

## CHAPTER IV

### Extremal Problems for Graph Parameters

An “extremal problem” asks for the minimum or maximum value of a function over a class of objects. In graph theory, we use extremal problem for finding an extreme value of some graph parameter over some class of graphs.

Let  $f$  be a graph parameter and  $\mathcal{J} \subseteq \mathcal{G}(m, n)$ . Proving that  $a$  is the minimum of  $f(G)$  of graph in a class  $\mathcal{J}$  requires showing two things:

1.  $f(G) \geq a$  for all  $G \in \mathcal{J}$ .
2.  $f(G) = a$  for some  $G \in \mathcal{J}$ .

The proof of the bound must apply to every  $G \in \mathcal{J}$ . For equality, it suffices to obtain an example in  $\mathcal{J}$  with the desired value of  $f$ .

Changing “ $\geq$ ” to “ $\leq$ ” yields the criteria for a maximum.

We divide the interpolation theorems for the six graph parameters of graphs of order  $n$  and size  $m$  into two parts. The first part is obtained from the fact that  $\omega$ ,  $\alpha_0$ ,  $\beta_0$ ,  $\chi$ ,  $\alpha_1$ , and  $\beta_1$  are interpolation graph parameters over  $\mathcal{G}(m, n)$  and  $\mathcal{CG}(m, n)$ . From Theorems 3.12 and 3.13, we can conclude that  $\omega$ ,  $\alpha_0$ ,  $\beta_0$ ,  $\chi$ ,  $\alpha_1$ , and  $\beta_1$  are interpolation graph parameters over  $\mathcal{G}(m, n)$  and  $\mathcal{CG}(m, n)$ , respectively. In the second part that concerns extremal problems, we will show that for  $f \in \{\omega, \alpha_0, \beta_0, \chi, \alpha_1, \beta_1\}$  there exist integers  $a, b, A$ , and  $B$  where

$$\begin{aligned}
a(f) &= \min\{f(G) : G \in \mathcal{G}(m, n)\}, \\
b(f) &= \max\{f(G) : G \in \mathcal{G}(m, n)\}, \\
A(f) &= \min\{f(G) : G \in \mathcal{CG}(m, n)\}, \text{ and} \\
B(f) &= \max\{f(G) : G \in \mathcal{CG}(m, n)\}, \text{ respectively.}
\end{aligned}$$

To achieve the minimum value of a graph parameter  $f$  of  $G \in \mathcal{G}(m, n)$ , we will use the technique of finding a lower bound that is the best possible. If we can construct a graph  $G$  such that  $f(G)$  is equal to this lower bound then the minimum value is equal to this bound. Otherwise, we will prove that it cannot construct a graph  $G$  with  $f(G)$  equal to this bound then construct a graph  $G$  whose  $f(G)$  is almost equal to this bound. This means that we obtain the minimum value. To achieve the maximum value, we will determine in the same way of finding the minimum value.

The following notation we often use in our results. For the real number  $r$ ,  $\lfloor r \rfloor$  denotes the greatest integer less than or equal to  $r$  and  $\lceil r \rceil$  denotes the smallest integer greater than or equal to  $r$ .

## 4.1 The Clique Number

### 4.1.1 The Clique Number in $\mathcal{G}(m, n)$

We consider the clique number of graphs with prescribed order and size. That is to determine  $\max\{\omega(G) : G \in \mathcal{G}(m, n)\}$  and  $\min\{\omega(G) : G \in \mathcal{G}(m, n)\}$ , simply written  $\max(\omega; m, n)$  and  $\min(\omega; m, n)$ , respectively. Note that  $\max(\omega; 0, n) = \min(\omega; 0, n) = 1$ ,  $\max(\omega; 1, 2) = \min(\omega; 1, 2) = 2$ , and for the complete graph  $K_n$ ,  $\omega(K_n) = n$ . Therefore, we shall determine  $\max(\omega; m, n)$  and  $\min(\omega; m, n)$  when  $n \geq 3$ . To achieve the  $\max(\omega; m, n)$ , we use the technique of finding a complete subgraph of  $G \in \mathcal{G}(m, n)$  whose vertex set is as large as possible.

Observe that for any graph  $G$  which has a largest clique of order  $k$  must have  $\binom{k}{2} \leq |E(G)| < \binom{k+1}{2}$ . In the following theorem, we consider  $\max(\omega; m, n)$ .

**Theorem 4.1.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq n - 1$  and  $\binom{k}{2} \leq m < \binom{k+1}{2}$ . Then  $\max(\omega; m, n) = k$ .

*Proof.* By above observation, if  $\binom{k}{2} \leq m < \binom{k+1}{2}$ , then  $\max(\omega; m, n) \leq k$ . We now construct a graph  $G \in \mathcal{G}(m, n)$  such that  $\omega(G) = k$ . Let  $G$  be a graph obtained from the complete graph  $K_k$  with  $V(K_k) = \{v_1, v_2, \dots, v_k\}$  by adding  $n - k$  vertices and joining each  $v_i$  for  $1 \leq i \leq m - \binom{k}{2}$  to one of the added vertices. We can see that the order of the largest clique in  $G$  is  $k$ . It is clear that  $G$  has order  $n$  and size  $m$  with  $\omega(G) = k$ . Thus  $\max(\omega; m, n) = k$ . □

To determine  $\min(\omega; m, n)$ , we shall apply Turán's Theorem. Turán [36] provided the complete  $r$ -partite graph of order  $n$ , the cardinalities of whose partite sets differ by at most 1, usually called the *Turán graph* and denoted by  $T_{n,r}$ . In Figure 4.1, we illustrate  $T_{8,3}$ ,  $T_{6,4}$ , and state the Turán's Theorem as follows:

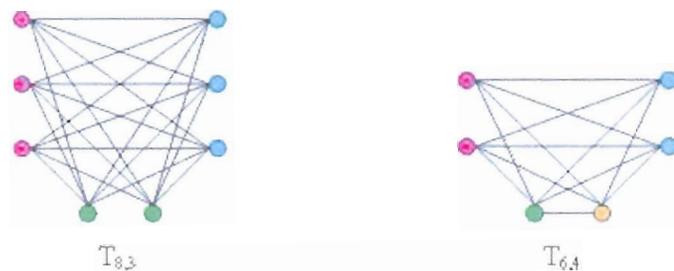


Figure 4.1: The Turán graphs

**Theorem 4.2.** (Turán's Theorem) Among the graphs of order  $n$  with no cliques of order  $r + 1$ ,  $T_{n,r}$  has the maximum number of edges. □

Let  $t_{n,r}$  and  $\bar{t}_{n,r}$  denote the edge number of  $T_{n,r}$  and  $\bar{T}_{n,r}$ , respectively. In order to use the Turán graph in our work, we would like to state the following facts.

1. If  $n = rq + t, 0 \leq t < r$ , then  $T_{n,r}$  consists of  $t$  partite sets of cardinality  $\lceil \frac{n}{r} \rceil$  and  $r - t$  partite sets of cardinality  $\lfloor \frac{n}{r} \rfloor$ .
2. Let  $G \in \mathcal{G}(m, n)$ . If  $\omega(G) \leq r$ , then  $m \leq t_{n,r}$ .
3.  $t_{n,r} = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$ , where  $a = \lfloor \frac{n}{r} \rfloor$ .

**Theorem 4.3.** Let  $n, m$ , and  $k \geq 2$  be positive integers where  $t_{n,k-1} < m \leq t_{n,k}$ . Then  $\min(\omega; m, n) = k$ .

*Proof.* By Turán's Theorem, every graph of order  $n$  with no cliques of order  $k+1$ ,  $T_{n,k}$  has the maximum number of edges. It follows that for a graph  $G \in \mathcal{G}(m, n)$  if  $m > t_{n,k-1}$ , then  $\omega(G) \geq k$ . Observe that if  $l < t_{n,k} - t_{n,k-1}$  edges are removed from  $T_{n,k}$ , then the result graph contains a maximum clique of order  $k$ . We can obtain a graph  $H \in \mathcal{G}(m, n)$  where  $\omega(H) = k$  by deleting  $t_{n,k} - m$  edges from  $T_{n,k}$ . Thus  $\min(\omega; m, n) = k$ . □

The following example illustrates the clique number of some graphs in  $\mathcal{G}(m, 4)$ .

**Example 4.4.** According to Theorems 4.1 and 4.3, let  $n = 4$ . Then

$$\max(\omega; m, 4) = \begin{cases} 2 & \text{if } 1 \leq m < 3, \\ 3 & \text{if } 3 \leq m < 6, \\ 4 & \text{if } m = 6. \end{cases}$$

$$\min(\omega; m, 4) = \begin{cases} 2 & \text{if } 0 < m \leq 4, \\ 3 & \text{if } 4 < m \leq 5, \\ 4 & \text{if } 5 < m \leq 6. \end{cases}$$

Firstly, consider  $m = 2$ . We have  $\max(\omega; 2, 4) = \min(\omega; 2, 4) = 2$ . In Figure 4.2, we construct the graph  $G_1 \in \mathcal{G}(2, 4)$  with  $\omega(G_1) = 2$  from the complete graph  $K_2$  by adding 2 vertices and joining one vertex of  $K_2$  to one of the added vertices. The clique of  $G_1$  is indicated by their complete subgraph with red edge as shown in the figure.

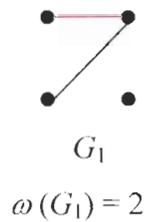


Figure 4.2: The clique number of a graph in  $\mathcal{G}(2, 4)$

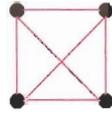
Secondly, consider  $m = 3$ . We have  $\max(\omega; 3, 4) = 3$  and  $\min(\omega; 3, 4) = 2$ . In Figure 4.3, we construct the graphs  $G_2 \in \mathcal{G}(3, 4)$  with  $\omega(G_2) = 2$  by deleting one edge from  $T_{4,2}$  and  $G_3 \in \mathcal{G}(3, 4)$  with  $\omega(G_3) = 3$  by adding one vertex to  $K_3$ . The clique of  $G_2$  and  $G_3$  are indicated by their complete subgraphs with red edges as shown in the figure.



Figure 4.3: The clique number of graphs in  $\mathcal{G}(3, 4)$

Finally, consider  $m = 6$ . We have  $\max(\omega; 6, 4) = \min(\omega; 6, 4) = 4$ . In Figure 4.4, we construct the graph  $G_4 \in \mathcal{G}(6, 4)$  with  $\omega(G_4) = 4$  as the complete graph

$K_4$ . The clique of  $G_4$  is indicated by their complete subgraph with red edges as shown in the figure.



$G_4$

$$\omega(G_4) = 4$$

Figure 4.4: The clique number of a graph in  $\mathcal{G}(6, 4)$

#### 4.1.2 The Clique Number in $\mathcal{CG}(m, n)$

In this section we determine the clique number of connected graphs with prescribed order and size. That is to determine  $\max\{\omega(G) : G \in \mathcal{CG}(m, n)\}$  and  $\min\{\omega(G) : G \in \mathcal{CG}(m, n)\}$ , simply written  $\text{Max}(\omega; m, n)$  and  $\text{Min}(\omega; m, n)$ , respectively. Observe that for any connected graph  $G$  of order  $n$  and size  $m$  which has a largest clique of order  $k$ , there must be a path from each vertex of its largest clique to every other  $n - k$  vertices. That is  $m \geq \binom{k}{2} + n - k$ . Every graph in  $\mathcal{CG}(m, n)$  has size at least  $n - 1$ . It is clear that for any positive integer  $n$ ,  $\text{Min}(\omega; m, n) = 2$  if and only if  $n - 1 \leq m \leq t_{n,2}$  and for the complete graph  $K_n$ ,  $\omega(K_n) = n$ . We obtain  $\text{Max}(\omega; m, n)$  and  $\text{Min}(\omega; m, n)$  in the following theorems.

**Theorem 4.5.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq n - 1$  and  $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - (k + 1)$ . Then  $\text{Max}(\omega; m, n) = k$ .

*Proof.* Let  $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - (k + 1)$ . By above observation,  $\text{Max}(\omega; m, n) \leq k$ . We now construct a graph  $H \in \mathcal{CG}(m, n)$  such that  $\omega(H) = k$ .

Since  $m < \binom{k+1}{2} + n - (k+1)$ , we have  $m - \binom{k}{2} - (n - k - 1) < k$ . The graph  $H$  can be obtained from  $K_k$  and  $P_{n-k}$  by adding  $m - \binom{k}{2} - (n - k - 1)$  edges to join some vertices of  $K_k$  to some vertices of  $P_{n-k}$ . We can see that the order of a largest clique in  $H$  is  $k$ . It is clear that  $H$  is a connected graph of order  $n$  and size  $m$  with  $\omega(H) = k$ . Thus  $\text{Max}(\omega; m, n) = k$ . □

**Theorem 4.6.** Let  $n, m$ , and  $k \geq 3$  be positive integers where  $t_{n,k-1} < m \leq t_{n,k}$ . Then  $\text{Min}(\omega; m, n) = k$ .

*Proof.* Since  $\mathcal{CG}(m, n)$  is a subset of  $\mathcal{G}(m, n)$ , thus  $\text{Min}(\omega; m, n) \geq \min(\omega; m, n)$ . By Theorem 4.3, if  $t_{n,k-1} < m \leq t_{n,k}$ , we have  $\text{Min}(\omega; m, n) \geq k$ . By deleting  $t_{n,k} - m$  edges from  $T_{n,k}$ , we can find a graph  $H$  in  $\mathcal{CG}(m, n)$  with  $\omega(H) = k$ . Thus  $\text{Min}(\omega; m, n) = k$ . □

The following example illustrates the clique number of some connected graphs in  $\mathcal{CG}(m, 5)$ .

**Example 4.7.** According to Theorems 4.5 and 4.6, let  $n = 5$ . Then

$$\text{Max}(\omega; m, 5) = \begin{cases} 2 & \text{if } 4 \leq m < 5, \\ 3 & \text{if } 5 \leq m < 7, \\ 4 & \text{if } 7 \leq m < 10, \\ 5 & \text{if } m = 10. \end{cases}$$

$$\text{Min}(\omega; m, 5) = \begin{cases} 3 & \text{if } 6 < m \leq 8, \\ 4 & \text{if } 8 < m \leq 9, \\ 5 & \text{if } 9 < m \leq 10. \end{cases}$$

Firstly, consider  $m = 7$ . We have  $\text{Max}(\omega; 7, 5) = 4$  and  $\text{Min}(\omega; 7, 5) = 3$ . In Figure 4.5, we construct the graphs  $G_1 \in \mathcal{CG}(7, 5)$  with  $\omega(G_1) = 3$  by deleting one

edge from  $T_{5,3}$  and  $G_2 \in \mathcal{CG}(7, 5)$  with  $\omega(G_2) = 4$  by adding one edge to  $K_4 \cup K_1$ . The clique of  $G_1$  and  $G_2$  are indicated by their complete subgraphs with red edges as shown in the figure.

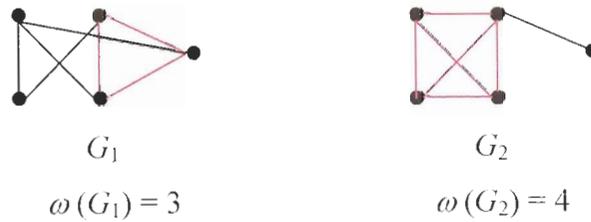


Figure 4.5: The clique number of connected graphs in  $\mathcal{CG}(7, 5)$

Secondly, consider  $m = 9$ . We have  $\text{Max}(\omega; 9, 5) = \text{Min}(\omega; 9, 5) = 4$ . In Figure 4.6, we construct the graph  $G_3 \in \mathcal{CG}(9, 5)$  with  $\omega(G_3) = 4$  as the graph  $T_{5,4}$ . The clique of  $G_3$  is indicated by their complete subgraph with red edges as shown in the figure.

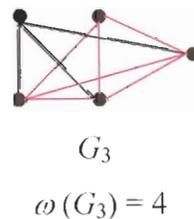


Figure 4.6: The clique number of a connected graph in  $\mathcal{CG}(9, 5)$

Finally, consider  $m = 10$ . We have  $\text{Max}(\omega; 10, 5) = \text{Min}(\omega; 10, 5) = 5$ . In Figure 4.7, we construct the graph  $G_4 \in \mathcal{CG}(10, 5)$  with  $\omega(G_4) = 5$  as the complete graph  $K_5$ . The clique of  $G_4$  is indicated by their complete subgraph with red edges as shown in the figure.

 $G_4$ 

$$\omega(G_4) = 5$$

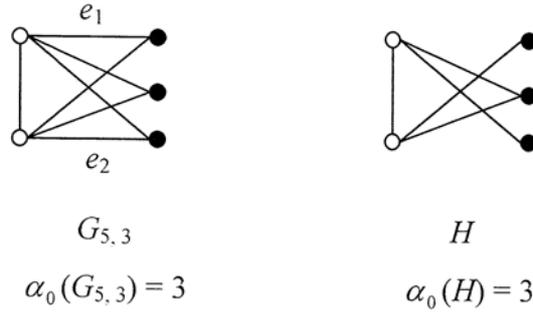
Figure 4.7: The clique number of a connected graph in  $\mathcal{CG}(10, 5)$ 

## 4.2 The Independence Number

### 4.2.1 The Independence Number in $\mathcal{G}(m, n)$

We consider the independence number of graphs with prescribed order and size. That is to determine  $\max\{\alpha_0(G) : G \in \mathcal{G}(m, n)\}$  and  $\min\{\alpha_0(G) : G \in \mathcal{G}(m, n)\}$ , simply written  $\max(\alpha_0; m, n)$  and  $\min(\alpha_0; m, n)$ , respectively. Note that  $\max(\alpha_0; 0, n) = \min(\alpha_0; 0, n) = n$ ,  $\max(\alpha_0; 1, 2) = \min(\alpha_0; 1, 2) = 1$ , and for the complete graph  $K_n$ ,  $\alpha_0(K_n) = 1$ .

We now introduce a graph concerning the bound of  $\max(\alpha_0; m, n)$ . Let  $G_{n,p}$  be a graph obtained from the complete graph  $K_{n-p}$  by adding  $p$  vertices and joining each added vertex to every vertex of  $K_{n-p}$ . It is clear that  $G_{n,p} \in \mathcal{G}(\binom{n}{2} - \binom{p}{2}, n)$  and  $\alpha_0(G_{n,p}) = p$ . Moreover, the graph  $G_{n,p}$  is connected, and for every  $i = 1, 2, \dots, p-1$ , there is a set  $X_i$  of  $i$  edges of  $G_{n,p}$  such that  $G_{n,p} - X_i$  is connected and  $\alpha_0(G_{n,p} - X_i) = p$ . For example, let  $X_2 = \{e_1, e_2\}$  be the subset of edges in  $G_{5,3}$ . The graph  $H = G_{5,3} - X_2$  can be shown in the following figure.

Figure 4.8: A graph  $H = G_{5,3} - X_2$ 

Let  $G \in \mathcal{G}(m, n)$ . By the fact that if  $\alpha_0(G) = p$  where  $2 \leq p \leq n - 1$ , then  $m \leq \binom{n}{2} - \binom{p}{2}$ . Then we have the formula for  $\max(\alpha_0; m, n)$  as shown in following theorem.

**Theorem 4.8.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq n - 1$  and  $\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}$ . Then  $\max(\alpha_0; m, n) = k$ .

*Proof.* Let  $\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}$  for  $2 \leq k \leq n - 1$ . By above observation,  $\max(\alpha_0; m, n) \leq k$ . Since  $m > \binom{n}{2} - \binom{k+1}{2}$ , we have  $\binom{n}{2} - \binom{k}{2} - m < k$ . We can construct a graph  $H$  of order  $n$  and size  $m$  with  $\alpha_0(H) = k$  by deleting some appropriate  $\binom{n}{2} - \binom{k}{2} - m$  edges from  $G_{n,k}$ . Thus  $\max(\alpha_0; m, n) = k$ . □

Some basic results concerning a lower bound of  $\alpha_0(G)$  is  $\alpha_0(G) \geq \frac{n}{\Delta(G)+1}$ . It was proved by Caro [4] that the lower bound of  $\alpha_0(G)$  can be improved by replacing  $\Delta(G)$  by  $\frac{2m}{n}$ , the average degree of vertices of  $G$ . Thus if  $G \in \mathcal{G}(m, n)$ , then  $\alpha_0(G) \geq \frac{n}{\frac{2m}{n}+1} = \frac{n^2}{2m+n}$ . Moreover, if  $\alpha_0(G) = p$ , then  $\frac{2m}{n} \geq \frac{n}{p} - 1$ .

**Lemma 4.9.** Let  $G$  be a graph of order  $n$  and  $\alpha_0(G) = p \geq 2$ . Then there exists a graph  $H$  of order  $n$ ,  $H$  is a union of  $p$  cliques and  $|E(H)| \leq |E(G)|$ .

*Proof.* Let  $G$  be a graph of order  $n$ . When  $n = 2$ ,  $G = H = \overline{K}_2$ . Suppose  $n > 2$ . We consider two cases.

*Case 1.* If there exists  $v \in V(G)$  such that  $\alpha_0(G-v) = p-1$ , then, by induction on  $n$ , there exists a graph  $H'$  of order  $n-1$ ,  $H'$  is a union of  $p-1$  cliques and  $|E(H')| \leq |E(G-v)|$ . Thus we can obtain a graph  $H$  of order  $n$  satisfying the desired property by adding a vertex to the graph  $H'$ .

*Case 2.* For every vertex  $v \in V(G)$ ,  $\alpha_0(G-v) = p$ . Let  $v \in V(G)$  with  $d_G(v) = \Delta(G) \geq \frac{2m}{n}$ . Thus, by induction on  $n$ , there exists a graph  $H'$  of order  $n-1$ ,  $H'$  is a union of  $p$  cliques and  $|E(H')| \leq |E(G-v)|$ . Since  $H'$  is a graph of order  $n-1$  and it is a union of  $p$  cliques, there must be at least one component  $C'$  of  $H'$  having order at most  $\lfloor \frac{n-1}{p} \rfloor$ . Hence  $\frac{2m}{n} \geq \lfloor \frac{n-1}{p} \rfloor$ . Let  $C$  be a clique obtained from  $C'$  by adding a vertex  $v$  and edges joining  $v$  to every vertex of  $C'$ . Thus we can construct a graph  $H$  from  $H'$  by replacing  $C'$  by  $C$ .

□

Let  $\mathcal{H}(n, p)$  be the set of graphs of order  $n$  each of which is a union of  $p$  cliques. We now investigate a graph  $H \in \mathcal{H}(n, p)$  of which  $|E(H)|$  is minimum.

Recall the Turán graph  $T_{n,r}$  which all partite sets have size  $\lfloor \frac{n}{r} \rfloor$  or  $\lceil \frac{n}{r} \rceil$ . Let  $G \in \mathcal{G}(m, n)$ . If  $\omega(G) \leq r$ , then  $m \leq t_{n,r}$  and for a fixed  $n$ , by using elementary arithmetic, we get  $t_{n,r-1} < t_{n,r}$  for all  $2 \leq r \leq n$ . In fact  $t_{n,r} - t_{n,r-1} \geq \binom{a+1}{2}$ , where  $a = \lfloor \frac{n}{r} \rfloor$ . Thus we have the following results.

**Lemma 4.10.** Let  $H \in \mathcal{H}(n, p)$ . Then  $|E(H)|$  is minimum if and only if  $H = \bar{T}_{n,p}$ .

□

**Corollary 4.11.** Let  $n, m$ , and  $2 \leq k \leq n-1$  be positive integers where  $\bar{t}_{n,k} \leq m < \bar{t}_{n,k-1}$ . Then  $\min(\alpha_0; m, n) = k$ .

□

The following example illustrates the independence number of some graphs in  $\mathcal{G}(m, 6)$ .

**Example 4.12.** According to Theorem 4.8 and Corollary 4.11, let  $n = 6$ . Then

$$\max(\alpha_0; m, 6) = \begin{cases} 5 & \text{if } 0 < m \leq 5, \\ 4 & \text{if } 5 < m \leq 9, \\ 3 & \text{if } 9 < m \leq 12, \\ 2 & \text{if } 12 < m \leq 14. \end{cases}$$

$$\min(\alpha_0; m, 6) = \begin{cases} 5 & \text{if } 1 \leq m < 2, \\ 4 & \text{if } 2 \leq m < 3, \\ 3 & \text{if } 3 \leq m < 6, \\ 2 & \text{if } 6 \leq m < 15. \end{cases}$$

Firstly, consider  $m = 5$ . We have  $\max(\alpha_0; 5, 6) = 5$  and  $\min(\alpha_0; 5, 6) = 3$ . In Figure 4.9, we construct the graphs  $G_1 \in \mathcal{G}(5, 6)$  with  $\alpha_0(G_1) = 3$  by adding 2 edges to the graph  $\bar{T}_{6,3}$  and  $G_3 \in \mathcal{G}(5, 6)$  with  $\alpha_0(G_3) = 5$  as the graph  $G_{6,5}$ . Since  $\alpha_0$  is an interpolation graph parameter over  $\mathcal{G}(m, n)$ , there exists a graph  $G_2 \in \mathcal{G}(5, 6)$  with  $\alpha_0(G_2) = 4$ . The independent set of  $G_1$ ,  $G_2$ , and  $G_3$  are indicated by their black vertices as shown in the figure.

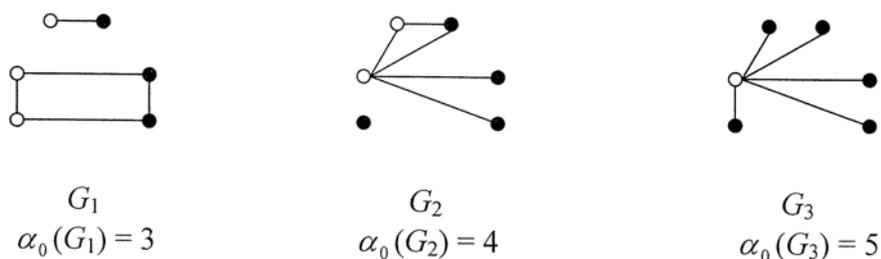


Figure 4.9: The independence number of graphs in  $\mathcal{G}(5, 6)$

Secondly, consider  $m = 9$ . We have  $\max(\alpha_0; 9, 6) = 4$  and  $\min(\alpha_0; 9, 6) = 2$ . In Figure 4.10, we construct the graphs  $G_4 \in \mathcal{G}(9, 6)$  with  $\alpha_0(G_4) = 2$  by adding 3 edges to the graph  $\bar{T}_{6,2}$  and  $G_6 \in \mathcal{G}(9, 6)$  with  $\alpha_0(G_6) = 4$  as the graph  $G_{6,4}$ .

Since  $\alpha_0$  is an interpolation graph parameter over  $\mathcal{G}(m, n)$ , there exists a graph  $G_5 \in \mathcal{G}(9, 6)$  with  $\alpha_0(G_5) = 3$ . The independent set of  $G_4, G_5$ , and  $G_6$  are indicated by their black vertices as shown in the figure.

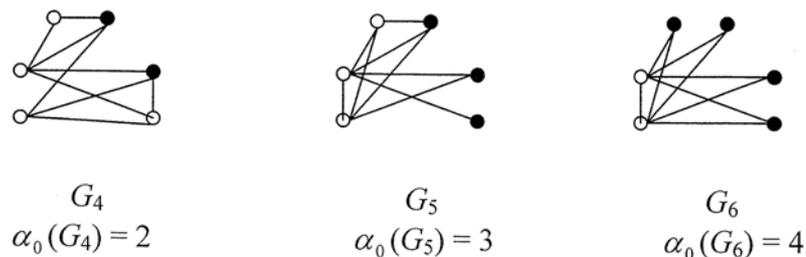


Figure 4.10: The independence number of graphs in  $\mathcal{G}(9, 6)$

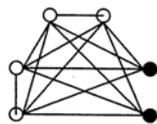
Thirdly, consider  $m = 10$ . We have  $\max(\alpha_0; 10, 6) = 3$  and  $\min(\alpha_0; 10, 6) = 2$ . In Figure 4.11, we construct the graphs  $G_7 \in \mathcal{G}(10, 6)$  with  $\alpha_0(G_7) = 2$  by adding 4 edges to the graph  $\bar{T}_{6,2}$  and  $G_8 \in \mathcal{G}(10, 6)$  with  $\alpha_0(G_8) = 3$  by deleting 2 edges from the graph  $G_{6,3}$ . The independent set of  $G_7$ , and  $G_8$  are indicated by their black vertices as shown in the figure.



Figure 4.11: The independence number of graphs in  $\mathcal{G}(10, 6)$

Finally, consider  $m = 13$ . We have  $\max(\alpha_0; 13, 6) = \min(\alpha_0; 13, 6) = 2$ . In Figure 4.12, we construct the graph  $G_9 \in \mathcal{G}(13, 6)$  with  $\alpha_0(G_9) = 2$  as the graph

$G_{6,2}$ . The independent set of  $G_9$  is indicated by their black vertices as shown in the figure.



$$G_9$$

$$\alpha_0(G_9) = 2$$

Figure 4.12: The independence number of a graph in  $\mathcal{G}(13, 6)$

### 4.2.2 The Independence Number in $\mathcal{CG}(m, n)$

In this section we determine the independence number of connected graphs with prescribed order and size. That is to determine  $\max\{\alpha_0(G) : G \in \mathcal{CG}(m, n)\}$  and  $\min\{\alpha_0(G) : G \in \mathcal{CG}(m, n)\}$ , simply written  $\text{Max}(\alpha_0; m, n)$  and  $\text{Min}(\alpha_0; m, n)$ , respectively. Note that  $\alpha_0(C_n) = \lfloor \frac{n}{2} \rfloor$ ,  $\alpha_0(P_n) = \lceil \frac{n}{2} \rceil$ , and for the complete graph  $K_n$ ,  $\alpha_0(K_n) = 1$ . Since the independence number of the star of order  $n$  is  $n - 1$ ,  $\text{Max}(\alpha_0; n - 1, n) = n - 1$ . The analogous result of Theorem 4.8 to the case of connected graphs can be obtained in the following corollary.

**Corollary 4.13.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq n - 2$  and  $\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}$ . Then  $\text{Max}(\alpha_0; m, n) = k$ .

□

In order to determine the value of  $\text{Min}(\alpha_0; m, n)$ , we first observe the following facts.

1.  $\text{Min}(\alpha_0; n - 1, n) = \lceil \frac{n}{2} \rceil$ .

2.  $\text{Min}(\alpha_0; n, n) = \lfloor \frac{n}{2} \rfloor$ .
3. Since for a positive integer  $k < m$ ,  $\text{Min}(\alpha_0; k + 1, n) \leq \text{Min}(\alpha_0; k, n)$ , it follows that for  $m \geq n + 1$ ,  $\text{Min}(\alpha_0; m, n) \leq \frac{n}{2}$ .

For integers  $n$  and  $p$  where  $p \leq \frac{n}{2}$  and  $H \in \mathcal{H}(n, p)$ , we call a component of  $H$  a *basic clique* of  $H$ . Thus, in this case,  $H$  must contain a basic clique of order at least 2. Further, if  $H$  contains an isolated vertex as its basic clique, then  $H$  must contain a basic clique of order at least 3. Thus for any  $H \in \mathcal{H}(n, p)$ , we can construct a connected graph  $H^*$  from  $H$  by adding  $p - 1$  edges to the  $p$  basic cliques of  $H$  in such a way that  $\alpha_0(H^*) = \alpha_0(H) = p$ . Note that  $H^*$  may not be unique, and  $|E(H^*)| = |E(H)| + p - 1$ . Let  $\mathcal{H}^*(n, p)$  be the set of graphs  $H^*$  which is obtained from  $H \in \mathcal{H}(n, p)$  in this manner. Thus we have the following theorem.

**Theorem 4.14.** Let  $G$  be a connected graph of order  $n$  and  $\alpha_0(G) = p \leq \frac{n}{2}$ . Then there exists a graph  $H \in \mathcal{H}^*(n, p)$  such that  $|E(H)| \leq |E(G)|$ .

*Proof.* We will proceed by induction on  $n$ . The result holds for  $n = 1$ . Suppose that for  $n \geq 2$ , there is a vertex  $v$  of  $G$  such that  $\alpha_0(G - v) = p - 1$  and  $G - v$  contains  $k$  components,  $G_1, G_2, \dots, G_k$ , of order  $n_1, n_2, \dots, n_k$ , respectively. Thus the vertex  $v$  must be contained in every maximum independent set of  $G$  and  $d_G(v) \geq k$ . If  $\alpha_0(G_i) = p_i$ , for  $i = 1, 2, \dots, k$ , then  $\sum_{i=1}^k p_i = p - 1$  and  $\alpha_0(G[V(G_i) \cup \{v\}]) = p_i + 1$ . Thus  $n_i \geq 2$ , for all  $i = 1, 2, \dots, k$ . By induction, there exists a graph  $H'_i \in \mathcal{H}^*(n_i, p_i)$  and  $|E(H'_i)| \leq |E(G_i)|$ . We can easily construct a graph  $H$  with the desired property.

Suppose that for every vertex  $v$  of  $G$ ,  $\alpha_0(G - v) = p$ . Let  $v$  be a vertex of  $G$  such that  $d_G(v) = \Delta(G)$ . We consider two cases.

*Case 1.* If  $G - v$  is connected, then there exists a graph  $H' \in \mathcal{H}^*(n - 1, p)$  such that  $|E(H')| \leq |E(G - v)|$ . Let  $X$  be the set of  $p - 1$  edges of  $H'$  such that  $H' - X \in \mathcal{H}(n - 1, p)$ . Thus  $H' - X$  contains  $p$  disjoint cliques. There exists a component  $C'$  of  $H' - X$  such that  $C'$  is a clique of order at most  $\lfloor \frac{n-1}{p} \rfloor$ . Since

$d_G(v) \geq \frac{n}{p} - 1$ , we can construct a clique  $C$  where  $V(C) = V(C' \cup \{v\})$  and  $E(C) = E(C') \cup \{vx : x \in V(C')\}$ . It is clear that a graph  $H$  which is obtained from  $H'$  by replacing  $C'$  by  $C$  has the desired property.

*Case 2.* Suppose that  $G-v$  is disconnected and  $G-v$  consists of  $k$  components,  $G_1, G_2, \dots, G_k$ , with  $|V(G_i)| = n_i$  and  $\alpha_0(G_i) = p_i$  for  $i = 1, 2, \dots, k$  and  $n_1 \leq n_2 \leq \dots \leq n_k$ . Thus  $\sum_{i=1}^k p_i = p$ . Put  $G'_i = G[V(G_i) \cup \{v\}]$ . Since  $\alpha_0(G'_i) \in \{\alpha_0(G_i), \alpha_0(G_i) + 1\}$  and  $\alpha_0(G_i) + \alpha_0(G_j) \leq \alpha_0(G'_i) + \alpha_0(G'_j) \leq \alpha_0(G_i) + \alpha_0(G_j) + 1$  for all  $i, j$  with  $1 \leq i < j \leq k$ , it follows that there is at most one  $i$  such that  $\alpha_0(G'_i) = \alpha_0(G_i) + 1$  and in this case  $n_i \geq 2$ .

*Case 2.1.* Suppose that there exists  $i$  such that  $\alpha_0(G'_i) = \alpha_0(G_i) + 1$  and  $\alpha_0(G'_j) = \alpha_0(G_j)$  for all  $i \neq j$ . Put  $G' = G - V(G_i)$ . Thus  $G'$  is a connected graph of order  $n - n_i$  with  $\alpha_0(G') = p - \alpha_0(G_i)$ . By induction, there is a graph  $H' \in \mathcal{H}^*(n - n_i, p - \alpha_0(G_i))$  where  $|E(H')| \leq |E(G')|$  and a graph  $H'' \in \mathcal{H}^*(n_i, \alpha_0(G_i))$  where  $|E(H'')| \leq |E(G_i)|$ . We can construct a graph  $H$  from  $H'$  and  $H''$  with the desired property by adding an appropriate edge.

*Case 2.2.* Suppose that  $\alpha_0(G'_i) = \alpha_0(G_i)$  for all  $i = 1, 2, \dots, k$ . Let  $G_i$  be a nontrivial component of  $G - v$  with  $\alpha_0(G_i) \leq \frac{n_i}{2}$ . If  $G_i$  is complete, by  $\alpha_0(G'_i) = \alpha_0(G_i) = 1$ , then  $G'_i$  is also complete. In this case  $\alpha_0(G'_i - vu) = \alpha_0(G_i) + 1$  for some  $vu \in E(G'_i)$ . Thus  $G - vu$  satisfies the condition in the Case 2.1. Suppose that  $G_i$  is not complete. Then we may assume that  $v$  is adjacent to only one vertex of  $G_i$ , otherwise we can remove some edges which join  $v$  to  $V(G_i)$ . Thus  $G_i$  has order at least 4. Furthermore,  $G_i$  can not be a path. By induction, there exists a graph  $H_i \in \mathcal{H}^*(n_i, p_i)$  such that  $|E(H_i)| \leq |E(G_i)|$ . Let  $C$  be a nontrivial basic clique of  $H_i$  and  $u \in V(C)$ . Let  $H'_i$  be a graph of which  $V(H'_i) = V(H_i) \cup \{v\}$  and  $E(H'_i) = E(H_i) \cup \{vu\}$ . The graph  $G'$  obtained from  $G$  by replacing  $G'_i$  by  $H'_i$  satisfies the condition in the Case 2.1. Thus the proof is complete. □

**Corollary 4.15.** Let  $n, m$ , and  $2 \leq k \leq \lceil \frac{n}{2} \rceil$  be positive integers where  $\bar{t}_{n,k} + k - 1 \leq m < \bar{t}_{n,k-1} + k - 2$ . Then  $\text{Min}(\alpha_0; m, n) = k$ . □

The following example illustrates the independence number of some connected graphs in  $\mathcal{CG}(m, 6)$ .

**Example 4.16.** According to Corollaries 4.13 and 4.15, let  $n = 6$ . Then

$$\text{Max}(\alpha_0; m, 6) = \begin{cases} 4 & \text{if } 5 < m \leq 9, \\ 3 & \text{if } 9 < m \leq 12, \\ 2 & \text{if } 12 < m \leq 14. \end{cases}$$

$$\text{Min}(\alpha_0; m, 6) = \begin{cases} 3 & \text{if } 5 \leq m < 7, \\ 2 & \text{if } 7 \leq m < 15. \end{cases}$$

Firstly, consider  $m = 6$ . We have  $\text{Max}(\alpha_0; 6, 6) = 4$  and  $\text{Min}(\alpha_0; 6, 6) = 3$ . In Figure 4.13, we construct the graphs  $G_1 \in \mathcal{CG}(6, 6)$  with  $\alpha_0(G_1) = 3$  by adding 3 edges to the graph  $\overline{T}_{6,3}$  and  $G_2 \in \mathcal{CG}(6, 6)$  with  $\alpha_0(G_2) = 4$  by deleting 3 edges from  $G_{6,4}$ . The independent set of  $G_1$  and  $G_2$  are indicated by their black vertices as shown in the figure.

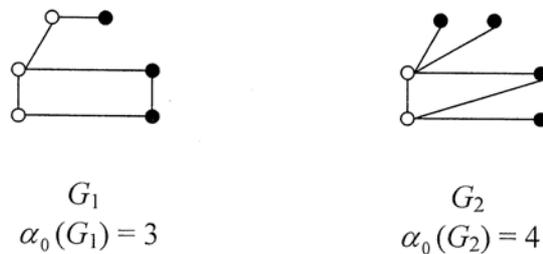


Figure 4.13: The independence number of connected graphs in  $\mathcal{CG}(6, 6)$

Secondly, consider  $m = 9$ . We have  $\text{Max}(\alpha_0; 9, 6) = 4$  and  $\text{Min}(\alpha_0; 9, 6) = 2$ . In Figure 4.14, we construct the graphs  $G_3 \in \mathcal{CG}(9, 6)$  with  $\alpha_0(G_3) = 2$  by adding 3 edges to the graph  $\overline{T}_{6,2}$  and  $G_5 \in \mathcal{CG}(9, 6)$  with  $\alpha_0(G_5) = 4$  as the graph  $G_{6,4}$ .

Since  $\alpha_0$  is an interpolation graph parameter over  $\mathcal{CG}(m, n)$ , there exists a graph  $G_4 \in \mathcal{CG}(9, 6)$  with  $\alpha_0(G_4) = 3$ . The independent set of  $G_3, G_4$ , and  $G_5$  are indicated by their black vertices as shown in the figure.

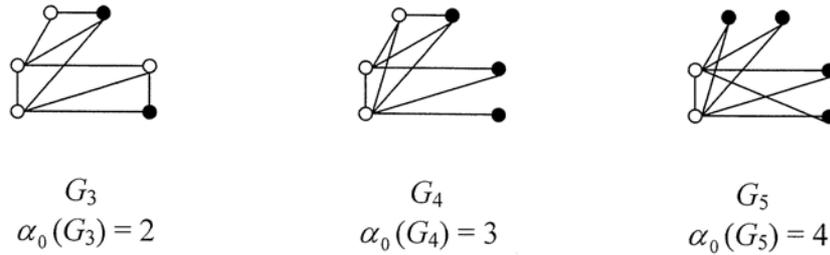


Figure 4.14: The independence number of connected graphs in  $\mathcal{CG}(9, 6)$

Thirdly, consider  $m = 11$ . We have  $\text{Max}(\alpha_0; 11, 6) = 3$  and  $\text{Min}(\alpha_0; 11, 6) = 2$ . In Figure 4.15, we construct the graphs  $G_6 \in \mathcal{CG}(11, 6)$  with  $\alpha_0(G_6) = 2$  by adding 5 edges to the graph  $\bar{T}_{6,2}$  and  $G_7 \in \mathcal{CG}(11, 6)$  with  $\alpha_0(G_7) = 3$  by deleting one edge from  $G_{6,3}$ . The independent set of  $G_6$  and  $G_7$  are indicated by their black vertices as shown in the figure.

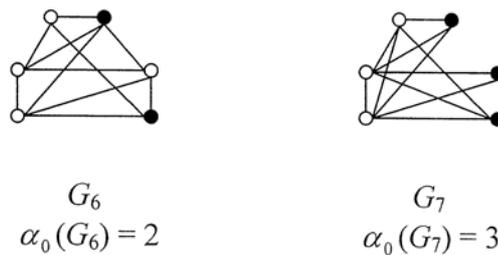
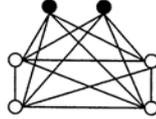


Figure 4.15: The independence number of connected graphs in  $\mathcal{CG}(11, 6)$

Finally, consider  $m = 14$ . We have  $\text{Max}(\alpha_0; 14, 6) = \text{Min}(\alpha_0; 14, 6) = 2$ . In Figure 4.16, we construct the graph  $G_8 \in \mathcal{CG}(14, 6)$  with  $\alpha_0(G_8) = 2$  as the graph

$G_{6,2}$ . The independent set of  $G_8$  is indicated by their black vertices as shown in the figure.



$$G_8$$

$$\alpha_0(G_8) = 2$$

Figure 4.16: The independence number of a connected graph in  $\mathcal{CG}(14, 6)$

### 4.3 The Vertex Covering Number

#### 4.3.1 The Vertex Covering Number in $\mathcal{G}(m, n)$

We consider the vertex covering number of graphs with prescribed order and size. That is to determine  $\max\{\beta_0(G) : G \in \mathcal{G}(m, n)\}$  and  $\min\{\beta_0(G) : G \in \mathcal{G}(m, n)\}$ , simply written  $\max(\beta_0; m, n)$  and  $\min(\beta_0; m, n)$ , respectively. It is easy to see that  $\max(\beta_0; 1, 2) = \min(\beta_0; 1, 2) = 1$  and for the complete graph  $K_n$ ,  $\beta_0(K_n) = n - 1$ . Theorem 3.6 gives the fact that for any graph of order  $n$ ,  $\alpha_0(G) + \beta_0(G) = n$ . Therefore, we shall determine  $\max(\beta_0; m, n)$  and  $\min(\beta_0; m, n)$  when  $n \geq 3$ .

**Corollary 4.17.** Let  $n, m$ , and  $k$  be positive integers where  $n \geq 3$ ,  $2 \leq k \leq n - 1$ , and  $\bar{t}_{n,k} \leq m < \bar{t}_{n,k-1}$ . Then  $\max(\beta_0; m, n) = n - k$ .

*Proof.* By Theorem 3.6, we have  $\max(\beta_0; m, n) = n - \min(\alpha_0; m, n)$ . Since  $\min(\alpha_0; m, n) = k$  for  $\bar{t}_{n,k} \leq m < \bar{t}_{n,k-1}$ , (Corollary 4.11), hence  $\max(\beta_0; m, n) = n - k$ .

□

By Theorem 3.6 and Corollary 4.8, we shall determine  $\min(\beta_0; m, n)$ .

**Corollary 4.18.** Let  $n, m$ , and  $k$  be positive integers where  $n \geq 3$ ,  $2 \leq k \leq n - 1$  and  $\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}$ . Then  $\min(\beta_0; m, n) = n - k$ .

*Proof.* By Theorem 3.6, we have  $\min(\beta_0; m, n) = n - \max(\alpha_0; m, n)$ . Since  $\max(\alpha_0; m, n) = k$  for  $\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}$ , (Theorem 4.8), hence  $\min(\beta_0; m, n) = n - k$ .

□

The following example illustrates the vertex covering number of some graphs in  $\mathcal{G}(m, 6)$ .

**Example 4.19.** According to Corollaries 4.17 and 4.18, let  $n = 6$ . Then

$$\max(\beta_0; m, 6) = \begin{cases} 1 & \text{if } 1 \leq m < 2, \\ 2 & \text{if } 2 \leq m < 3, \\ 3 & \text{if } 3 \leq m < 6, \\ 4 & \text{if } 6 \leq m < 15. \end{cases}$$

$$\min(\beta_0; m, 6) = \begin{cases} 1 & \text{if } 0 < m \leq 5, \\ 2 & \text{if } 5 < m \leq 9, \\ 3 & \text{if } 9 < m \leq 12, \\ 4 & \text{if } 12 < m \leq 14. \end{cases}$$

Firstly, consider  $m = 5$ . We have  $\max(\beta_0; 5, 6) = 3$  and  $\min(\beta_0; 5, 6) = 1$ . As shown in Figure 4.9, the graphs  $G_1, G_2$ , and  $G_3$  has  $\beta_0(G_1) = 3$ ,  $\beta_0(G_2) = 2$ , and  $\beta_0(G_3) = 1$ . Since  $\beta_0$  is an interpolation graph parameter over  $\mathcal{G}(m, n)$ , there exists a graph  $G_2 \in \mathcal{G}(5, 6)$  with  $\beta_0(G_2) = 2$ . The vertex cover of  $G_1, G_2$ , and  $G_3$  are indicated by their white vertices.

Secondly, consider  $m = 9$ . We have  $\max(\beta_0; 9, 6) = 4$  and  $\min(\beta_0; 9, 6) = 2$ . As shown in Figure 4.10, the graphs  $G_4, G_5$ , and  $G_6$  has  $\beta_0(G_4) = 4$ ,  $\beta_0(G_5) = 3$ ,

and  $\beta_0(G_6) = 2$ . Since  $\beta_0$  is an interpolation graph parameter over  $\mathcal{G}(m, n)$ , there exists a graph  $G_5 \in \mathcal{G}(9, 6)$  with  $\beta_0(G_5) = 3$ . The vertex cover of  $G_4, G_5$ , and  $G_6$  are indicated by their white vertices.

Thirdly, consider  $m = 10$ . We have  $\max(\beta_0; 10, 6) = 4$  and  $\min(\beta_0; 10, 6) = 3$ . As shown in Figure 4.11, the graphs  $G_7$  and  $G_8$  has  $\beta_0(G_7) = 4$  and  $\beta_0(G_8) = 3$ . The vertex cover of  $G_7$  and  $G_8$  are indicated by their white vertices.

Finally, consider  $m = 13$ . We have  $\max(\beta_0; 13, 6) = \min(\beta_0; 13, 6) = 4$ . As shown in Figure 4.12, the graph  $G_9$  has  $\beta_0(G_9) = 4$ . The vertex cover of  $G_9$  is indicated by their white vertices.

### 4.3.2 The Vertex Covering Number in $\mathcal{CG}(m, n)$

In this section we determine the vertex covering number of connected graphs with prescribed order and size. That is to determine  $\max\{\beta_0(G) : G \in \mathcal{CG}(m, n)\}$  and  $\min\{\beta_0(G) : G \in \mathcal{CG}(m, n)\}$ , simply written  $\text{Max}(\beta_0; m, n)$  and  $\text{Min}(\beta_0; m, n)$ , respectively. We can see that  $\text{Min}(\beta_0; n-1, n) = 1$ . Since the vertex covering number of the star of order  $n$  is 1 and for the complete graph  $K_n$ ,  $\beta_0(K_n) = n-1$ . Next, we determine formula for  $\text{Max}(\beta_0; m, n)$  and  $\text{Min}(\beta_0; m, n)$  as follows.

**Corollary 4.20.** Let  $n, m$ , and  $k \geq 2$  be positive integers where  $2 \leq k \leq \frac{n}{2}$  and  $\bar{t}_{n,k} + k - 1 \leq m < \bar{t}_{n,k-1} + k - 2$ . Then  $\text{Max}(\beta_0; m, n) = n - k$ .

*Proof.* By Theorem 3.6, we have  $\text{Max}(\beta_0; m, n) = n - \text{Min}(\alpha_0; m, n)$ . Since  $\text{Min}(\alpha_0; m, n) = k$  for  $\bar{t}_{n,k} + k - 1 \leq m < \bar{t}_{n,k-1} + k - 2$ , (Corollary 4.15), hence  $\text{Max}(\beta_0; m, n) = n - k$ .

□

By Theorem 3.6 and Corollary 4.13, we shall determine  $\text{Min}(\beta_0; m, n)$ .

**Corollary 4.21.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq n - 2$  and  $\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}$ . Then  $\text{Min}(\beta_0; m, n) = n - k$ .

*Proof.* By Theorem 3.6, we have  $\text{Min}(\beta_0; m, n) = n - \text{Max}(\alpha_0; m, n)$ . Since  $\text{Max}(\alpha_0; m, n) = k$  for  $\binom{n}{2} - \binom{k+1}{2} < m \leq \binom{n}{2} - \binom{k}{2}$ , (Corollary 4.13), hence  $\text{Min}(\beta_0; m, n) = n - k$ . □

The following example illustrates the vertex covering number of some connected graphs in  $\mathcal{CG}(m, 6)$ .

**Example 4.22.** According to Corollaries 4.20 and 4.21, let  $n = 6$ . Then

$$\text{Max}(\beta_0; m, 6) = \begin{cases} 3 & \text{if } 5 \leq m < 7, \\ 4 & \text{if } 7 \leq m < 15. \end{cases}$$

$$\text{Min}(\beta_0; m, 6) = \begin{cases} 2 & \text{if } 5 < m \leq 9, \\ 3 & \text{if } 9 < m \leq 12, \\ 4 & \text{if } 12 < m \leq 14. \end{cases}$$

Firstly, consider  $m = 6$ . We have  $\text{Max}(\beta_0; 6, 6) = 3$  and  $\text{Min}(\beta_0; 6, 6) = 2$ . As shown in Figure 4.13, the graphs  $G_1$  and  $G_2$  has  $\beta_0(G_1) = 3$  and  $\beta_0(G_2) = 2$ . The vertex cover of  $G_1$  and  $G_2$  are indicated by their white vertices.

Secondly, consider  $m = 9$ . We have  $\text{Max}(\beta_0; 9, 6) = 4$  and  $\text{Min}(\beta_0; 9, 6) = 2$ . As shown in Figure 4.14, the graphs  $G_3, G_4$  and  $G_5$  has  $\beta_0(G_3) = 4$ ,  $\beta_0(G_4) = 3$ , and  $\beta_0(G_5) = 2$ . Since  $\beta_0$  is an interpolation graph parameter over  $\mathcal{CG}(m, n)$ , there exists a graph  $G_4 \in \mathcal{CG}(9, 6)$  with  $\beta_0(G_4) = 3$ . The vertex cover of  $G_3, G_4$ , and  $G_5$  are indicated by their white vertices.

Thirdly, consider  $m = 11$ . We have  $\text{Max}(\beta_0; 11, 6) = 4$  and  $\text{Min}(\beta_0; 11, 6) = 3$ . As shown in Figure 4.15, the graphs  $G_6$  and  $G_7$  has  $\beta_0(G_6) = 4$  and  $\beta_0(G_7) = 3$ . The vertex cover of  $G_6$  and  $G_7$  are indicated by their white vertices.

Finally, consider  $m = 14$ . We have  $\text{Max}(\beta_0; 14, 6) = \text{Min}(\beta_0; 14, 6) = 4$ . As shown in Figure 4.16, the graph  $G_8$  has  $\beta_0(G_8) = 4$ . The vertex cover of  $G_8$  is indicated by their white vertices.

## 4.4 The Chromatic Number

### 4.4.1 The Chromatic Number in $\mathcal{G}(m, n)$

We consider the chromatic number of graphs with prescribed order and size. That is to determine  $\max\{\chi(G) : G \in \mathcal{G}(m, n)\}$  and  $\min\{\chi(G) : G \in \mathcal{G}(m, n)\}$ , simply written  $\max(\chi; m, n)$  and  $\min(\chi; m, n)$ , respectively. An obvious bound of  $\chi(G)$  is  $\chi(G) \geq \omega(G)$ ; equality holds for  $K_n$ . Since each color class is an independent set,  $\chi(G) \geq \frac{n}{\alpha_0(G)}$ . Brooks [2] observed that every graph  $G$  which is not an odd cycle may be colored by  $\Delta(G) + 1$  colors. Erdős and Gallai [9] showed that any regular graph on  $n$  vertices has the chromatic number  $k \leq \frac{3n}{5}$  unless the graph is complete. It is easy to see that  $\max(\chi; 0, n) = \min(\chi; 0, n) = 1$ ,  $\max(\chi; 1, 2) = \min(\chi; 1, 2) = 2$ , and for the complete graph  $K_n$ ,  $\chi(K_n) = n$ . We now determine  $\max(\chi; m, n)$  and  $\min(\chi; m, n)$ . To achieve the value of  $\max(\chi; m, n)$ , we will use the technique of finding a complete subgraph of  $G \in \mathcal{G}(m, n)$  whose vertex set is as large as possible.

**Theorem 4.23.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq n - 1$  and  $\binom{k}{2} \leq m < \binom{k+1}{2}$ . Then  $\max(\chi; m, n) = k$ .

*Proof.* Observe that every  $k$ -chromatic graph has at least  $\binom{k}{2}$  edges. Thus for  $n, m$ , and  $k$  satisfying conditions of the Theorem,  $\max(\chi; m, n) \leq k$ . Let  $\binom{k}{2} \leq m < \binom{k+1}{2}$ . We now construct a graph  $H \in \mathcal{G}(m, n)$  such that  $\chi(H) = k$ . Since

$m < \binom{k+1}{2}$ , we have  $m - \binom{k}{2} < \binom{k+1}{2} - \binom{k}{2} = k$ . So  $H$  can be obtained from the graph  $K_k$  with  $V(K_k) = \{v_1, v_2, \dots, v_k\}$  by adding  $n - k$  vertices and joining each  $v_i$  for  $1 \leq i \leq m - \binom{k}{2}$  to one of the added vertices. It is clear that  $H$  has order  $n$  and size  $m$  with  $\chi(H) = k$ . Thus  $\max(\chi; m, n) = k$ .

□

**Theorem 4.24.** Let  $n, m$ , and  $k \geq 2$  be positive integers where  $t_{n,k-1} < m \leq t_{n,k}$ . Then  $\min(\chi; m, n) = k$ .

*Proof.* The proof follows from Theorem 4.3 and the fact that  $\chi(G) \geq \omega(G)$ . We have  $\min(\chi; m, n) \geq k$  for  $t_{n,k-1} < m \leq t_{n,k}$ . We can obtain a graph  $H \in \mathcal{G}(m, n)$  where  $\chi(H) = k$  by deleting some appropriate  $t_{n,k} - m$  edges from  $T_{n,k}$  which each partite set is a color class. Thus  $\min(\chi; m, n) = k$ .

□

The following example illustrates the chromatic number of some graphs in  $\mathcal{G}(m, 4)$ .

**Example 4.25.** According to Theorems 4.23 and 4.24, let  $n = 4$ . Then

$$\max(\chi; m, 4) = \begin{cases} 2 & \text{if } 1 \leq m < 3, \\ 3 & \text{if } 3 \leq m < 6, \\ 4 & \text{if } m = 6. \end{cases}$$

$$\min(\chi; m, 4) = \begin{cases} 2 & \text{if } 0 < m \leq 4, \\ 3 & \text{if } 4 < m \leq 5, \\ 4 & \text{if } 5 < m \leq 6. \end{cases}$$

Firstly, consider  $m = 2$ . We have  $\max(\chi; 2, 4) = \min(\chi; 2, 4) = 2$ . In Figure 4.17, we construct the graph  $G_1 \in \mathcal{G}(2, 4)$  with  $\chi(G_1) = 2$  from the complete graph  $K_2$  by adding 2 vertices and joining one vertex of  $K_2$  to one of the added vertices.

The color classes of  $G_1$  are indicated by black and white vertices as shown in the figure.

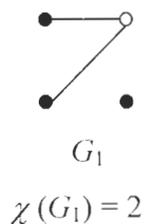


Figure 4.17: The chromatic number of a graph in  $\mathcal{G}(2, 4)$

Secondly, consider  $m = 3$ . We have  $\max(\chi; 3, 4) = 3$  and  $\min(\chi; 3, 4) = 2$ . In Figure 4.18, we construct the graphs  $G_2 \in \mathcal{G}(3, 4)$  with  $\chi(G_2) = 2$  by deleting one edge from  $T_{4,2}$  and  $G_3 \in \mathcal{G}(3, 4)$  with  $\chi(G_3) = 3$  by adding one vertex to  $K_3$ . The color classes of  $G_2$  are indicated by black and white vertices. The color classes of  $G_3$  are indicated by black, white, and red vertices as shown in the figure.

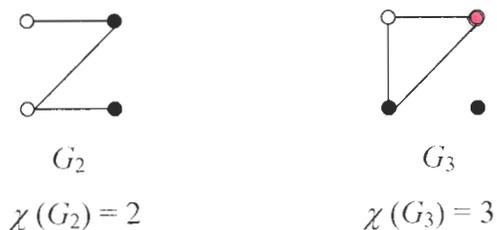
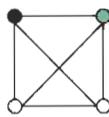


Figure 4.18: The chromatic number of graphs in  $\mathcal{G}(3, 4)$

Finally, consider  $m = 6$ . We have  $\max(\chi; 6, 4) = \min(\chi; 6, 4) = 4$ . In Figure 4.19, we construct the graph  $G_4 \in \mathcal{G}(6, 4)$  with  $\chi(G_4) = 4$  as the complete graph  $K_4$ . The color class of  $G_4$  is indicated by black, white, green, and red vertices as shown in the figure.

 $G_4$ 

$$\chi(G_4) = 4$$

Figure 4.19: The chromatic number of a graph in  $\mathcal{G}(6, 4)$ 

#### 4.4.2 The Chromatic Number in $\mathcal{CG}(m, n)$

In this section we determine the chromatic number of connected graphs with prescribed order and size. That is to determine  $\max\{\chi(G) : G \in \mathcal{CG}(m, n)\}$  and  $\min\{\chi(G) : G \in \mathcal{CG}(m, n)\}$ , simply written  $\text{Max}(\chi; m, n)$  and  $\text{Min}(\chi; m, n)$ , respectively. Caccetta and Pullman [3], showed that if  $k > 1$ , then for every  $n \geq \frac{5k}{3}$ , there exists a connected, regular,  $k$ -chromatic graph on  $n$  vertices. Note that some trees have the chromatic number equal 2, it follows that  $\text{Min}(\chi; m, n) = 2$  if and only if  $n - 1 \leq m \leq t_{n,2}$  and for the complete graph  $K_n$ ,  $\chi(K_n) = n$ .

If the connectivity is concerned, one can ask for the minimum size among connected  $k$ -chromatic graph with  $n$  vertices. This question was answered in [38] that the minimum size of such a graph is at least  $\binom{k}{2} + n - k$ . Further,  $\text{Max}(\chi; m, n)$  can be obtained in the similar way of finding  $\text{Max}(\omega; m, n)$ .

**Theorem 4.26.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq n - 1$  and  $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - (k + 1)$ . Then  $\text{Max}(\chi; m, n) = k$ .

*Proof.* By above observation, if  $m < \binom{k+1}{2} + n - (k + 1)$ , then  $\text{Max}(\chi; m, n) \leq k$ . We now construct a graph  $H \in \mathcal{CG}(m, n)$  such that  $\chi(H) = k$ . Let  $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - (k + 1)$ . Since  $m < \binom{k+1}{2} + n - (k + 1)$ , we have  $m - \binom{k}{2} - n + k < k - 1$ . That is  $m - \binom{k}{2} - n + k + 1 < k$ . Then we can obtain the graph  $H$  by adding  $m - \binom{k}{2} - n + k + 1$  edges to join some vertices of  $K_k$  to some vertices of  $P_{n-k}$ . We can see that the order of a largest clique in  $H$  is  $k$ . It is clear that  $H$  is a connected

graph of order  $n$  and size  $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - (k+1)$  with  $\chi(H) = k$ . Thus  $\text{Max}(\chi; m, n) = k$ .

□

**Theorem 4.27.** Let  $n, m$ , and  $k \geq 3$  be positive integers where  $t_{n,k-1} < m \leq t_{n,k}$ . Then  $\text{Min}(\chi; m, n) = k$ .

*Proof.* The proof follows from Theorem 4.6 and the fact that  $\chi(G) \geq \omega(G)$ . We have  $\text{Min}(\chi; m, n) \geq k$  for  $t_{n,k-1} < m \leq t_{n,k}$ . We can obtain a connected graph  $H \in \mathcal{G}(m, n)$  where  $\chi(H) = k$  by deleting some appropriate  $t_{n,k} - m$  edges from  $T_{n,k}$  which each partite set is a color class. Thus  $\text{Min}(\chi; m, n) = k$ .

□

The following example illustrates the chromatic number of some connected graphs in  $\mathcal{CG}(m, 5)$ .

**Example 4.28.** According to Theorems 4.26 and 4.27, let  $n = 5$ . Then

$$\text{Max}(\chi; m, 5) = \begin{cases} 2 & \text{if } 4 \leq m < 5, \\ 3 & \text{if } 5 \leq m < 7, \\ 4 & \text{if } 7 \leq m < 10, \\ 5 & \text{if } m = 10. \end{cases}$$

$$\text{Min}(\chi; m, 5) = \begin{cases} 3 & \text{if } 6 < m \leq 8, \\ 4 & \text{if } 8 < m \leq 9, \\ 5 & \text{if } 9 < m \leq 10. \end{cases}$$

Firstly, consider  $m = 7$ . We have  $\text{Max}(\chi; 7, 5) = 4$  and  $\text{Min}(\chi; 7, 5) = 3$ . In Figure 4.20, we construct the graphs  $G_1 \in \mathcal{CG}(7, 5)$  with  $\chi(G_1) = 3$  by deleting one edge from  $T_{5,3}$  and  $G_2 \in \mathcal{CG}(7, 5)$  with  $\chi(G_2) = 4$  by adding one edge to  $K_4 \cup K_1$ . The color classes of  $G_1$  is indicated by black, white, and red vertices. The color

classes of  $G_2$  is indicated by black, white, red, and blue vertices as shown in the figure.

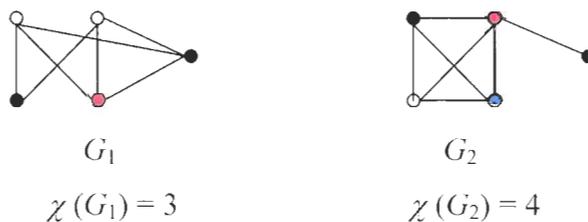


Figure 4.20: The chromatic number of connected graphs in  $\mathcal{CG}(7, 5)$

Secondly, consider  $m = 9$ . We have  $\text{Max}(\chi; 9, 5) = \text{Min}(\chi; 9, 5) = 4$ . In Figure 4.21, we construct the graph  $G_3 \in \mathcal{CG}(9, 5)$  with  $\chi(G_3) = 4$  as the graph  $T_{5,4}$ . The color classes of  $G_3$  is indicated by black, white, red, and blue vertices as shown in the figure.

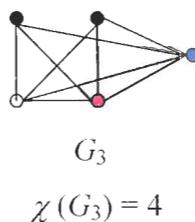


Figure 4.21: The chromatic number of a connected graph in  $\mathcal{CG}(9, 5)$

Finally, consider  $m = 10$ . We have  $\text{Max}(\chi; 10, 5) = \text{Min}(\chi; 10, 5) = 5$ . In Figure 4.22, we construct the graph  $G_4 \in \mathcal{CG}(10, 5)$  with  $\chi(G_4) = 5$  as the complete graph  $K_5$ . The color classes of  $G_3$  is indicated by black, white, red, green and blue vertices as shown in the figure.

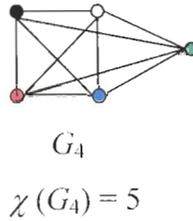


Figure 4.22: The chromatic number of a connected graph in  $\mathcal{CG}(10, 5)$

## 4.5 The Matching Number

### 4.5.1 The Matching Number in $\mathcal{G}(m, n)$

We consider the matching number of graphs with prescribed order and size. That is to determine  $\max\{\alpha_1(G) : G \in \mathcal{G}(m, n)\}$  and  $\min\{\alpha_1(G) : G \in \mathcal{G}(m, n)\}$ , simply written  $\max(\alpha_1; m, n)$  and  $\min(\alpha_1; m, n)$ , respectively. König-Egerváry's Theorem states that  $\alpha_1(G) = \beta_0(G)$  for every bipartite graphs  $G$  with  $\delta(G) \geq 1$ . Note that  $\max(\alpha_1; 0, n) = \min(\alpha_1; 0, n) = 0$  and for  $n = 2, 3$  and  $m \geq 1$ ,  $\max(\alpha_1; m, n) = \min(\alpha_1; m, n) = 1$ . Therefore, we shall determine  $\max(\alpha_1; m, n)$  and  $\min(\alpha_1; m, n)$  when  $n \geq 4$  and  $m \geq 1$ .

**Theorem 4.29.** For  $n \geq 4$ , then

1.  $\max(\alpha_1; m, n) = m$  if  $1 \leq m < \lfloor \frac{n}{2} \rfloor$ ,
2.  $\max(\alpha_1; m, n) = \lfloor \frac{n}{2} \rfloor$  if  $\lfloor \frac{n}{2} \rfloor \leq m \leq \binom{n}{2}$ .

*Proof.* 1. It is clear that  $\max(\alpha_1; m, n) \leq m$ . We can construct a graph  $G \in \mathcal{G}(m, n)$  with  $\alpha_1(G) = m$  by adding  $n - 2m$  vertices to  $mK_2$ .

2. It is clear that  $\max(\alpha_1; m, n) \leq \lfloor \frac{n}{2} \rfloor$ . We can construct a graph  $H \in \mathcal{G}(m, n)$  with  $\alpha_1(H) = \lfloor \frac{n}{2} \rfloor$  by adding  $m - \lfloor \frac{n}{2} \rfloor$  edges to  $\lfloor \frac{n}{2} \rfloor K_2$  if  $n$  is even. If  $n$  is odd, a graph  $H$  can be obtained by adding  $m - \lfloor \frac{n}{2} \rfloor$  edges to  $\lfloor \frac{n}{2} \rfloor K_2 \cup K_1$ .

□

We now introduce some notation concerning the bound of  $\alpha_1$ . For a graph  $F$  of order  $k$  and integer  $n$  which  $n \geq k$ , the extremal number  $ex(n; F)$  of  $F$  is a maximum number of edges in a graph of order  $n$  that does not contains  $F$  as a subgraph. Notice that if  $G$  is a graph of order  $n \geq k$  and size  $ex(n; F) + 1$ , then  $G$  contains  $F$  as a subgraph. We define  $EX(n; F)$  analogous to  $ex(n; F)$  if that graph  $G$  is connected. To determine  $\min(\alpha_1; m, n)$ , we need the following theorem.

**Theorem 4.30.** [21] For positive integers  $n$  and  $k$  with  $n \geq 2k$ ,

$$ex(n; kK_2) = \max \left\{ (k-1)n - \binom{k}{2}, \binom{2k-1}{2} \right\}.$$

□

An equivalent statement of Theorem 4.30 is that for a positive integer  $k$ , every graph of order  $n \geq 2k$  and size at least

$$ex(n; kK_2) + 1 = \max \left\{ 1 + (k-1)n - \binom{k}{2}, 1 + \binom{2k-1}{2} \right\}$$

contains a matching of size  $k$ . We determine  $\min(\alpha_1; m, n)$  in the following corollary.

**Corollary 4.31.** For  $n \geq 4$ , then

1.  $\min(\alpha_1; m, n) = 1$  if  $1 \leq m \leq n - 1$ ,
2.  $\min(\alpha_1; m, n) = k$  if  $ex(n; kK_2) < m \leq ex(n; (k+1)K_2)$  for  $4 \leq 2k \leq n$ .

*Proof.* 1. It is clear that  $\min(\alpha_1; m, n) \geq 1$  for  $m \geq 1$ . We can construct a graph  $G \in \mathcal{G}(m, n)$  with  $\alpha_1(G) = 1$  by adding  $n - 1 - m$  vertices to  $K_{1,m}$ .

2. Let  $G \in \mathcal{G}(m, n)$  where  $4 \leq 2k \leq n$ . By Theorem 4.30, observe that  $ex(n; kK_2) < m \leq ex(n; (k+1)K_2)$  if and only if  $\alpha_1(G) = k$ . Therefore,  $\min(\alpha_1; m, n) = k$ .

□

Next, we consider the matching number of the graph  $G_{n,n-k}$  in order to show in our example as follows:

Let  $n$  and  $k$  be positive integers, notice that  $kn - \binom{k+1}{2} \geq \binom{2k+1}{2}$  if and only if  $k \leq \frac{2n-3}{5}$ . Let  $G \in \mathcal{G}(m, n)$  such that  $\alpha_1(G) = k$  and  $n \geq 2k$ . By Theorem 4.30, we have  $m \leq kn - \binom{k+1}{2}$  if  $2 \leq k \leq \lfloor \frac{2n-3}{5} \rfloor$ , and  $m \leq \binom{2k+1}{2}$ , if  $\lfloor \frac{2(n+1)}{5} \rfloor \leq k \leq \lfloor \frac{n}{2} \rfloor$ . Recall the graph  $G_{n,p}$  such that  $p = n - k$  for  $2 \leq k \leq \lfloor \frac{2n-3}{5} \rfloor$ . Observe that  $|E(G_{n,n-k})| = \binom{n}{2} - \binom{n-k}{2} = kn - \binom{k+1}{2}$ . Let  $S$  be a maximum independent set of  $G_{n,n-k}$  with  $|S| = n - k$  and  $M$  be a maximum matching of  $G_{n,n-k}$ . Since  $n \geq 2k$ ,  $k < n - k$ . Thus  $M$  consists of edges which join  $k$  vertices of  $K_k$  to  $k$  vertices of  $S$ . That is  $\alpha_1(G_{n,n-k}) = k$  for  $2 \leq k \leq \lfloor \frac{2n-3}{5} \rfloor$ .

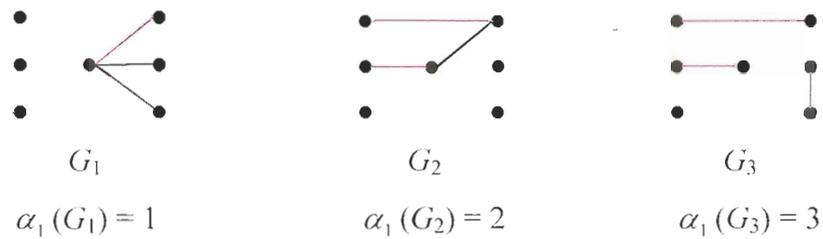
The following example illustrates the matching number of some graphs in  $\mathcal{G}(m, 7)$ .

**Example 4.32.** According to Theorem 4.29 and Corollary 4.31, let  $n = 7$ . Then

$$\max(\alpha_1; m, 7) = \begin{cases} m & \text{if } m < 3, \\ 3 & \text{if } m \geq 3. \end{cases}$$

$$\min(\alpha_1; m, 7) = \begin{cases} 1 & \text{if } 1 \leq m \leq 6, \\ 2 & \text{if } 6 < m \leq 11, \\ 3 & \text{if } 11 < m \leq 21. \end{cases}$$

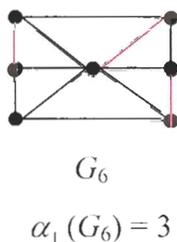
Firstly, consider  $m = 3$ . Then  $\max(\alpha_1; 3, 7) = 3$  and  $\min(\alpha_1; 3, 7) = 1$ . In Figure 4.23, we construct the graphs  $G_1 \in \mathcal{G}(3, 7)$  with  $\alpha_1(G_1) = 1$  as the graph  $K_{1,3} \cup \overline{K_3}$  and  $G_3 \in \mathcal{G}(6, 7)$  with  $\alpha_1(G_3) = 3$  as the graph  $3K_2 \cup K_1$ . Since  $\alpha_1$  is an interpolation graph parameter over  $\mathcal{G}(m, n)$ , there exists the graph  $G_2 \in \mathcal{G}(3, 7)$  with  $\alpha_1(G_2) = 2$ . The independent edges of  $G_1, G_2$ , and  $G_3$  are indicated by their red edges as shown in the figure.

Figure 4.23: The matching number of graphs in  $\mathcal{G}(3, 7)$ 

Secondly, consider  $m = 8$ . Then  $\max(\alpha_1; 8, 7) = 3$  and  $\min(\alpha_1; 8, 7) = 2$ . In Figure 4.24, we construct the graphs  $G_4 \in \mathcal{G}(8, 7)$  with  $\alpha_1(G_4) = 2$  by deleting 3 edges from one vertex of degree 6 in  $G_{7,5}$  and  $G_5 \in \mathcal{G}(8, 7)$  with  $\alpha_1(G_5) = 3$  by adding 5 edges to the graph  $3K_2 \cup K_1$ . The independent edges of  $G_4$  and  $G_5$  are indicated by their red edges as shown in the figure.

Figure 4.24: The matching number of graphs in  $\mathcal{G}(8, 7)$ 

Finally, consider  $m = 12$ . Then  $\max(\alpha_1; 12, 7) = \min(\alpha_1; 12, 7) = 3$ . In Figure 4.25, we construct the graph  $G_6 \in \mathcal{G}(12, 7)$  with  $\alpha_1(G_6) = 3$  by adding 9 edges to the graph  $3K_2 \cup K_1$ . The independent edges of  $G_6$  is indicated by its red edges as shown in the figure.

Figure 4.25: The matching number of a graph in  $\mathcal{G}(12, 7)$ 

### 4.5.2 The Matching Number in $\mathcal{CG}(m, n)$

In this section we determine the matching number of connected graphs with prescribed order and size. That is to determine  $\max\{\alpha_1(G) : G \in \mathcal{CG}(m, n)\}$  and  $\min\{\alpha_1(G) : G \in \mathcal{CG}(m, n)\}$ , simply written  $\text{Max}(\alpha_1; m, n)$  and  $\text{Min}(\alpha_1; m, n)$ , respectively. Note that the matching number of a graph of order  $n$  cannot exceed  $\lfloor \frac{n}{2} \rfloor$  and  $\alpha_1(P_n) = \lfloor \frac{n}{2} \rfloor$ ,  $\text{Max}(\alpha_1; n-1, n) = \lfloor \frac{n}{2} \rfloor$ . Since the matching number of the star of order  $n$  is 1,  $\text{Min}(\alpha_1; n-1, n) = 1$ . For the complete graph  $K_n$ ,  $\alpha_1(K_n) = \lfloor \frac{n}{2} \rfloor$ . Note that for  $n = 3$  and  $m \geq 2$ ,  $\text{Max}(\alpha_1; m, 3) = \text{Min}(\alpha_1; m, 3) = 1$ . Therefore, we shall determine  $\text{Max}(\alpha_1; m, n)$  and  $\text{Min}(\alpha_1; m, n)$  when  $n \geq 4$  and  $m \geq n$ .

**Theorem 4.33.** For positive integers  $n \geq 4$  and  $n \leq m \leq \binom{n}{2}$ ,

$$\text{Max}(\alpha_1; m, n) = \lfloor \frac{n}{2} \rfloor.$$

*Proof.* It is clear that for  $m = n$ , we have  $\alpha_1(C_n) = \lfloor \frac{n}{2} \rfloor$  and for  $m = \binom{n}{2}$ ,  $\alpha_1(K_n) = \lfloor \frac{n}{2} \rfloor$ . By the fact that for any connected graph  $G$ ,  $\alpha_1(G) \leq \lfloor \frac{n}{2} \rfloor$  and  $\text{Max}(\alpha_1; k, n) \leq \text{Max}(\alpha_1; k+1, n)$  for a fixed integer  $n$  and  $n \leq k < \binom{n}{2}$ . We can conclude that  $\text{Max}(\alpha_1; m, n) = \lfloor \frac{n}{2} \rfloor$ . □

Next, we introduce an odd component of a graph as a component whose number of vertices is odd. We denote by  $O(G)$  the number of odd components of a graph  $G$ .

To determine the maximum number of edges of a connected graph with prescribed the matching number, we need the Tutte-Berge's formula as stated in the following theorem.

**Theorem 4.34.** [1] Let  $G$  be a graph of order  $n$  and  $S \subseteq V(G)$ . Then  $\alpha_1(G) = \frac{1}{2}(n - d)$ , where  $d = \max\{O(G - S) - |S|\}$ .

□

By the above theorem, we have some observations as follows. Let  $\alpha_1(G) = k$ . Obviously,  $n \geq 2k$ . Then there exists a subset  $S$  of  $V(G)$  such that

$$k = \alpha_1(G) = \frac{1}{2}(n - O(G - S) + |S|).$$

1. If  $|S| > k$ , then  $2k = n - O(G - S) + |S| > n - O(G - S) + k$ . Thus  $O(G - S) > n - k$ . Since each odd component of  $G - S$  contains at least one element,  $n \geq O(G - S) + |S| > n - k + k > n$ . This is a contradiction. Thus  $1 \leq |S| \leq k$ .

2. Let  $O(G - S) = q$  and  $n_1, n_2, \dots, n_q$  be the order of the  $q$  odd components of  $G - S$ . Thus  $n_1 + n_2 + \dots + n_q + |S| \leq n$ . In order to obtain a graph with maximum size, we may assume that  $n_1 + n_2 + \dots + n_q + |S| = n$ . Thus

$$|E(G)| \leq \binom{s}{2} + \sum_{i=1}^q \binom{n_i}{2} + (n - s)s$$

where  $\binom{n}{k} = 0$  for  $n < k$  and  $s$  is the cardinality of  $S$ .

We can investigate a simple bound for  $|E(G)|$  as shown in the following lemma.

**Lemma 4.35.** Let  $G \in \mathcal{CG}(m, n)$  and  $\alpha_1(G) = k$ . Then

$$m \leq \binom{s}{2} + \binom{2k - 2s + 1}{2} + (n - s)s.$$

*Proof.* Consider the  $q$  odd components of  $G$  each of which has the odd number of vertices  $n_1, n_2, \dots, n_q$  such that  $n_1 + n_2 + \dots + n_q + s = n$ . With out loss of

generality, let  $n_1 \geq n_2 \geq \dots \geq n_q \geq 1$ . Then we perform the following procedure.

Step 1. For  $1 \leq i \leq q$ , if there exist some  $i$ 's where  $n_i \geq 3$ , then let  $t$  be a largest integer such that  $n_t \geq 3$ . That is  $n_1 \geq n_2 \geq \dots \geq n_t > 1 = \dots = 1$ .

Step 2. If  $t \geq 2$ , then we define the cardinality of the  $q$  odd components as follows;

$$\begin{aligned} n_1 &:= n_1 + 2, \\ n_j &:= n_j \quad \text{for } 2 \leq j \leq t-1, \\ n_t &:= n_t - 2. \end{aligned}$$

We see that  $\binom{n_1+2}{2} + \binom{n_t-2}{2} > \binom{n_1}{2} + \binom{n_t}{2}$  for  $2 \leq t \leq q$ .

Step 3. Repeat Step 1, 2 whenever  $n_2 > 1$ , otherwise stop.

The procedure terminates when  $n_2 = 1$  and yields  $q$  components where exactly one component has order  $n - s - q + 1$  and each of the rest  $q - 1$  components has one vertex.

We have the maximum number of edges of a graph with the matching number  $k$  is  $\binom{s}{2} + \binom{n-s-q+1}{2} + (n-s)s$ . Since  $q = n - 2k + s$ ,  $m \leq \binom{s}{2} + \binom{2k-2s+1}{2} + (n-s)s$ .  $\square$

To determine the bound for size of a graph with matching number  $k$ , we also need to consider some characteristic of the graph  $G_{n,n-k}$

**Theorem 4.36.** For positive integers  $n$  and  $k$  with  $n \geq 2k$ . Every connected graph of order  $n$  whose matching number  $k$  has size at most

$$EX(n; (k+1)K_2) = \max \left\{ kn - \binom{k+1}{2}, \binom{2k}{2} + n - 2k \right\}.$$

*Proof.* By the result of Lemma 4.35,

if  $s = 1$ , we have that the size of graph is  $m \leq \binom{2k-1}{2} + n - 1 = \binom{2k}{2} + n - 2k$ .

If  $s = k > 1$ , then  $m \leq \binom{k}{2} + (n-k)k = kn - \binom{k+1}{2}$ .

Now observe that if  $\binom{2k}{2} + n - 2k - [kn - \binom{k+1}{2}] \geq 0$ , then  $n \leq \frac{5}{2}k$ . This means that if  $n > \frac{5}{2}k$ , then  $EX(n; (k+1)K_2) = kn - \binom{k+1}{2}$ . We claim that if  $n \leq \frac{5}{2}k$ , then  $EX(n; (k+1)K_2) = \binom{2k}{2} + n - 2k$ .

Suppose that  $s = i$  where  $1 < i < k$ . We have  $m \leq \binom{i}{2} + \binom{2k-2i+1}{2} + (n-i)i$ .

Consider the difference of size when  $s = k$  and  $s = i$ ,

$$kn - \binom{k+1}{2} - \left[ \binom{i}{2} + \binom{2k-2i+1}{2} + (n-i)i \right] = \frac{1}{2}(k-i)(2n-5k).$$

This means that if  $\frac{1}{2}(k-i)(2n-5k) \geq 0$ , then  $n \geq \frac{5}{2}k$ . In addition, if  $n = \frac{5}{2}k$ , then, of course,  $kn - \binom{k+1}{2} = \binom{2k}{2} + n - 2k$ . Thus

$$EX(n; (k+1)K_2) \leq \max \left\{ kn - \binom{k+1}{2}, \binom{2k}{2} + n - 2k \right\}.$$

We have that the graph  $G_{n,n-k}$  has order  $n$ , size  $kn - \binom{k+1}{2}$ , and  $\alpha_1(G_{n,n-k}) = k$ . The graph obtained from  $K_{2k} \cup \overline{K}_{n-2k}$  by adding  $n - 2k$  edges which join one vertex of  $K_{2k}$  to each vertex of  $\overline{K}_{n-2k}$  has order  $n$ , size  $\binom{2k}{2} + n - 2k$ , and the matching number  $k$ . Moreover, clearly both graphs are connected. Therefore,  $EX(n; (k+1)K_2) \geq \max \left\{ kn - \binom{k+1}{2}, \binom{2k}{2} + n - 2k \right\}$ , which yields the desired result. □

**Corollary 4.37.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq \frac{n}{2}$  and  $EX(n; kK_2) < m \leq EX(n; (k+1)K_2)$ . Then  $\text{Min}(\alpha_1; m, n) = k$ .

*Proof.* Let  $G \in \mathcal{CG}(m, n)$  where  $4 \leq 2k \leq n$ . By Theorem 4.36, observe that  $EX(n; kK_2) < m \leq EX(n; (k+1)K_2)$  if and only if  $\alpha_1(G) = k$ . Therefore,  $\text{Min}(\alpha_1; m, n) = k$ . □

The following example illustrates the matching number of some connected graphs in  $\mathcal{CG}(m, 8)$ .

**Example 4.38.** According to Theorem 4.33 and Corollary 4.37, let  $n = 8$ . Then

$$\text{Max}(\alpha_1; m, 8) = 4 \quad \text{if } 8 \leq m \leq 28.$$

$$\text{Min}(\alpha_1; m, 8) = \begin{cases} 2 & \text{if } 7 < m \leq 13, \\ 3 & \text{if } 13 < m \leq 18, \\ 4 & \text{if } 18 < m \leq 28. \end{cases}$$

Firstly, consider  $m = 8$ . Then  $\text{Max}(\alpha_1; 8, 8) = 4$  and  $\text{Min}(\alpha_1; 8, 8) = 2$ . In Figure 4.26, we construct the connected graph  $G_1 \in \mathcal{CG}(8, 8)$  with  $\alpha_1(G_1) = 2$  by deleting 5 edges from one vertex of degree 7 in  $G_{8,6}$ . The connected graph  $G_3 \in \mathcal{CG}(8, 8)$  with  $\alpha_1(G_3) = 4$ , can be obtained from the graph  $4K_2$  by adding 4 edges in order to have a path connecting every two vertices. Since  $\alpha_1$  is an interpolation graph parameter over  $\mathcal{CG}(m, n)$ , there exists the graph  $G_2 \in \mathcal{CG}(8, 8)$  with  $\alpha_1(G_2) = 3$ . The independent edges of  $G_1, G_2$ , and  $G_3$  are indicated by their red edges as shown in the figure.

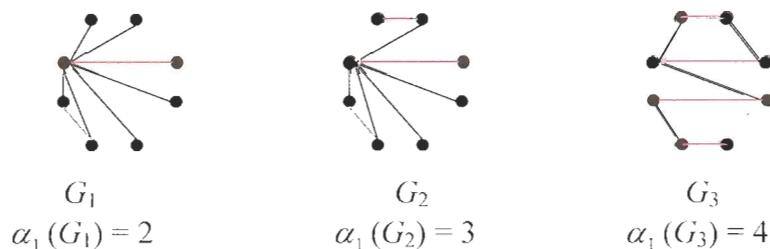


Figure 4.26: The matching number of graphs in  $\mathcal{CG}(8, 8)$

Secondly, consider  $m = 18$ . Then  $\text{Max}(\alpha_1; 18, 8) = 4$  and  $\text{Min}(\alpha_1; 18, 8) = 3$ . In Figure 4.27, we obtain the connected graph  $G_4 \in \mathcal{CG}(18, 8)$  with  $\alpha_1(G_4) = 3$  as the graph  $G_{8,5}$ . The connected graph  $G_5 \in \mathcal{CG}(18, 8)$  with  $\alpha_1(G_5) = 4$ , can be obtained from the graph  $4K_2$  by adding 14 edges in order to have a path connecting every two vertices. The independent edges of  $G_4$  and  $G_5$  are indicated by their red edges as shown in the figure.

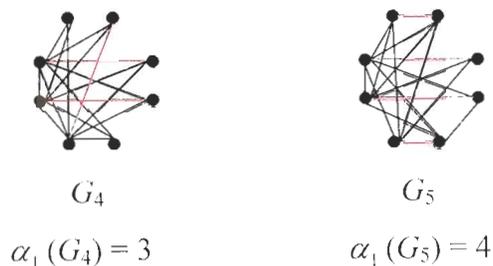


Figure 4.27: The matching number of graphs in  $\mathcal{CG}(18, 8)$

Finally, consider  $m = 19$ . Then  $\text{Max}(\alpha_1; 19, 8) = \text{Min}(\alpha_1; 19, 8) = 4$ . The connected graph  $G_6 \in \mathcal{CG}(19, 8)$  with  $\alpha_1(G_6) = 4$ , can be obtained from the graph  $4K_2$  by adding 15 edges in order to have a path connecting every two vertices. The independent edges of  $G_6$  is indicated by its red edges as shown in the figure.

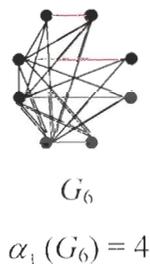


Figure 4.28: The matching number of a graph in  $\mathcal{CG}(19, 8)$

## 4.6 The Edge Covering Number

### 4.6.1 The Edge Covering Number in $\mathcal{G}(m, n)$

We consider the edge covering number of graphs with prescribed order and size. That is to determine  $\max\{\beta_1(G) : G \in \mathcal{G}(m, n)\}$  and  $\min\{\beta_1(G) : G \in \mathcal{G}(m, n)\}$ , simply written  $\max(\beta_1; m, n)$  and  $\min(\beta_1; m, n)$ , respectively. Note that for  $m \geq 2$ ,

$\max(\beta_1; m, 3) = \min(\beta_1; m, 3) = 2$  and for the complete graph  $K_n, \beta_1(K_n) = \lceil \frac{n}{2} \rceil$ . The edge covering number of graphs is well defined when  $\delta(G) \geq 1$ , that is  $m \geq \lceil \frac{n}{2} \rceil$  and each component of  $G$  contains at least 2 vertices. Theorem 3.10 gives the fact that for any graph of order  $n$  and  $\delta(G) \geq 1, \alpha_1(G) + \beta_1(G) = n$ . So we can apply Theorem 4.34 to determine the edge covering number. Therefore, we shall determine  $\max(\beta_1; m, n)$  and  $\min(\beta_1; m, n)$  when  $n \geq 4$  and  $\delta(G) \geq 1$ .

**Corollary 4.39.** For  $4 \leq 2k \leq n$ , then

1.  $\max(\beta_1; m, n) = m$  if  $\lceil \frac{n}{2} \rceil \leq m \leq n - 1$ ,
2.  $\max(\beta_1; m, n) = n - k$  if  $EX(n; kK_2) < m \leq EX(n; (k + 1)K_2)$ .

*Proof.* 1. It is clear that for  $m = n - 1$ , we have  $\beta_1(K_{1,m}) = m$ . Consider  $m = \lceil \frac{n}{2} \rceil$ . If  $n$  is even, then we have  $\beta_1(mK_2) = m$ . If  $n$  is odd, let  $H$  be the graph obtained from  $\frac{n-1}{2}K_2$  by adding a new vertex  $v$  and adding the edge  $uv$  where  $u$  is a vertex of  $\frac{n-1}{2}K_2$ . We can see that  $H \in \mathcal{G}(m, n)$  and  $\beta_1(H) = \frac{n-1}{2} + 1 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil = m$ . Since  $\max(\beta_1; k, n) \leq \max(\beta_1; k + 1, n)$  for a fixed integer  $n$  and  $\lceil \frac{n}{2} \rceil \leq k < n - 1$ ,  $\max(\beta_1; m, n) = m$  for  $\lceil \frac{n}{2} \rceil \leq m \leq n - 1$ .

2. Since the edge covering number is well defined when  $\delta(G) \geq 1$ , there exists a graph  $G$  of order  $n$  such that  $\alpha_1(G) + \beta_1(G) = n$  whose edge set has cardinality at most  $EX(n; kK_2)$ . It is clear that  $\max(\beta_1; m, n) = n - \text{Min}(\alpha_1; m, n)$ . That is if  $EX(n; kK_2) < m \leq EX(n; (k + 1)K_2)$ , then  $\max(\beta_1; m, n) = n - k$ .

□

**Corollary 4.40.** For  $n \geq 4$  and  $\delta(G) \geq 1$ , then  $\min(\beta_1; m, n) = \lceil \frac{n}{2} \rceil$ .

*Proof.* By Theorem 3.10, we have  $\min(\beta_1; m, n) = n - \max(\alpha_1; m, n)$ .

For  $m > \lceil \frac{n}{2} \rceil$ ,  $\max(\alpha_1; m, n) = \lfloor \frac{n}{2} \rfloor$ . Hence  $\min(\beta_1; m, n) = \lceil \frac{n}{2} \rceil$ .

□

The following example illustrates the edge covering number of some graphs in  $\mathcal{G}(m, 7)$ .

**Example 4.41.** According to Corollaries 4.39 and 4.40, let  $n = 7$ . Then

$$\max(\beta_1; m, 7) = \begin{cases} m & \text{if } 4 \leq m \leq 6, \\ 5 & \text{if } 6 < m \leq 11, \\ 4 & \text{if } 11 < m \leq 21. \end{cases}$$

$$\min(\beta_1; m, 7) = 4.$$

Firstly, consider  $m = 6$ . Then  $\max(\beta_1; 6, 7) = 6$  and  $\min(\beta_1; 6, 7) = 4$ . In Figure 4.29, we construct the graphs  $G_1 \in \mathcal{G}(6, 7)$  with  $\beta_1(G_1) = 4$  by Theorem 4.29 in order to have  $\delta(G_1) \geq 1$  and  $G_3 \in \mathcal{G}(6, 7)$  with  $\beta_1(G_3) = 6$  by Corollary 4.39. Since  $\beta_1$  is an interpolation graph parameter over  $\mathcal{G}(m, n)$ , there exists the graph  $G_2 \in \mathcal{G}(6, 7)$  with  $\beta_1(G_2) = 5$ . The red edges of  $G_1, G_2$ , and  $G_3$  are covered all vertices of  $G_1, G_2$ , and  $G_3$ , respectively.

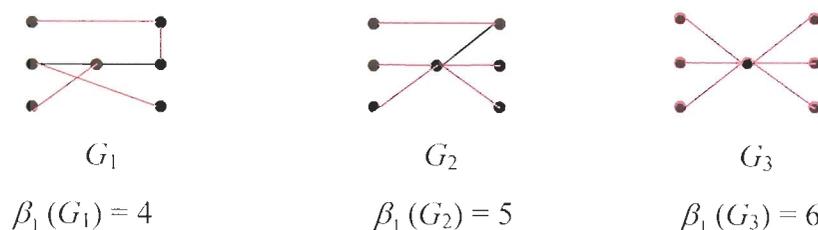
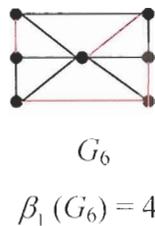


Figure 4.29: The edge covering number of graphs in  $\mathcal{G}(6, 7)$

Secondly, consider  $m = 8$ . Then  $\max(\beta_1; 8, 7) = 5$  and  $\min(\beta_1; 8, 7) = 4$ . In Figure 4.30, we construct the graphs  $G_4 \in \mathcal{G}(8, 7)$  with  $\beta_1(G_4) = 4$  by Theorem 4.29 in order to have  $\delta(G_4) \geq 1$  and  $G_5 \in \mathcal{G}(8, 7)$  with  $\beta_1(G_5) = 5$  by deleting 3 edges from one vertex of degree 6 in  $G_{7,5}$ . The red edges of  $G_4$  and  $G_5$  are covered all vertices of  $G_4$  and  $G_5$ , respectively.

Figure 4.30: The edge covering number of graphs in  $\mathcal{G}(8, 7)$ 

Finally, consider  $m = 12$ . Then  $\max(\beta_1; 12, 7) = \min(\beta_1; 12, 7) = 4$ . In Figure 4.31, we obtain the graph  $G_6 \in \mathcal{G}(12, 7)$  with  $\beta_1(G_6) = 4$  by Theorem 4.29 in order to have  $\delta(G_1) \geq 1$ . The red edges of  $G_6$  are covered all vertices of  $G_6$ .

Figure 4.31: The edge covering of a graph in  $\mathcal{G}(12, 7)$ 

#### 4.6.2 The Edge Covering Number in $\mathcal{CG}(m, n)$

In this section we determine the edge covering number of connected graphs with prescribed order and size. That is to determine  $\max\{\beta_1(G) : G \in \mathcal{CG}(m, n)\}$  and  $\min\{\beta_1(G) : G \in \mathcal{CG}(m, n)\}$ , simply written  $\text{Max}(\beta_1; m, n)$  and  $\text{Min}(\beta_1; m, n)$ , respectively. Since the edge covering number of the star of order  $n$  is  $n - 1$ ,  $\text{Max}(\beta_1; n - 1, n) = n - 1$ . Note that  $\text{Min}(\beta_1; n - 1, n) = \lceil \frac{n}{2} \rceil$ , as the edge covering number of  $P_n$  is  $\lceil \frac{n}{2} \rceil$ . We shall give formulas for  $\text{Max}(\beta_1; m, n)$  and  $\text{Min}(\beta_1; m, n)$  when  $n \geq 4$  and  $n \leq m \leq \binom{n}{2}$ . We can see that for  $n \geq 2$ ,  $n - 1 \geq \lceil \frac{n}{2} \rceil$ , so  $\beta_1(G)$

is well defined for all  $G \in \mathcal{CG}(m, n)$ .

**Corollary 4.42.** Let  $n, m$ , and  $k$  be positive integers where  $2 \leq k \leq \frac{n}{2}$  and  $EX(n; kK_2) < m \leq EX(n; (k+1)K_2)$ . Then  $\text{Max}(\beta_1; m, n) = n - k$ .

*Proof.* By Theorem 3.10, we have  $\text{Max}(\beta_1; m, n) = n - \text{Min}(\alpha_1; m, n)$ .

For  $EX(n; kK_2) < m \leq EX(n; (k+1)K_2)$ ,  $\text{Min}(\alpha_1; m, n) = k$ . Hence  $\text{Max}(\beta_1; m, n) = n - k$ .

□

**Corollary 4.43.** For  $n \geq 4$ , then  $\text{Min}(\beta_1; m, n) = \lceil \frac{n}{2} \rceil$ .

*Proof.* By Theorem 3.10, we have  $\text{Min}(\beta_1; m, n) = n - \text{Max}(\alpha_1; m, n)$ .

For  $m > n - 1$ ,  $\text{Max}(\alpha_1; m, n) = \lfloor \frac{n}{2} \rfloor$ . Hence  $\text{Min}(\beta_1; m, n) = \lceil \frac{n}{2} \rceil$ .

□

The following example illustrates the edge covering number of some connected graphs in  $\mathcal{CG}(m, 8)$ .

**Example 4.44.** According to Corollaries 4.42 and 4.43, let  $n = 8$ . Then

$$\text{Max}(\beta_1; m, 8) = \begin{cases} 6 & \text{if } 7 < m \leq 13, \\ 5 & \text{if } 13 < m \leq 18, \\ 4 & \text{if } 18 < m \leq 28. \end{cases}$$

$$\text{Min}(\beta_1; m, 8) = 4.$$

Firstly, consider  $m = 8$ . Then  $\text{Max}(\beta_1; 8, 8) = 6$  and  $\text{Min}(\beta_1; 8, 8) = 4$ . In Figure 4.32, we construct the connected graphs  $G_1 \in \mathcal{CG}(8, 8)$  with  $\beta_1(G_1) = 4$  by adding 4 edges in order to have a path connecting every two vertices to the graph  $4K_2$  and  $G_3 \in \mathcal{CG}(8, 8)$  with  $\beta_1(G_3) = 6$  by deleting 5 edges from one vertex of degree 7 in  $G_{8,6}$ . Since  $\beta_1$  is an interpolation graph parameter over  $\mathcal{CG}(m, n)$ , there

exists the graph  $G_2 \in \mathcal{CG}(8, 8)$  with  $\beta_1(G_2) = 5$ . The red edges of  $G_1, G_2$ , and  $G_3$  are covered all vertices of  $G_1, G_2$ , and  $G_3$ , respectively.

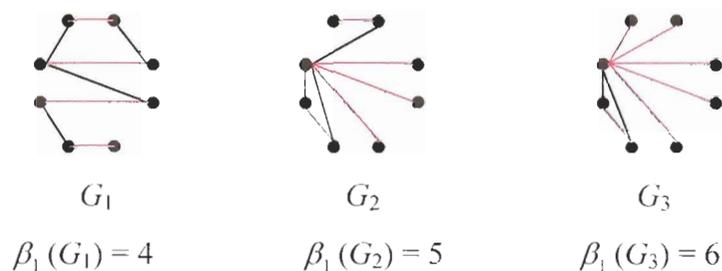


Figure 4.32: The edge covering number of graphs in  $\mathcal{CG}(8, 8)$

Secondly, consider  $m = 18$ . Then  $\text{Max}(\beta_1; 18, 8) = 5$  and  $\text{Min}(\beta_1; 18, 8) = 4$ . In Figure 4.33, we construct the connected graph  $G_4 \in \mathcal{CG}(18, 8)$  with  $\beta_1(G_4) = 4$  by adding 14 edges in order to have a path connecting every two vertices to the graph  $4K_2$ . The connected graph  $G_5 \in \mathcal{CG}(18, 8)$  with  $\beta_1(G_5) = 5$ , can be obtained from  $G_{8,5}$ . The red edges of  $G_4$  and  $G_5$  are covered all vertices of  $G_4$  and  $G_5$ , respectively.



Figure 4.33: The edge covering number of graphs in  $\mathcal{CG}(18, 8)$

Finally, consider  $m = 19$ . Then  $\text{Max}(\beta_1; 19, 8) = \text{Min}(\beta_1; 19, 8) = 4$ . In Figure 4.34, we obtain the graph  $G_6 \in \mathcal{CG}(19, 8)$  with  $\beta_1(G_6) = 4$  by adding 15 edges in order to have a path connecting every two vertices to the graph  $4K_2$ . The red edges of  $G_{10}$  are covered all vertices of  $G_6$ .



$G_6$

$$\beta_1(G_6) = 4$$

Figure 4.34: The edge covering of a graph in  $\mathcal{G}(19, 8)$