CHAPTER III
INTERPOLATION THEOREMS

In the study of interpolation of graph parameter f over $\mathcal{J}$, we may consider two parts. The first part, we consider whether or not a given graph parameter f is an interpolation graph parameter over $\mathcal{J}$. If it is, in the second part we shall develop techniques to find minimum and maximum value of the graph parameter f on $\mathcal{J}$.

In this chapter, we now prove interpolation theorems for six graph parameters, namely the clique number, the independence number, the vertex covering number, the chromatic number, the matching number, and the edge covering number over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$.

3.1 Interpolation Theorems

For any non negative integers m and n, it is shown in Corollaries 1.6 and 1.8 that the $\mathcal{G}(m, n)$-graph is connected and the subgraph of the $\mathcal{G}(m, n)$-graph induced by $\mathcal{CG}(m, n)$ is also connected. Let f be the graph parameter. Then we have the following theorems.

Theorem 3.1. Let $G \in \mathcal{G}(m, n)$ and t be a jumping transformation on G, if $|f(G) - f(G^t)| \leq 1$, then f is an interpolation graph parameter over $\mathcal{G}(m, n)$.

□

Theorem 3.2. Let $G \in \mathcal{CG}(m, n)$ and t be a jumping transformation on G, if $|f(G) - f(G^t)| \leq 1$, then f is an interpolation graph parameter over $\mathcal{CG}(m, n)$.

□

Theorem 3.3. Let $\mathcal{J} \subseteq \mathcal{G}(m,n)$ and the subgraph of the $\mathcal{G}(m, n)$-graph induced by $\mathcal{J}$ be connected. For a graph $G \in \mathcal{J}$ and a jumping transformation t on G, if $|f(G) - f(G^t)| \leq 1$, then f is an interpolation graph parameter over $\mathcal{J}$.

□

3.2 Interpolation Graph Parameters

We now prove interpolation theorems for the graph parameters ω, α_0, β_0, χ, α_1, and β_1 over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$.

Theorem 3.4. Let $G \in \mathcal{G}(m,n)$ and t be a jumping transformation on G. Then $|\omega(G) - \omega(G^t)| \leq 1$.

Proof. For a subgraph H of G, we can see that $\omega(H) \leq \omega(G)$. If $e \in E(G)$, then $\omega(G)-1 \leq \omega(G-e) \leq \omega(G)$. Let $f \in E(\overline{G})$, $\omega(G-e) \leq \omega(G-e+f) \leq \omega(G-e)+1$. Hence $\omega(G) - 1 \leq \omega(G - e + f) \leq \omega(G) + 1$. That is $-1 \leq \omega(G) - \omega(G^t) \leq 1$. Thus $|\omega(G) - \omega(G^t)| \leq 1$.

□

Theorem 3.5. Let $G \in \mathcal{G}(m,n)$ and t be a jumping transformation on G. Then $|\alpha_0(G) - \alpha_0(G^t)| \leq 1$.

Proof. If $e \in E(G)$, then $\alpha_0(G) \leq \alpha_0(G - e) \leq \alpha_0(G) + 1$. Also, if $f \in E(\overline{G - e})$, $\alpha_0(G - e) - 1 \leq \alpha_0(G - e + f) \leq \alpha_0(G - e)$. Hence $\alpha_0(G) - 1 \leq \alpha_0(G - e + f) \leq \alpha_0(G) + 1$. That is $-1 \leq \alpha_0(G) - \alpha_0(G^t) \leq 1$. Thus $|\alpha_0(G) - \alpha_0(G^t)| \leq 1$.

□

Gallai [12] proposed a result concerning the relationship between α_0 and β_0 as follows.

Theorem 3.6. ([12]) For any graph G of order n, $\alpha_0(G) + \beta_0(G) = n$.

□

We exploit the above theorem to show the interpolation property of β_0.

Corollary 3.7. Let $G \in \mathcal{G}(m,n)$ and t be a jumping transformation on G. Then $|\beta_0(G) - \beta_0(G^t)| \leq 1$.

Proof. By Theorem 3.6, we have $\beta_0(G) = n - \alpha_0(G)$. Since $|\beta_0(G) - \beta_0(G^t)| = |(n - \alpha_0(G)) - (n - \alpha_0(G^t))| \leq 1$, that is $|\beta_0(G) - \beta_0(G^t)| \leq 1$.

□

The chromatic number also has the interpolation property as follows.

Theorem 3.8. Let $G \in \mathcal{G}(m,n)$ and t be a jumping transformation on G. Then $|\chi(G) - \chi(G^t)| \leq 1$.

Proof. For a subgraph H of G, we can see that $\chi(H) \leq \chi(G)$. If $e \in E(G)$, then $\chi(G)-1 \leq \chi(G-e) \leq \chi(G)$. Let $f \in E(\overline{G})$, $\chi(G-e) \leq \chi(G-e+f) \leq \chi(G-e)+1$. Hence $\chi(G) - 1 \leq \chi(G - e + f) \leq \chi(G) + 1$. That is $-1 \leq \chi(G) - \chi(G^t) \leq 1$. Thus $|\chi(G) - \chi(G^t)| \leq 1$.

□

The following theorems show the interpolation property of α_1 and β_1, respectively.

Theorem 3.9. Let $G \in \mathcal{G}(m,n)$ and t be a jumping transformation on G. Then $|\alpha_1(G) - \alpha_1(G^t)| \leq 1$.

Proof. For a subgraph H of G, we can see that $\alpha_1(H) \leq \alpha_1(G)$. If $e \in E(G)$, then $\alpha_1(G) - 1 \leq \alpha_1(G - e) \leq \alpha_1(G)$. Let $f \in E(\overline{G})$, $\alpha_1(G - e) \leq \alpha_1(G - e + f) \leq \alpha_1(G - e) + 1$. Hence $\alpha_1(G) - 1 \leq \alpha_1(G - e + f) \leq \alpha_1(G) + 1$. That is $-1 \leq \alpha_1(G) - \alpha_1(G^t) \leq 1$. Thus $|\alpha_1(G) - \alpha_1(G^t)| \leq 1$.

□

Norman and Rabin [24] also showed the relationship between α_1 and β_1 as follows.

Theorem 3.10. ([24]) For any graph G of order n and $\delta(G) \geq 1$, $\alpha_1(G) + \beta_1(G) = n$.

The interpolation property of $\beta_1(G)$ is based on the above theorem.

Theorem 3.11. Let $G \in \mathcal{G}(m, n, ; \delta \geq 1)$ and t be a jumping transformation on G. Then $|\beta_1(G) - \beta_1(G^t)| \leq 1$.

Proof. By Theorem 3.10, we have $\beta_1(G) = n - \alpha_1(G)$. Since $|\beta_1(G) - \beta_1(G^t)| = |(n - \alpha_1(G)) - (n - \alpha_1(G^t))| \leq 1$, that is $|\beta_1(G) - \beta_1(G^t)| \leq 1$.

□

Combining the above results and the fact that the $\mathcal{G}(m,\ n)$-graph, the $\mathcal{CG}(m,\ n)$-graph, and the $\mathcal{G}(m, n; \delta \geq 1)$-graph are connected, we conclude the following theorems.

Theorem 3.12. The graph parameters ω, α_0, β_0, χ, α_1, and β_1 are interpolation graph parameters over $\mathcal{G}(m,\ n)$.

□

Theorem 3.13. The graph parameters ω, α_0, β_0, χ, α_1, and β_1 are interpolation graph parameters over $\mathcal{CG}(m,\ n)$.

□

Theorem 3.14. Let $f \in \{\omega, \alpha_0, \beta_0, \chi, \alpha_1, \beta_1\}$. Then there exist integers $a = a(f)$ and $b = b(f)$ in which there is a graph G of order n and size m, with $f(G) = c$ if and only if c is an integer satisfying $a \leq c \leq b$.

□

Theorem 3.15. Let $f \in \{\omega, \alpha_0, \beta_0, \chi, \alpha_1, \beta_1\}$. Then there exist integers $A = A(f)$ and $B = B(f)$ in which there is a connected graph G of order n and size m, with $f(G) = C$ if and only if C is an integer satisfying $A \leq C \leq B$.

□