the poles of the conductivity. Note that  $\Re\sigma = 0$ , except at the poles of  $\Im\sigma$  where  $\Re\sigma$  diverges as a  $\delta$ -function.

This expression is unphysical, as we have already explained (the probe limit breaks down in the limit  $b \rightarrow \infty$ ), but it is useful for computational purposes. We shall use it as a starting point to calculate the conductivity of a physical system at low temperatures.

At low temperatures, we can calculate the first-order correction analytically by considering the  $b \rightarrow \infty$  wave equation (60) as the zeroth-order equation. Then, for the first-order correction  $\delta\mathcal{A}$  to the potential at low temperatures, we obtain from (58),

$$-\delta\mathcal{A}'' + [z - \hat{\omega}^2]\delta\mathcal{A} = -\frac{1}{3(1-z^3)}\mathcal{H}_1\mathcal{A}, \tag{68}$$

where

$$\begin{aligned} \mathcal{H}_1 = & z[9z - 2(1+z+z^2)i\omega] \frac{d}{dz} + 3b^3z(2F_1(z) + z^3) \\ & - (1+2z+3z^2)i\omega + \frac{z^2(1-15z-12z^2-10z^3)}{3(1+z+z^2)}\omega^2. \end{aligned} \tag{69}$$

The first-order potential leads to quasinormal modes which are zeroes of  $\mathcal{A} + \delta\mathcal{A}$ . Thus the real frequencies (66) of the (unphysical)  $b \rightarrow \infty$  limit get shifted at low temperatures away from the real axis. We obtain  $\hat{\omega} \rightarrow \hat{\omega} + \delta\hat{\omega}$ , where

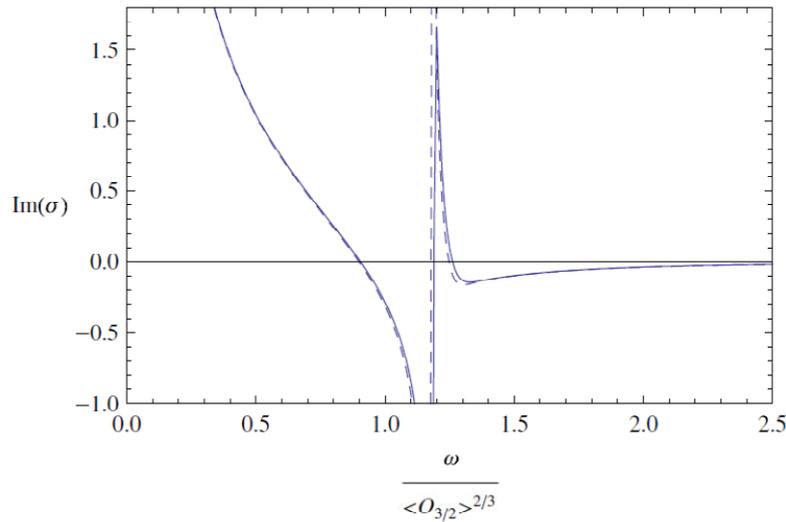
$$\delta\hat{\omega} = \frac{\pi \text{Bi}(-\hat{\omega}^2)}{3\hat{\omega} \text{Ai}'(-\hat{\omega}^2)} \int_0^1 \frac{dz}{1-z^3} \text{Ai}(bz - \hat{\omega}^2) \mathcal{H}_1 \text{Ai}(bz - \hat{\omega}^2). \tag{70}$$

This first-order expression is valid for low frequencies. As we heat up the system, most modes disappear and we are left with a finite number of quasinormal modes. Their number decreases as we increase the temperature. Conversely, as we cool down the system, (70) becomes increasingly accurate for an increasing number of modes. These modes shift toward the real axis ( $\delta\hat{\omega} \rightarrow 0$ ) as we lower the temperature. In the low temperature regime (21), the larger the  $q$  is the more spikes one obtains for  $\sigma(\omega)$ .

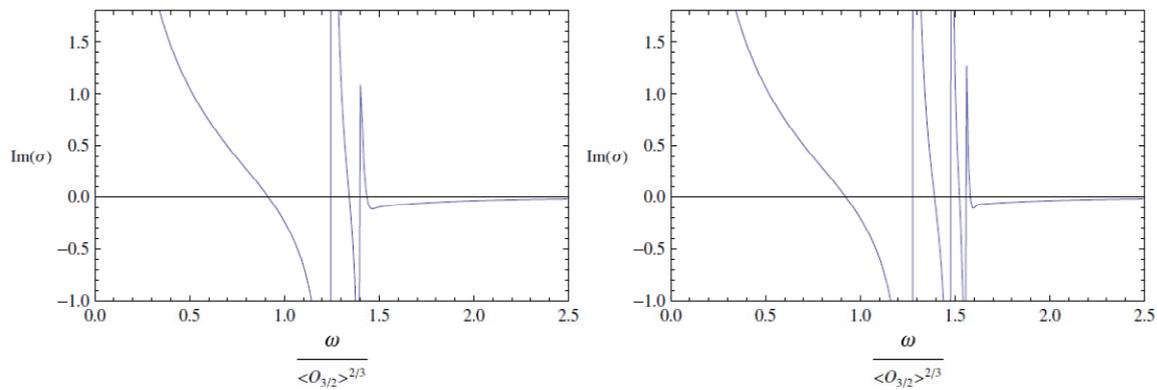
This shifting of quasinormal modes can be seen in plots of the conductivity. As the mode frequency approaches the real axis, the corresponding spike in the plot of the imaginary part of the conductivity becomes more pronounced. To demonstrate this, we calculated the conductivity using the first-order approximation (32) to the scalar field. In figure 7, we show the imaginary part of the conductivity at temperature  $T/T_c \approx 0.1$  and compare with the exact numerical solution. The agreement is very good even at such high temperature at which only one quasinormal mode is left. Unfortunately, this is the low temperature limit attained by numerical analysis as numerical instabilities prohibit one from lowering the temperature further. Using our analytic results, we see in figure 8 the emergence of an increasing number of poles as we lower the temperature to  $T/T_c \approx 0.06$  and  $0.04$ . Finally in figure 9, we compare the lower temperature ( $T/T_c \approx 0.01$ ) result with the  $b \rightarrow \infty$  limit analytic expression (67), thus demonstrating convergence.

For  $\Delta > 3/2$ , the potential is

$$V = b^{2\Delta} z^{2\Delta-2} (1-z^3) F(bz), \tag{71}$$



**Figure 7.** The imaginary part of the conductivity in  $d = 3$  using expression (32) for the scalar field (dotted line) compared with the exact numerical solution (solid line) at  $\frac{T}{T_c} \approx 0.1$



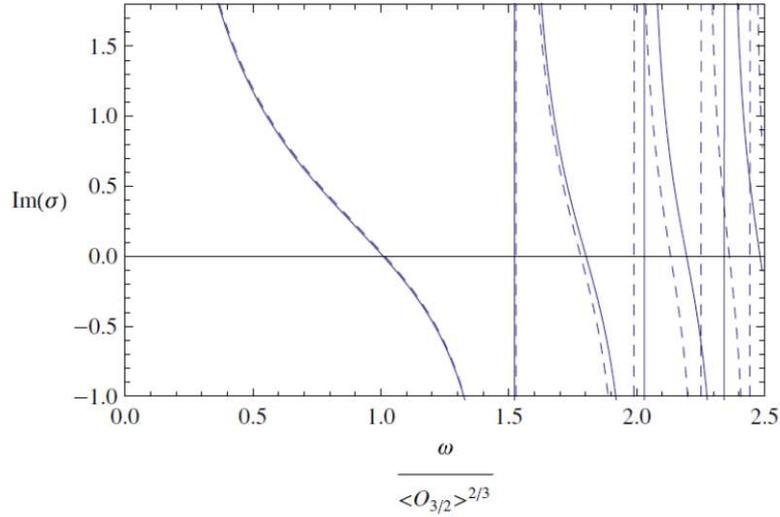
**Figure 8.** The imaginary part of the conductivity versus frequency in  $d = 3$  using expression (32) for  $F$  at  $\frac{T}{T_c} \approx 0.05$  (left),  $0.04$  (right). As the temperature decreases, poles move toward the real axis.

with  $F$  given approximately by (47). It attains a maximum of order  $b^{2(2-\Delta)}$  for  $\Delta < 2$ . Therefore, at zero temperature it has infinite height. However, the width becomes infinitely narrow leading to a finite tower of poles for the conductivity (quasinormal modes). In the zero temperature limit, the number of modes increases as one approaches the BF bound and decreases away from it. For  $\Delta \geq 2$ , the height of the potential becomes finite at zero temperature. It turns out that the potential is too narrow to possess bound states, so no poles exist for  $\Delta \geq 2$ .

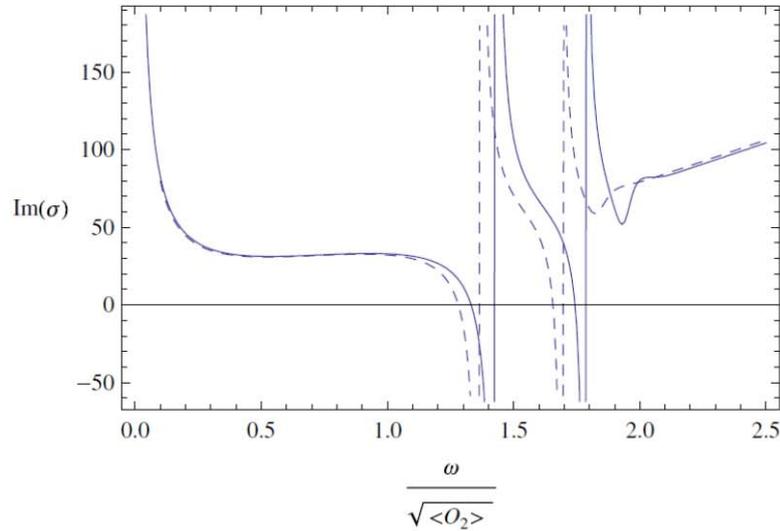
### 5.2. Four dimensions

The  $d = 4$  case is similar. Working as in the  $d = 3$  case, in the (unphysical)  $b \rightarrow \infty$  limit, the wave equation for  $\Delta = 2$  (at the BF bound) in the probe limit reduces to

$$A'' - \frac{1}{z}A' - [b^4z^2 - \omega^2]A = 0, \tag{72}$$



**Figure 9.** Comparison of the imaginary part of the conductivity in  $d = 3$  using expression (32) for  $F$  at  $\frac{T}{T_c} \approx 0.01$  (dotted line) and the  $b \rightarrow \infty$  limit (67) (solid line).



**Figure 10.** The imaginary part of the conductivity in  $d = 4$  using expression (32) for the scalar field (dotted line) compared with the exact numerical solution (solid line) at  $\frac{T}{T_c} \approx 0.17$

whose acceptable solution can be written in terms of a Whittaker function,

$$A = W_{\frac{\hat{\omega}^2}{4}, \frac{1}{2}}(b^2 z^2) \tag{73}$$

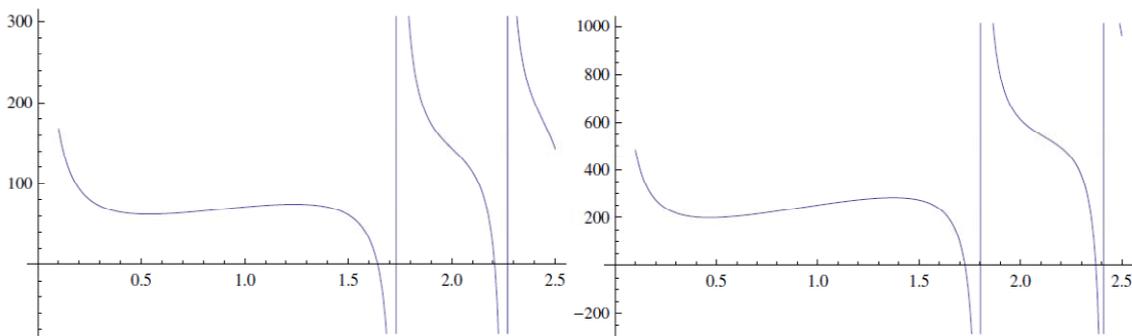
(The other solution diverges as  $z \rightarrow \infty$ ). At the boundary ( $z \rightarrow 0$ ), it has a logarithmic divergence which we need to subtract before we can calculate QNMs and the conductivity [4]. The conductivity is then given by

$$\sigma(\hat{\omega}) = \frac{2}{i\hat{\omega}} \frac{A_2}{A_0} + \frac{i\hat{\omega}}{2}, \tag{74}$$

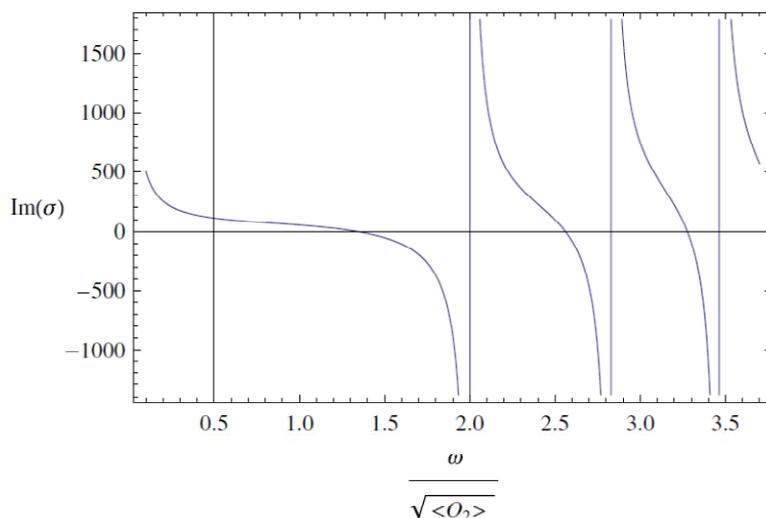
where

$$A(z) = A_0 + A_2 b^2 z^2 - A_0 \frac{\hat{\omega}^2}{2} b^2 z^2 \ln(b^2 z^2) + \dots, \tag{75}$$

with an arbitrarily chosen cutoff.



**Figure 11.** The imaginary part of the conductivity versus frequency in  $d = 4$  using expression (32) for  $F$  at  $\frac{T}{T_c} \approx 0.1$  (left),  $0.04$  (right).



**Figure 12.** The imaginary part of the conductivity in the (unphysical) limit  $b \rightarrow \infty$  in  $d = 4$  (77).

Using the expansion for small arguments,

$$W_{\frac{\hat{\omega}^2}{4}, \frac{1}{2}}(b^2 z^2) = -\frac{2}{\hat{\omega}^2 \Gamma(-\hat{\omega}^2/4)} \left\{ 1 - [1 + \hat{\omega}^2(2\gamma - 1 + \ln(b^2 z^2) + \psi(1 - \hat{\omega}^2/4))] \frac{b^2 z^2}{2} \right\} + \dots, \tag{76}$$

we deduce the limit conductivity as  $q, b \rightarrow \infty$ ,

$$\sigma(\hat{\omega}) = \frac{1}{i\hat{\omega}} + i\hat{\omega} \left[ 2\gamma - \frac{1}{2} + \psi(1 - \hat{\omega}^2/4) \right]. \tag{77}$$

We have a pole at  $\omega = 0$  as expected and an infinite tower of real poles determined by the poles of the digamma function. The poles have real frequencies

$$\hat{\omega} = \frac{\omega_n}{\langle \mathcal{O} \rangle^{1/2}} = 2\sqrt{n}, \quad n = 0, 1, 2, \dots \tag{78}$$

These poles are unphysical, but are good approximations to the quasinormal modes at low temperatures, in the physical regime (21). As we increase the temperature, these modes move away from the real axis. At any given temperature we have a finite number of such modes.

As we lower the temperature, their number increases and the quasinormal modes approach poles (78) on the real axis. To demonstrate this, we have calculated the conductivity using the first-order approximation (32) to the scalar field. In figure 10, we compare the imaginary part of the conductivity in  $d = 4$  with numerical results at temperature  $T/T_c \approx 0.17$  and find good agreement between them. As we go to lower temperatures, numerical instabilities arise and it is no longer possible to compare our analytic results with their numerical counterparts. We find convergence to the  $b \rightarrow \infty$  limit (77), but much slower than in  $d = 3$ . In figure 11, we show the imaginary part of the conductivity at  $T/T_c \approx 0.1$  and 0.04. As we lower the temperature, the number of poles increases and the poles shift to the right on the real axis approaching the limit values (78), which correspond to the  $b \rightarrow \infty$  limit of the conductivity shown in figure 12.

## 6. Conclusion

We discussed holographic superconductors in the probe limit when the scalar hair of the dual black hole is near the BF bound. Using the analytic tools developed in [14], we analyzed the low temperature regime trying to go beyond the range of applicability of numerical techniques which fail at very low temperatures. Thus undeterred by numerical instabilities, we found that at low temperatures the condensate diverges as  $|\ln T|^\delta$ , where  $\delta$  depends on the dimension of spacetime (equation (46)). This signals the breakdown of the probe limit at low temperatures even at the BF bound. The divergence is very mild which explains why it was missed in earlier numerical analyses. Although the probe limit at zero temperature cannot be a physical state for scaling dimensions  $\Delta \leq \frac{d}{2}$ , it is still useful to analyze it, because it is a good approximation to physical states at low temperatures and thus a good starting point for computational purposes.

We calculated the low temperature conductivity at the BF bound in the probe limit and found exact limit analytic expressions in  $d = 3, 4$ . Thus, we showed that as we lower the temperature, the conductivity has an increasing number of spikes (quasinormal modes) which approach the infinite tower of real poles determined by the zeroes of the Airy function in  $d = 3$  (equation (66)) and the Gamma function in  $d = 4$  (equation (78)). As we heat up the system, the number of spikes decreases and their positions move off of the real axis.

The probe limit we studied here can be used as a zeroth-order contribution to a perturbative expansion in  $1/q^2$ . Our results can be extended in a systematic way to include back reaction to the bulk metric in order to analyze physical states. This will greatly facilitate the probe of the zero temperature limit and shed light on the ground state of the system, as numerical methods fail due to numerical instabilities. Work in this direction is in progress.

## Acknowledgments

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## References

- [1] Maldacena J M 1998 *Adv. Theor. Math. Phys.* **2** 231  
Maldacena J M 1999 *Int. J. Theor. Phys.* **38** 1113 (arXiv:hep-th/9711200)
- [2] Gubser S S 2008 *Phys. Rev. D* **78** 065034 (arXiv:0801.2977 [hep-th])
- [3] Hartnoll S A, Herzog C P and Horowitz G T 2008 *Phys. Rev. Lett.* **101** 031601 (arXiv:0803.3295 [hep-th])
- [4] Horowitz G T and Roberts M M 2008 *Phys. Rev. D* **78** 126008 (arXiv:0810.1077 [hep-th])

- [5] Franco S, Garcia-Garcia A and Rodriguez-Gomez D 2009 arXiv:[0906.1214](#) [hep-th]
- [6] Hartnoll S A and Kovtun P 2007 *Phys. Rev. D* **76** 066001 (arXiv:[0704.1160](#) [hep-th])
- [7] Hartnoll S A, Kovtun P K, Muller M and Sachdev S 2007 *Phys. Rev. B* **76** 144502 (arXiv:[0706.3215](#)[cond-mat.str-el])
- [8] Hartnoll S A and Herzog C P 2007 *Phys. Rev. D* **76** 106012 (arXiv:[0706.3228](#) [hep-th])
- [9] Hartnoll S A and Herzog C P 2008 *Phys. Rev. D* **77** 106009 (arXiv:[0801.1693](#) [hep-th])
- [10] Hartnoll S A 2009 *Class. Quantum Grav.* **26** 224002 (arXiv:[0903.3246](#) [hep-th])
- [11] Herzog C P 2009 *J. Phys. A* **42** 343001 (arXiv:[0904.1975](#) [hep-th])
- [12] Horowitz G T 2010 arXiv:[1002.1722](#) [hep-th]
- [13] Horowitz G T and Roberts M M 2009 *J. High Energy Phys.* **JHEP11(2009)015** (arXiv:[0908.3677](#) [hep-th])
- [14] Siopsis G and Therrien J 2010 *J. High Energy Phys.* **JHEP05(2010)013** (arXiv:[1003.4275](#) [hep-th])
- [15] Faulkner T, Horowitz G T, McGreevy J, Roberts M M and Vegh D 2009 arXiv:[0911.3402](#) [hep-th]
- [16] Breitenlohner P and Freedman D Z 1982 *Ann. Phys.* **144** 249
- [17] Hertog T and Maeda K 2005 Stability and thermodynamics of AdS black holes with scalar hair *Phys. Rev. D* **71** 024001 (arXiv:[hep-th/0409314](#))
- [18] Koutsoumbas G, Papantonopoulos E and Siopsis G 2009 *J. High Energy Phys.* **JHEP07(2009)026** (arXiv:[0902.0733](#) [hep-th])

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Quasinormal Modes of Flat AdS Reissner–Norstrom Black Holes with the Maxwell Field Background in 3+1 Dimensions, Ortira Chapkeaw and **Suphot Musiri**, Siam Physics Congress 2011

## Quasinormal Modes of Flat AdS Reissner-Nordstrom Black Holes with the Maxwell Field Background in 3+1 Dimensions

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### Abstract

The quasinormal modes(QNM) of the 3+1 dimensional anti de Sitter(AdS) Reissner-Nordstrom(RN) black holes interacting with the scalar field are analytically calculated for the zero order perturbation. To simplify the problem we study for the case of the sectional curvature  $k=0$ . For the zero and small perturbative charge case our result is consistent with the numerical result.

**Keywords:** Quasinormal modes, Quasinormal frequencies, AdS black holes, Reissner-Nordstrom black holes, Black hole phase transition

### Introduction

Quasinormal modes of black holes are the solutions to the Einstein wave equation, where the boundary conditions are only the ingoing wave at the horizon of the black holes and only outgoing or decayed wave at the infinity [1]. These conditions allow only certain number of the wave frequencies and cause the frequencies becoming complex number.

Quasinormal modes are the wave that the black holes respond to the perturbation. The frequencies are inversely proportional to time that the system uses to recoil back to the equilibrium. The wave carries information of black hole property and space-time geometry, in some cases also including the perturbative particles.

The correspondence between anti de Sitter space-time and the conformal field theory (AdS/CFT) [2] proposes that the general relativity in  $d+1$  dimensional AdS space-time is equivalent to the quantum field in  $d$  dimensions at the boundary of the system. This allows us to study a microscopic system with strong interactions through studying general relativity of a corresponding AdS space-time.

The AdS black holes have recently become more intense research topics, due to the arrival of the AdS/CFT correspondence. In many cases, for example, large-mass black holes in spherical-symmetric space-times, the perturbation is stable. However, in [3] the black hole solution in 3+1 dimensions with the sectional curvature  $k=-1$ , hyperbolic symmetry, and perturbed by a scalar field is found. When the scalar field is turned off, the black hole changes to an AdS black hole, called this phenomenon, black hole phase transition.

In this work, the quasinormal modes of charged AdS black holes, Reissner-Nordstrom(RN) [4] perturbed by a small charge is analytically calculated for the zero-order perturbation. We consider for the

case of the sectional curvature  $k=0$  and in 3+1 dimensions. Let  $Q$  be the black hole charge which is the source of electromagnetic field,  $F_{\mu\nu}$  in this space-time and let  $\psi$  be the charge scalar field that perturbs the system.

The Lagrangian of the fields and the interaction can be written as [5]

$$L = -\frac{1}{4}F_{\mu\nu}^2 - \left| \partial_{\mu}\psi - iqA_{\mu}\psi \right|^2 - m^2|\psi|^2 \quad (1)$$

where  $A_{\mu}$  the potentials of  $F_{\mu\nu}$  and  $m$  is the mass of the charge  $q$ .

We arrange this article as following, the first section is the anti de Sitter Reissner-Nordstrom black holes, which describes some properties of these black holes. The second section is the quasinormal modes of AdS RN black hole perturbed by a charge scalar field with  $k=0$  in 3+1 dimensions, which presents our calculation. The third section is the result and conclusion, which discusses the result, compares with the numerical work. The forth section is conclusion, where some further suggestion is given.

### Anti de Sitter Reissner-Nordstrom Black Holes

#### The wave equations

The metric of  $d$ -dimensional AdS RN space-times is [6]

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2h_{ij}dx^i dx^j$$

where

$$f = k - \frac{2M}{r^{d-3}} + \frac{Q^2}{4r^{2d-6}} + \frac{r^2}{L^2}$$

$M$  and  $Q$  are the black hole mass and charge respectively.  $L$  is the AdS radius.  $k$  is the sectional

curvature number, 1 for spherical, 0 for flat and -1 for hyperbolic symmetry. The Hawking temperature can be calculated from the equation,  $T = f'(r_+)/4\pi$ , where  $r_+$  is the horizon of the black hole.

$$T = \frac{r_+^{-2d-1}}{16\pi L^2} \left[ 4(d-1)r_+^{2d+2} - (d-3)L^2(Q^2 r_+^6 - 4kr_+^{2d}) \right]$$

The field potential,  $A_\mu$ , in this system can be written in the form of the potential,  $\Phi(r)$  as

$$A = \Phi(r)dt$$

By varying  $\psi$  and  $\Phi$  in eq(1), the wave equations of  $\psi$  and  $\Phi$  can be obtained from eq(1). The simple solution of the potential,  $\Phi$ , for  $\psi = 0$  is

$$\Phi = \sqrt{\frac{d-2}{2(d-3)}} \left( \frac{Q}{r^{d-3}} - \frac{Q}{r_+^{d-3}} \right)$$

The wave equation of the scalar field  $\psi$

$$\left[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) - m_{eff}^2 \right] \psi = 0 \quad (2)$$

where  $m_{eff}^2 = m^2 + g^{tt} q^2 \Phi^2$ . The wave equation is separable when its ansatz is

$$\psi = e^{-i\omega t} r^{-(2-d)/2} R(r) S(x_i) \quad (3)$$

where  $S(x_i)$  is a harmonic function with eigenvalue  $\lambda^2 = l(l+d-3)$ ,  $l=0,1,2,3,\dots$ . To understand the boundary conditions at the horizon and the infinity, the tortoise variable,  $dr_* = dr/f(r)$ , is introduced into the radial wave equation

$$\frac{d^2 R(r)}{dr_*^2} + [\omega^2 - V(r)] R(r) = 0 \quad (4)$$

where the effective potential  $V(r)$  is

$$V(r) = \frac{(d-2)(d-4)}{4r^2} f^2 + \frac{\lambda^2}{r^2} f + m_{eff}^2 f + \frac{d-2}{2r} f f'$$

At the horizon,  $V(r \rightarrow r_+) \Rightarrow 0$ , only the ingoing wave is allowed in this region  $e^{-i\omega r_*} \approx (z-1)^{\alpha_1}$ , whereas in the far away region  $V(r \rightarrow \infty) \Rightarrow \infty$ , the wave must be decayed.

### Quasinormal Modes of AdS RN Black Holes

In our calculation, we set  $k=0$  and  $d=4$ . To solve the wave equation in region,  $r_+ \leq r < \infty$ , let define

a new variable  $z = \frac{r_+}{r}$ . The wave equation (4) changes to

$$z^2(z-1)(z-a_2)(z-a_3)(z-a_4) \times \frac{d}{dz} (z-1)(z-a_2)(z-a_3)(z-a_4) \frac{dR}{dz} + K(z)R = 0 \quad (5)$$

where

$$K = (2r_+/Q)^2 (\omega r_+)^2 z^2 - \lambda^2 (2r_+/Q)^2 z^2 (z-1)(z-a_2)(z-a_3)(z-a_4) - m^2 r_+^2 (2r_+/Q)^2 (z-1)(z-a_2)(z-a_3)(z-a_4) + 4q^2 r_+^2 (2r_+/Q)^2 z^2 (z-1)^2 - 2(z-1)^2 (z-a_2)^2 (z-a_3)^2 (z-a_4)^2 + z(z-1)(z-a_2)^2 (z-a_3)^2 (z-a_4)^2 + z(z-1)^2 (z-a_2)(z-a_3)^2 (z-a_4)^2 + z(z-1)^2 (z-a_2)^2 (z-a_3)(z-a_4)^2 + z(z-1)^2 (z-a_2)^2 (z-a_3)^2 (z-a_4)$$

and the new parameters in the equation are

$$a_2 = A + B - p/3$$

$$a_3 = -(A+B)/2 + i\sqrt{3}/2(A-B)$$

$$a_4 = -(A+B)/2 - i\sqrt{3}/2(A-B)$$

$$A, B = \left[ \begin{array}{c} -\frac{1}{2} \left( \frac{2p^3}{3^3} - \frac{p^2}{3} + p \right) \\ \pm \sqrt{\frac{1}{4} \left( \frac{2p^3}{3^3} - \frac{p^2}{3} + p \right)^2 + \frac{1}{3^3} (p - \frac{p^2}{3})^3} \end{array} \right]^{1/3}$$

$$p = -\frac{4r_+^2}{Q^2 L^2}$$

To simplify the equation, we let

$$R = z^{\alpha_0} (z-1)^{\alpha_1} (z-a_2)^{\alpha_2} (z-a_3)^{\alpha_3} (z-a_4)^{\alpha_4} F(z)$$

where

$$\alpha_0 = \frac{1}{2} \pm \frac{1}{2} \sqrt{9 + 4m^2 L^2}$$

$$\alpha_1 = i \frac{\omega r_+ (2r_+/Q)^2}{(1-a_2)(1-a_3)(1-a_4)}$$

$$\alpha_2 = i \left[ \frac{(\omega r_+)^2 (2r_+/Q)^4}{(a_2-1)^2 (a_2-a_3)^2 (a_2-a_4)^2} + \frac{4(qr_+)^2 (2r_+/Q)^2}{(a_2-a_3)^2 (a_2-a_4)^2} \right]^{1/2}$$

$$\alpha_3 = i \left[ \frac{(\omega r_+)^2 (2r_+ / Q)^4}{(a_3 - 1)^2 (a_3 - a_2)^2 (a_3 - a_4)^2} + \frac{4(qr_+)^2 (2r_+ / Q)^2}{(a_3 - a_2)^2 (a_3 - a_4)^2} \right]^{1/2}$$

$$\alpha_4 = i \left[ \frac{(\omega r_+)^2 (2r_+ / Q)^4}{(a_4 - 1)^2 (a_4 - a_2)^2 (a_4 - a_3)^2} + \frac{4(qr_+)^2 (2r_+ / Q)^2}{(a_4 - a_2)^2 (a_4 - a_3)^2} \right]^{1/2}$$

We take the factor

$$z^{\alpha_0+1} (z-1)^{\alpha_1+1} (z-a_2)^{\alpha_2+2} (z-a_3)^{\alpha_3+2} (z-a_4)^{\alpha_4+2}$$

out of the wave equation (5) and change variable to  $y = 1 - z$ , near the horizon. We are going to calculate the zero order solution, where the frequency and charge,  $q$ , are small. Then the wave equation is reduced into the form of

$$y(1-y) \frac{d^2 F}{dy^2} + (c - (a+b+1)y) \frac{dF}{dy} - JF = 0$$

which is a hypergeometric function where

$$a, b = \alpha_0 + \alpha_1 + \frac{1+2\alpha_2}{1-a_2} + \frac{1+2\alpha_3}{1-a_3} + \frac{1+2\alpha_4}{1-a_4}$$

$$\pm \sqrt{(\alpha_0 + \alpha_1 + \frac{1+2\alpha_2}{1-a_2} + \frac{1+2\alpha_3}{1-a_3} + \frac{1+2\alpha_4}{1-a_4})^2 - J}$$

(6)

$$J \approx 1 + \alpha_0(1+2\alpha_1) - \frac{4m^2 r_+^4 / Q^2}{(1-a_2)(1-a_3)(1-a_4)}$$

$$+ \frac{\alpha_1 + \alpha_2}{1-a_2} + \frac{\alpha_1 + \alpha_3}{1-a_3} + \frac{\alpha_1 + \alpha_4}{1-a_4}$$

We keep only the linear terms of  $\alpha$ , the small frequency and small charge in  $J$ , as the zero-order perturbation. At the horizon,  $y=0$ , the wave is only ingoing into black hole. Then the solution is

$$R(y) = y^{\alpha_1} F(a, b; c; y)$$

From the property of the hypergeometric functions,

$$F(a, b; c; y) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-y)$$

$$+ (1-y)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b-c+1; 1-y)$$

(7)

In this far away zone the solution behavior is in the form of

$$R(z) \approx (\text{constant}) z^{\alpha_{0+}} + (\text{constant}) z^{\alpha_{0-}} \quad (8)$$

The solution must be finite in this area. However  $\alpha_{0-} = 1/2 - 1/2\sqrt{9 + 4m^2 L^2} < 0$  for  $m^2 L^2 > -2$

this causes  $z^{\alpha_{0-}} \rightarrow \infty$  as  $z \rightarrow 0$ . To eliminate the divergent term we set the constant in the front of the second term in eq(7) or eq(8) to be zero by letting the arguments of gamma function dominators be negative integer number,  $n = 0, 1, 2, 3, \dots$ , i.e.,

$$a = -n, \quad b = -n$$

or in term of  $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$  from eq(6),

$$n^2 + n(\alpha_0 + \alpha_1 + \frac{1+2\alpha_2}{1-a_2} + \frac{1+2\alpha_3}{1-a_3} + \frac{1+2\alpha_4}{1-a_4})$$

$$+ J = 0 \quad (9)$$

## Results and Discussion

The approximated quasinormal frequencies could be solved from Eq(9).

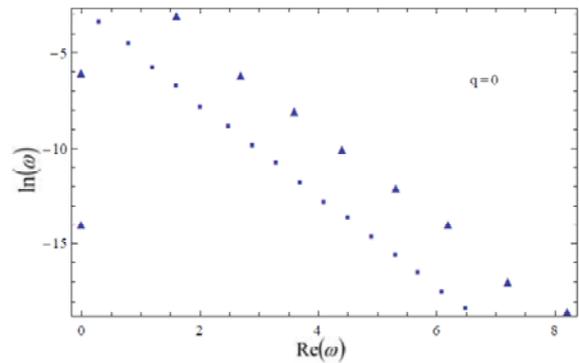


Figure 1 The frequency plot between the real part on the x-axis and the imaginary part on the y-axis where  $q=0$ ,  $L=1.1$ ,  $r_+=Q=1$  and  $m^2 L^2=4$ . The markers  $\blacksquare$  and  $\blacktriangle$  represent our result and numerical result [6] respectively.

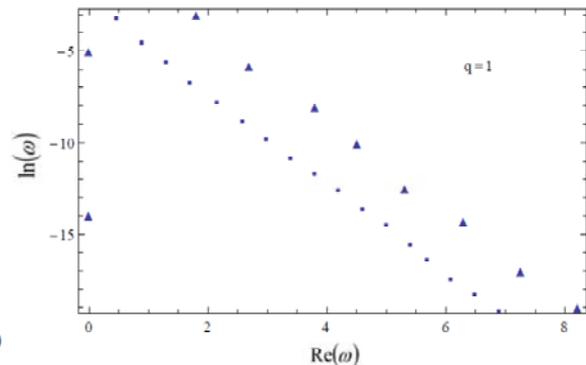
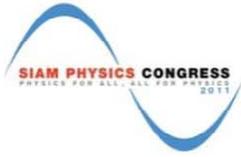


Figure 2 The frequency plot between the real part on the x-axis and the imaginary part on the y-axis where  $q=1$ ,  $L=1.1$ ,  $r_+=Q=1$  and  $m^2 L^2=4$ . The markers  $\blacksquare$  and  $\blacktriangle$  represent our result and numerical result [6] respectively.

Our result is consistent with the numerical work. The slopes of the both graphs in figure 1 and 2 are approximately -0.4. The space between the allowed



frequencies is equally separated when the number  $n$  is large. The quasinormal frequencies are a discrete set of complex number with the equally spacing. This result is similar to those in many literatures, e.g. [7].

### Conclusions

Our work aims to have better understanding analytically of the AdS black hole system, which is perturbed by a simply gauge field. We take  $k=0$  and  $d=4$  to simplify the problem. The zero-order perturbation gives the quasinormal modes and frequencies of the RN black holes interacting with the scalar field, where the frequencies and the charge  $q$  are taken to be small. Even though our work is in agreement with the numerical result, there are things that can be enhanced to improve the result. For example, we could find a better approximated zero order solution for large number of  $q$  and other parameters, or continue to calculate the first order perturbation, etc.

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### References

1. G. T. Horowitz and V. E. Hubeny, "Quasinormal Modes of AdS Black Holes and Approach to Thermal Equilibrium", *Physical Review D* **62** (2000), 024027.
2. J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", *Advances in Theoretical and Mathematical Physics* **2** (1998), 231.
3. C. Martinez, R. Troncoso and J. Zanelli, "Exact Black Hole Solution with a Minimally coupled Scalar Field", *Physical Review D* **70** (2004), 084035.
4. S. Chandrasekhar, "The Mathematical Theory of Black Holes", Oxford University Press, 1998.
5. S. S. Gubser, "Breaking an Abelian Gauge Symmetry near a Black Hole Horizon", *Physical Review D* **78** (2008), 065034.
6. X. He, B. Wang, R. Cai and C. Lin, "Signature of the Black Hole Phase Transition in Quasinormal Modes", arXiv:1002.2679.
7. J. Nataro and R. Schiappa, "On the Classification of Asymptotic Quasinormal Frequencies for d-Dimensional Black Holes and Quantum Gravity", *Advances in Theoretical and Mathematical Physics* **8** (2004), 1001.

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# ANALYTICAL CALCULATION OF QUASINORMAL MODES FOR NON-ROTATING KALUZA-KLEIN BLACK HOLE WITH SQUASHED HORIZONS

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**Abstract:** We study the Kaluza-Klein black holes, with squashed horizons. The system is perturbed by a massless scalar field. We analytically calculate the solution to the wave equation, which are called the quasinormal modes. We use approximation technique for the non-rotating case, where the frequencies to solution are a set of discrete complex number. Our analytical result is very well in agreement with the numerical work. Both results imply the existence of the stable ground state and the few lower excited states.

**Introduction:** The extra dimensions have become mandatory to sustain or generalize the principles or theories in many physics areas of research. The Kaluza-Klein theory has combined the general relativity and electrodynamics into a 5-dimensional theory. Black holes contained in the general relativity can be viewed as singularities that dictate the spacetime curvature and the behaviors of particle and fields in the systems. The fields in the black hole systems must satisfy the boundary conditions that only ingoing wave at the horizons of the black holes and only outgoing wave or the vanishing wave at the infinity are allowed. The waves that obey these boundary conditions are called quasinormal modes and their associated frequencies as quasinormal frequencies.<sup>1</sup> The gravitational wave signals from supermassive black holes are expected to be detected by Laser Interferometer Space Antenna (LISA).<sup>2</sup> In this work, the non-rotating Kaluza-Klein black holes with the squashed horizon are studied and the quasinormal modes of these systems are analytically calculated, where we expect the frequencies to be a set of discrete complex number.

In this introduction section we describe the Kaluza-Klein black hole system and the squashed shape of the horizon. The wave equation in this curved spacetime is considered. The five-dimensional rotating Kaluza-Klein black holes with the squashed horizons have the metric in the form of<sup>3</sup>

$$ds^2 = -dt^2 + \frac{\Sigma}{\Delta} k^2 dr^2 + \frac{r^2 + a^2}{4} [k(\sigma_1^2 + \sigma_2^2) + \sigma_3^2] + \frac{\mu}{r^2 + a^2} (dt - \frac{a}{2} \sigma_3)^2 \quad (1)$$

where

$$\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi, \quad \sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi, \quad \sigma_3 = d\psi + \cos \theta d\phi \quad (2)$$

$$\Sigma(r) = r^2(r^2 + a^2), \quad (3)$$

$$\Delta(r) = (r^2 + a^2)^2 - \mu r^2 \quad (4)$$

$$k(r) = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2}. \quad (5)$$

Parameters  $\mu$  and  $a$  correspond to black hole mass and angular momentum respectively. The variables have the range  $0 \leq r \leq r_\infty$ ,  $0 < \theta < \pi$ ,  $0 < \phi < 2\pi$ ,  $0 < \psi < 4\pi$ , where the radius has the upper limit  $r_\infty$ . The sphere of the horizon is squashed by the term  $k(r)$ . The

singularities of the metric are obtained by setting  $\Delta(r_{\pm}) = 0$ .  $r_+$  and  $r_-$  are outer and inner horizons respectively, which are depend on  $\mu$  and  $a$  as

$$r_{\pm} = \sqrt{\frac{(\mu - 2a^2) \pm \sqrt{\mu^2 - 4a^2\mu}}{2}} \quad (6)$$

Since the metric is apparently singular at  $r = r_{\infty}$ , the radial coordinate  $r$  is restricted within the range  $0 < r < r_{\infty}$ . Note that, if we take  $r_{\infty} \rightarrow \infty$  which causes  $k(r) \rightarrow 1$  therefore our metric reduces to five-dimensional Kerr black hole. Moreover, the shape of the event horizon is also characterized by the function  $k(r_+), k(r_-)$ .<sup>3</sup>

The metric above describes the space-time geometry of the rotating squashed Kaluza-Klein black hole which looks like a five-dimensional squashed black hole near the horizons, and like the Kaluza-Klein geometry at  $r \rightarrow r_{\infty}$ . To see the asymptotic behavior of this metric, let us define a new radial coordinate<sup>4</sup>

$$\rho = \rho_0 \frac{r^2}{r_{\infty}^2 - r^2} \quad (7)$$

where

$$\rho_0 = \sqrt{\frac{k_0(r_{\infty}^2 + a^2)}{4}}, \text{ and } k_0 = k(r=0) = \frac{(r_{\infty}^2 + a^2)^2 - \mu r_{\infty}^2}{r_{\infty}^4} \quad (8)$$

In a new radial coordinate  $\rho$  varies from 0 to  $\infty$  while  $r$  varies from 0 to  $r_{\infty}$ . By transforming (1) via a new coordinate (7) and take limit  $\rho \rightarrow \infty$ , the metric becomes

$$ds^2 = -dt^2 + d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2) + \frac{r_{\infty}^2 + a^2}{4}\sigma_3^2 + \frac{\mu}{r^2 + a^2}(dt - \frac{a}{2}\sigma_3)^2 \quad (9)$$

To remove the cross-term between  $dt$  and  $\sigma_3$ , let define new coordinates as

$$\begin{aligned} \tilde{\psi} &= \psi - \frac{2\mu a}{(r_{\infty}^2 + a^2)^2 + \mu a^2} t \\ \tilde{t} &= \sqrt{\frac{(r_{\infty}^2 + a^2)^2 - \mu a^2}{(r_{\infty}^2 + a^2)^2 + \mu a^2}} t \end{aligned} \quad (10)$$

and define a new notation  $\tilde{\sigma}_3 = d\tilde{\psi} + \cos\theta d\phi$ , and replace all these new coordinates and notation into (9). Then, the asymptotic structure of the rotating squashed Kaluza-Klein black hole is revealed

$$ds^2 = -d\tilde{t}^2 + d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2) + \frac{(r_{\infty}^2 + a^2)^2 + \mu a^2}{4(r_{\infty}^2 + a^2)} \tilde{\sigma}_3^2 \quad (11)$$

The first three terms on the RHS of (11) represent a four dimensional Minkowski spacetime while the rest is a twisted  $S^1$  bundle. The size of the compactified dimension at infinity is also obtained<sup>4</sup>

$$r_{\infty}'^2 = \frac{(r_{\infty}^2 + a^2)^2 + \mu a^2}{r_{\infty}^2 + a^2} \quad (12)$$

The size of the extra dimension depends on three parameters  $r_{\infty}, \mu$  and  $a$ . Note that, for  $a \rightarrow 0$  or  $r_{\infty}^2 \gg a^2$ , the radius of the compactified dimension (12) could be interpreted by  $r_{\infty}$ .

### Klein-Gordon equation in curved background

Our aim is to study a scalar field which evolves in the rotating squashed black hole spacetime. Hence, we need to construct equation of motion for a scalar field in curved background. An equation of motion for a real scalar field is so called Klein-Gordon equation.

To derive a Klein-Gordon equation in a curved background, let consider an action for a single scalar field in curved spacetime

$$s = \int \sqrt{-g} \left[ -\frac{1}{2} g^{\mu\nu} (\nabla_\mu \Phi)(\nabla_\nu \Phi) - \frac{1}{2} m^2 \Phi^2 \right] d^4x \quad (13)$$

Here  $m$  stands for mass of a scalar field, where  $g$  is determinant of the metric tensor. For a scalar field case, it is possible to replace the covariant derivative with an ordinary partial derivative. By varying this action with respect to the scalar field, we obtain an equation of motion for a scalar field in curved background.

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) - m^2 \Phi = 0 \quad (15)$$

This equation will be used as an important part of our calculation in the next section. Its solution describes the behavior of the scalar field in the curved spacetime. For simplicity in our work, we consider only for a massless scalar field.

**Methodology:** In this section we study the wave equation of the perturbed scalar field. The equation is separable. We analytically solve the radius wave equation. We obtain and present the frequency-constrain equation which correspond to the each quasinormal mode.

### Equation of motion for a real scalar field in a rotating squashed Kaluza-Klein black hole

We are going to calculate an equation of motion for a scalar particle in our particular metric (1). First, we have defined the proper time  $dt = Bd\tau$  and  $B \equiv \frac{(r_\infty^2 + a^2)^2}{2\rho_0 r_\infty^3}$  is a constant.<sup>5</sup>

The former metric (1) becomes

$$ds^2 = -B^2 d\tau^2 + \frac{\Sigma}{\Delta} k^2 dr^2 + \frac{r^2 + a^2}{4} [k(\sigma_1^2 + \sigma_2^2) + \sigma_3^2] + \frac{\mu}{r^2 + a^2} (Bd\tau - \frac{a}{2} \sigma_3)^2 \quad (16)$$

Therefore, we can calculate components of metric tensor  $g_{\mu\nu}$ , and its inverse  $g^{\mu\nu}$  as

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{\mu}{r^2 + a^2}\right) B^2 & 0 & 0 & -\frac{a\mu \cos\theta}{2(r^2 + a^2)} B & -\frac{a\mu}{2(r^2 + a^2)} B \\ 0 & \frac{\Sigma k^2}{\Delta} & 0 & 0 & 0 \\ 0 & 0 & \frac{k^2(r^2 + a^2)}{4} & 0 & 0 \\ -\frac{a\mu \cos\theta}{2(r^2 + a^2)} B & 0 & 0 & \frac{r^2 + a^2}{4} (k \sin^2 \theta + \cos^2 \theta) + \frac{\mu a^2 \cos^2 \theta}{4(r^2 + a^2)} & \frac{\cos\theta}{4} \left( \frac{(r^2 + a^2)^2 + \mu a^2}{r^2 + a^2} \right) \\ -\frac{a\mu}{2(r^2 + a^2)} B & 0 & 0 & \frac{\cos\theta}{4} \left( \frac{(r^2 + a^2)^2 + \mu a^2}{r^2 + a^2} \right) & \frac{(r^2 + a^2)^2 + a^2 \mu}{4(r^2 + a^2)} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -\left(\frac{(r^2+a^2)^2+a^2\mu}{(r^2+a^2)^2-r^2\mu}\right)\frac{1}{B^2} & 0 & 0 & 0 & -\left(\frac{2a\mu}{(r^2+a^2)^2-r^2\mu}\right)\frac{1}{B} \\ 0 & \frac{\Delta}{\Sigma k^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{k^2(r^2+a^2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{k(r^2+a^2)\sin^2\theta} & -\frac{4\cos\theta}{k(r^2+a^2)\sin^2\theta} \\ -\left(\frac{2a\mu}{(r^2+a^2)^2-r^2\mu}\right)\frac{1}{B} & 0 & 0 & -\frac{4\cos\theta}{k(r^2+a^2)\sin^2\theta} & \frac{4\cos^2\theta}{k(r^2+a^2)\sin^2\theta} + \frac{4(r^2+a^2-\mu)}{(r^2+a^2)^2-r^2\mu} \end{pmatrix}$$

where  $\sqrt{-g} = \frac{k^2 \sin\theta B}{8} \sqrt{(r^2+a^2)\Sigma}$ , In our calculation, we denote spacetime indices by  $(\tau, r, \theta, \phi, \psi) \rightarrow (0, 1, 2, 3, 4)$ . It is convenient to use a new radial coordinate  $\rho$  which defined by (7). Then, take the ansatz for a scalar field  $\Phi(\tau, \rho, \theta, \phi, \psi) = e^{-i\omega\tau} R(\rho) e^{im\phi + i\lambda\psi} S(\theta)$ , where  $S(\theta)$  is the spheroidal harmonics. After change to a new radial coordinate and put the ansatz into (15), we can separate the angular variable part from the radial and the time parts. For the angular part, it reads <sup>5</sup>

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left[ \sin\theta \frac{d}{d\theta} \right] S(\theta) - \left[ \frac{(m - \lambda \cos\theta)^2}{\sin^2\theta} - E_{lm\lambda} \right] S(\theta) = 0 \quad (17)$$

Here the eigenvalue of the angular equation is  $E_{lm\lambda} = l(l+1) - \lambda^2$ . The parameters  $l$  is the angular number of the variable  $\theta$  and  $\lambda$  is integer number for the 5<sup>th</sup> dimension represented by the angle variable,  $\psi$ . However, the time variable can be removed in the final step of the calculation. So, the only remaining part is the radial component which takes the form

$$\Theta \frac{d^2 R(\rho)}{d\rho^2} + \frac{d\Theta}{d\rho} \frac{dR(\rho)}{d\rho} + \left[ \frac{\tilde{N}^2}{\Theta} + \Lambda - l(l+1) + \lambda^2 \right] R(\rho) = 0 \quad (18)$$

and

$$\begin{aligned} \Theta(\rho) &= \frac{(r_\infty^2 + a^2)}{4r_\infty^4 \rho_0^2} \left[ (\rho r_\infty^2 + a^2(\rho + \rho_0))^2 - \mu\rho(\rho + \rho_0)r_\infty^2 \right], \\ \tilde{N}^2 &= \frac{\mu r_\infty^2 (\rho + \rho_0)^4}{N^4 (r_\infty^2 + a^2)^2} \left[ \omega - \frac{\lambda a N^2 (r_\infty^2 + a^2)}{\rho_0 r_\infty^3} \right]^2, \\ \Lambda &= \frac{4\rho_0^2 r_\infty^6 (\rho + \rho_0)^2 \omega^2}{N^2 (r_\infty^2 + a^2)^4} - \frac{4\lambda^2 (\rho + \rho_0)^2}{r_\infty^2 + a^2}, \\ N^2 &= \frac{\rho + \rho_0}{a^2} \cdot \frac{\rho + \frac{r_+^2}{r_\infty^2 + a^2} \rho_0}{\rho + \frac{r_+^2}{r_\infty^2 + a^2} \rho_0}, \\ \rho_+ &= \rho_0 \frac{r_+^2}{r_\infty^2 - r_+^2}, \\ r_\infty^2 &= \frac{(\mu - 2a^2) \pm \sqrt{\mu^2 - 4a^2\mu}}{2}. \end{aligned} \quad (19)$$

In order to obtain the quasinormal frequencies  $\omega$ , we have to solve (18) under certain boundary conditions as mentioned before. In this work, we limit ourselves to the case of non-rotation  $a = 0$  and the equation of motion (18) becomes

$$\rho(\rho - \rho_+)^2 \frac{d^2 R(\rho)}{d\rho^2} + (\rho - \rho_+)(2\rho - \rho_+) \frac{dR(\rho)}{d\rho} + \frac{\mu\omega^2 \rho(\rho + \rho_0)^2}{r_\infty^2} R(\rho) + (\rho - \rho_+) \left[ \frac{4\rho_0^2(\rho + \rho_0)\rho\omega^2}{r_\infty^2} - \frac{4\lambda^2(\rho + \rho_0)^2}{r_\infty^2} - E_{lm\lambda} \right] R(\rho) = 0 \quad (20)$$

To simplify the solution let us separate the singularity at horizon by writing the  $R(\rho)$  as,

$$R(\rho) = (\rho - \rho_+)^{\alpha} F(\rho)$$

where

$$\alpha \equiv -\frac{i\omega r_\infty}{2\left(\frac{r_\infty^2}{\mu} - 1\right)^{1/2}} \quad (21)$$

The minus sign of  $\alpha$  represents the incoming wave at the horizon. After substituted  $R(\rho)$  in equation (20) the wave equation changes to

$$\begin{aligned} &\rho(\rho - \rho_+) \frac{d^2 F(\rho)}{d\rho^2} + 2\alpha\rho \frac{dF(\rho)}{d\rho} + (2\rho - \rho_+) \frac{dF(\rho)}{d\rho} + (\alpha(\alpha - 1) + 2\alpha)F(\rho) + \\ &\frac{\mu\omega^2}{r_\infty^2} \left[ (\rho - \rho_+)^2 + 2(\rho - \rho_+)(\rho_+ + \rho_0) + \rho_+(\rho - \rho_+) + 2\rho_+(\rho_+ + \rho_0) + (\rho_+ + \rho_0)^2 \right] F(\rho) + \\ &\left[ \frac{4\rho_0^2(\rho_+ + \rho_0)\rho\omega^2}{r_\infty^2} - \frac{4\lambda^2(\rho + \rho_0)^2}{r_\infty^2} - E_{lm\lambda} \right] F(\rho) = 0 \end{aligned} \quad (22)$$

Let's define a new variable  $x = \frac{\rho}{\rho_+}$ , where  $x = 1$  at the horizon. The radius wave equation is reduced to

$$x(1-x) \frac{d^2 F(x)}{dx^2} + [1 - (2\alpha + 2)x] \frac{dF(x)}{dx} + [\alpha' + \beta'x + \gamma'x^2] F(x) = 0 \quad (23)$$

$$\alpha' = \frac{4\lambda^2 \rho_0^2}{r_\infty^2} + E_{lm\lambda} - \alpha$$

$$\beta' = -\frac{\mu\omega^2}{r_\infty^2} \rho_+(\rho_+ + \rho_0) - \rho_0\rho_+\omega^2 + \frac{8\lambda^2 \rho_+\rho_0}{r_\infty^2} \quad (24)$$

$$\gamma' = -\omega^2 \rho_+^2 + \frac{4\lambda^2 \rho_+^2}{r_\infty^2}$$

To further simplify the solution let us define  $F = e^{Cx} H(x)$ , where we choose  $C^2 = -\gamma'$ . Then the above equation, (23) change to

$$x(x-1) \frac{d^2 H}{dx^2} + [2Cx(x-1) - 1 + 2(1+\alpha)x] \frac{dH}{dx} + (\alpha' - C)H + [-C^2 + 2C + 2C\alpha + \beta']xH = 0 \quad (25)$$

Change the variable from  $x$  to  $v = x - 1$ , where at the horizon  $v = 0$ . Next divide the equation with  $-2Cv$  and the wave equation becomes

$$(-2Cv) \frac{d^2 H}{d(-2Cv)^2} + \left[ 1 + 2\alpha - (-2Cv) + 1 - \frac{1}{x} \right] \frac{dH}{d(-2Cv)} - \left[ 1 + \alpha + \frac{\beta' + \gamma'}{2C} \right] H - \frac{\alpha' - C}{2Cv} H = 0 \quad (26)$$

The wave equation, near the horizon,  $x = 1$ , can be approximated as

$$(-2Cv) \frac{d^2 H}{d(-2Cv)^2} + [1 + 2\alpha - (-2Cv)] \frac{dH}{d(-2Cv)} - \left[ \frac{1}{2} + \alpha + \frac{\alpha' + \beta' + \gamma'}{2C} \right] H = 0 \quad (27)$$

The solutions to the above wave equation, (27) are the confluent hypergeometric function. However we need only the ingoing solution to the horizon in order to satisfy the boundary condition

$$R = e^{Cx} (\rho - \rho_+)^{\alpha} F(\hat{a}; \hat{b}; -2Cv) \quad (28)$$

where  $\hat{a} = \frac{1}{2} + \alpha + \frac{\alpha' + \beta' + \gamma'}{2C}$  and  $\hat{b} = 1 + 2\alpha$ . In the region,  $x \rightarrow \infty$  the solution must be vanished in order to satisfy the other boundary condition where the potential in this region diverged. We can approximate the solution at the infinity by using the property of the confluent hypergeometric function

$$R(v \rightarrow \infty) \approx e^{Cx - i\pi\hat{a}} (-2C)^{-\hat{a}} v^{\alpha - \hat{a}} \frac{1}{\Gamma(\hat{b} - \hat{a})} + e^{-Cx} (-2C)^{\hat{a} - \hat{b}} v^{\alpha + \hat{a} - \hat{b}} \frac{1}{\Gamma(\hat{a})} \quad (29)$$

As  $v \rightarrow \infty$ , the first term decays, while the second term diverges. To get rid of the divergence, set the argument of the Gamma function to be negative integer.

$$\hat{a} = \frac{1}{2} + \alpha + \frac{\alpha' + \beta' + \gamma'}{2C} = -n, \quad n = 0, 1, 2, 3, \dots \quad (30)$$

The above equation, (30), is a constrain equation for the frequencies, where it can be written down in term of a dimensionless  $\omega\rho_+$  parameter as

$$\left( 2n + 1 - \frac{2i\omega\rho_+ r_\infty}{\sqrt{\mu}} \right) \sqrt{-(\omega\rho_+)^2 + \frac{\lambda^2 \mu}{r_\infty^2 (r_\infty^2 - \mu)}} - \frac{i\omega\rho_+ r_\infty}{\sqrt{\mu}} + (\omega\rho_+)^2 \left( 1 + \frac{r_\infty^2}{\mu} \right) - \frac{\lambda^2 \mu}{r_\infty^2 - \mu} - l(l+1) = 0 \quad (31)$$

The quasinormal frequencies can be solved from this equation. There are four roots to equation (31), but there is only one that gives positive real number and negative imaginary number, corresponding to the ingoing wave at the horizon and decaying wave at the far away region respectively. We will compare our result with numerical work in the next section

**Results, Discussion and Conclusion:** The quasinormal frequencies are obtained from equation (31). We put some specific parameters in order to compare with the numerical result as the following<sup>6</sup>

**Table 1** The quasinormal frequencies from WKB method and our analytical result, where  $l = 10$ ,  $\rho_0 / \rho_+ = 3$  or  $r_\infty / r_+ = 2$  and  $n = 2$

$\lambda$	WKB <sup>6</sup>	Analytical work
0	3.5149 - 0.15956i	3.4921 - 0.16667i
0.5	3.5336 - 0.15821i	3.4940 - 0.16643i
1	3.5898 - 0.15413i	3.4999 - 0.16572i
1.5	3.6842 - 0.14729i	3.5098 - 0.16455i
2	3.8178 - 0.13759i	3.5236 - 0.16291i
2.5	3.9924 - 0.12493i	3.5413 - 0.16083i
3	4.2103 - 0.10912i	3.5629 - 0.15832i
3.5	4.4747 - 0.08988i	3.5883 - 0.15539i

From both results the imaginary part is negative, causing the wave vanishing in the far away region. As  $\lambda$  increases, the real part increases while the imaginary part decreases.

Our frequencies change more slowly than WKB result when  $\lambda$  increases. To improve our frequency result, we can take our approximated solution as the zero order perturbation. We can continue to perform the first order. Also we can add the rotation  $a \neq 0$  to black hole to the problem in our future work.

**References:**

1. Horowitz, G., Hubeny, V.E. *Phys.Rev.D.* **2000**, 62, 024027.
2. Vallisneri, M., *Class. Quantum Grav.* **2009**, 26, 094024.
3. Ishihara, H. Matsuno, K., *Prog. Theor. Phys.* **2006**, 116, 417-422.
4. Wang, T., *Nucl. Phys. B* **2006**, 756, 86-99.
5. Chen, S., Wang, B., Su, R. *Phys. Rev. D.* **2008**, 77, 024039.
6. Ishihara, H., Kimura, M., Konoplya, R.A., Murata, K., Soda, J., Zhidenko, A., *Phys. Rev. D.* **2008**, 77, 084019.

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