CHAPTER II
LITERATURE REVIEW

This chapter presents some studies of interpolation theorems and extremal problems. Numerous researches have studied in interpolation theorems on various kinds of graph parameters and classes of graphs.

Harary and others answered the interpolation on various kinds of graph parameters over the set of all spanning trees of a given graph [14, 15, 16, 17]. Harary introduced a graph transformation and called it a *fundamental exchange* or an *edge exchange* as follows:

Let G be a graph of order $n \geq 3$. The *tree graph*, $\mathbf{T}(G)$, of G is defined by specifying $V(\mathbf{T}(G))$ to be the set of all spanning trees of G, two vertices $T_1, T_2 \in V(\mathbf{T}(G))$ are adjacent in $\mathbf{T}(G)$ if and only if T_1 and T_2 have exactly $n-2$ edges on common. It was proved that $\mathbf{T}(G)$ is connected in [16].

Harary used fundamental exchanges to transform a spanning tree to another spanning tree of the same graph. In other words, let f be a graph parameter. If S and T are adjacent in $\mathbf{T}(G)$ and $|f(S) - f(T)| \leq 1$, then f is an interpolation graph parameter over $\mathbf{T}(G)$.

This chapter consists of two sections. Section 2.1 introduces bounds for the six graph parameters namely ω, α_0, β_0, χ, α_1, and β_1. Section 2.2 reviews the interpolation theorems on various kinds of graph parameters and classes of graphs.

2.1 Some Bounds for Graph Parameters

Following results establish the bounds and some related bounds of some graph parameters.

Corollary 2.1. [27] Let G be a graph with degree sequence $d = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. Then

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n - d_i},$$

where $\omega(G)$ is the clique number of G.

□

Theorem 2.2. [33] Let G be a graph of order n with maximum degree $\Delta(G)$. Then

$$\omega(G) \geq \frac{n}{n - \Delta(G)}.$$

□

Corollary 2.3. [4] Let G be a graph with degree sequence $d = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$. Then

$$\alpha_0(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1},$$

where $\alpha_0(G)$ is the independence number of G.

□

Theorem 2.4. [35] Let G be a graph of order n with maximum degree $\Delta(G)$. Then

$$\alpha_0(G) \geq \frac{n}{\Delta(G) + 1}.$$

□

Theorem 2.5. [35] Let G be a K_4-free graph of order n, $\Delta(G) = 3$. Then

$$\alpha_0(G) \geq \frac{n}{3}.$$

□

Theorem 2.6. [35] Let G be a $K_{\Delta(G)+1}$-free graph of order n with $\Delta(G) \geq 4$. Then $\alpha_0(G) \geq \frac{n}{\Delta(G)}$ or there exists an independent set S of G and $G - S$ is a $K_{\Delta(G)+1}$-free graph.

□

Theorem 2.7. [10] Let G be a graph of order n with maximum degree Δ. Then

$$\alpha_0(G) \geq \frac{2n}{\omega(G) + \Delta(G) + 1}.$$

□

In 1963, Erdős and Gallai [9] proved that any regular graph on n vertices has chromatic number $k \leq \frac{3n}{5}$ unless the graph is complete. Commenting on their result in a personal communication, Erdős wrote, "probably such a graph exists for every $k \leq \frac{3n}{5}$, except possibly for trivial exceptional cases".

Caccetta and Pullman [3] confirmed and strengthened their conjecture by showing that if $k > 1$, then for every $n \geq \frac{5k}{3}$, there exists a connected regular k-chromatic graph on n vertices.

In 1956, Nordhaus and Gaddum [23] defined graph parameters $\chi + \overline{\chi}$ and $\chi\overline{\chi}$ as: $(\chi + \overline{\chi})(G) := \chi(G) + \chi(\overline{G})$ and $(\chi\overline{\chi})(G) := \chi(G)\chi(\overline{G})$, for any graph G.

The authors proved the following theorem.

Theorem 2.8. [23] Let G be a graph of order n. Then

1. $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n - 1$,

2. $n \leq \chi(G)\chi(\overline{G}) \leq (\frac{1}{2}(n + 1))^2$.

□

Landon Rabern [20] provided their results which is a generalization of the Nordhaus-Gaddum upper bound as follows:

Theorem 2.9. [20] Let G be a graph of order n. Then for any induced subgraph H of G,

$$\chi(G) \leq \chi(H) + \frac{1}{2}(\omega(G) + n - |V(H)| - 1).$$

□

Corollary 2.10. [20] Let G be a graph of order n. Then

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \beta_0(G)}{2} \right\rceil.$$

□

The following result due to Finck [11] establishes the existence of a graph with prescribed the chromatic number.

Theorem 2.11. [11] Every pair of positive integers p and q with $p + q \leq n + 1$ and $pq \geq n$ there exists a graph G of order n such that $\chi(G) = p$ and $\chi(\overline{G}) = q$.

□

The following theorems and corollaries concern the improvement of the bound for $\chi(G)$.

Theorem 2.12. [6] For every graph G of order n,

$$\chi(G) \geq \omega(G) \quad \text{and} \quad \chi(G) \geq \frac{n}{\alpha_0(G)}.$$

□

Theorem 2.13. [6] For every graph G, $\chi(G) \leq 1 + \Delta(G)$.

□

Theorem 2.14. (Brooks's Theorem) [6] For every connected graph G that is not an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.

□

Theorem 2.15. [6] For every graph G,

$$\chi(G) \leq 1 + \max\{\delta(H)\},$$

where the maximum is taken over all induced subgraphs H of G.

□

Corollary 2.16. [5] Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then

$$\chi(G) \leq \max\{\min\{i, d(v_i) + 1\}\}.$$

□

Corollary 2.17. [5] For every graph G,

$$\chi(G) \leq 1 + \ell(G),$$

where $\ell(G)$ denotes the length of a longest path in G.

□

Corollary 2.18. [5] If G is a chordal graph, then $\chi(\mathrm{G}) = \omega(\mathrm{G})$.

□

Next we turn our attention to the matching and edge covering numbers. Not as much is know about lower bound for the cardinality of matching of graphs. Every 4-connected triangulated planar graphs has a matching of size $\lfloor \frac{n}{2} \rfloor$, because it has a Hamiltonian cycle [37]. Nishizeki and Baybars [22] gave bounds for planar graphs as a function of the minimum degree and the connectivity. In particular, the authors proved the following theorem.

Theorem 2.19. [22] For any 3-connected planar graphs of order $n \geq 22$, then

$$\alpha_1(G) \geq \frac{1}{3}(n+4).$$

$\square$

König and Egerváry [19] provided that in a bipartite graph we can certify optimality of a matching by a vertex cover.

Theorem 2.20. (König-Egerváry's Theorem) [19] If G is a bipartite graph, then

$$\alpha_1(G) = \beta_0(G).$$

$\square$

The improvement of bounds for the matching number is shown as follows.

Theorem 2.21. [5] If G is a connected cubic graph of order n containing fewer than $3(k+1)$ bridges, then $\alpha_1(G) \geq \frac{n-2k}{2}$.

$\square$

Theorem 2.22. [5] If G is a connected cubic graph of order n all of whose bridges lie on r edge-disjoint paths of G, then $\alpha_1(G) \geq \frac{n}{2} - \lfloor \frac{2r}{3} \rfloor$.

$\square$

Theorem 2.23. [5] Let G be a graph of order n without isolated vertices. Then

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \alpha_1(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Furthermore, these bounds are sharp.

$\square$

David Sitton [7] provided some result about the possible number of edges in a maximum matching of a complete multipartite graph as stated in the following theorems.

Theorem 2.24. [7] If $K_{l,m,n}$ is a nontrivial complete 3-partite graph with $l \leq m \leq n$ (and $n < l+m$), then the size of a maximum matching is $\lfloor \frac{l+m+n}{2} \rfloor$.

$\square$

Theorem 2.25. [7] Let $K_{m_1,m_2,\ldots,m_n}$ be a complete multipartite graph with m_i vertices in the i^{th} part, labelled so that $m_1 \leq m_2 \leq \cdots \leq m_n$. If $m_n \geq m_1 + m_2 + \cdots + m_{n-1}$, then the size of a maximum matching is $m_1 + m_2 + \cdots + m_{n-1}$.

□

Theorem 2.26. [7] Give any complete multipartite graph of order n, $K_{m_1,m_2,\ldots,m_n}$, where $m_1 \leq m_2 \leq \cdots \leq m_n$ and $m_n \leq m_1 + m_2 + \cdots + m_{n-1}$, then the size of a maximum matching is $\lfloor \frac{n}{2} \rfloor$.

□

Theorem 2.27. [7] Give any complete multipartite graph of order n, $K_{m_1,m_2,\ldots,m_n}$, with m_n vertices in the maximum part, the size of a maximum matching is

$$\min\{\sum_{i=1}^{n-1} m_i, \left\lfloor \frac{1}{2} \sum_{i=1}^{n} m_i \right\rfloor\}.$$

□

Corollary 2.28. [5] Let G be a graph of order n without isolated vertices. Then

$$\left\lceil \frac{n}{2} \right\rceil \leq \beta_1(G) \leq \left\lfloor \frac{n\Delta(G)}{1 + \Delta(G)} \right\rfloor.$$

Furthermore, these bounds are sharp.

□

Acyclic graph is a graph containing no cycle as its subgraph. An acyclic graph is called a *forest*. Therefore, each component of an acyclic graph is a tree. Since a tree is connected, every two vertices in a tree are connected by a unique path. Let G be a graph and $F \subseteq V(G)$, F is called an *induced forest* of G, if the induced subgraph $G[F]$ of G contains no cycle. For a graph G, we define, $I(G)$ as:

$$I(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

Corollary 2.29. [27] If G is a graph of order n with maximum degree $\Delta = \Delta(G) \geq 1$, then $I(G) \geq \frac{2n}{\Delta+1}$.

□

Theorem 2.30. [27] Let G be a graph with degree sequence $d = (d_1, d_2, \ldots, d_n)$, $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$. Then

$$I(G) \geq 2\sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

□

2.2 Interpolation Theorems of Some Graph Parameters

Punnim has studied interpolation theorems on various kinds of graph parameters over $\mathcal{R}(d)$. The author has used the switching transformation(σ) to transform a realization of a given graphic degree sequence d to another realization of d. Let f be a graph parameter and d be a graphic degree sequence. In [8], it has been shown that the $\Sigma(d)$-graph is connected. It follows that for a graph G of degree sequence d and a switching σ if $|f(G) - f(G^\sigma)| \leq 1$, then f is an interpolation graph parameter over $\mathcal{R}(d)$. Punnim proved in [26, 33, 34] that the chromatic number(χ), the clique number(ω), and the matching number(α_1), respectively, are interpolation graph parameters over $\mathcal{R}(d)$.

Punnim has considered interpolation of graph parameters into two parts. The first part is to consider whether or not a given graph parameter f interpolates over $\mathcal{J} \subseteq \mathcal{G}$. If it is, we will consider the second part that is finding minimum and maximum value of the graph parameter f on $\mathcal{J}$.

In [33], Punnim obtained the clique number is an interpolation graph parameter over $\mathcal{R}(d)$ and also found its minimum and maximum value.

Theorem 2.31. [33] Let G be a graph and σ be a switching on G. Then

$$|\omega(G) - \omega(G^{\sigma})| \leq 1$$

$\square$

Theorem 2.32. [33] Let $d = r^n$ be a graphic degree sequence with $r + 2 \leq n \leq 2r + 1$. Then $\max(\omega, r^n) = \lfloor \frac{n}{2} \rfloor$.

$\square$

Theorem 2.33. [33] For any $r \geq 6$ and odd integer s such that $5 \leq s < r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if} \quad t = 0, \\ q+1 & \text{if} \quad 1 \leq t \leq s-2, \\ q+2 & \text{if} \quad t = s-1. \end{cases}$$

$\square$

Theorem 2.34. [33] For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\omega, r^{r+s}) = \begin{cases} q & \text{if} \quad t = 0, \\ q+1 & \text{if} \quad t \geq 2. \end{cases}$$

$\square$

It is clear that $\alpha_0(G)$ is a graph parameter and $\alpha_0(G) = \omega(\overline{G})$, for any graph G. Observe that for a graph G and a switching σ on G, $\overline{G}^{\sigma} = \overline{G^{\sigma}}$. Thus the interpolation result for α_0 in $\mathcal{R}(d)$ follows directly from the graph parameter ω.

Punnim showed in [32] the following results that the independence number is an interpolation graph parameter over $\mathcal{R}(d)$.

Theorem 2.35. [32] Let G be a graph and σ be a switching on G. Then

$$|\alpha_0(G) - \alpha_0(G^\sigma)| \leq 1$$

$\square$

In [35], Samanmoo found the minimum and maximum value of the independence number of regular graphs established in the following list of theorems.

Theorem 2.36. [35] Let $d = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence where $d_1 + 1 \leq n \leq 2d_1 + 1$. Then $\min(\alpha_0, d) = 1$ if and only if $\mathcal{R}(d) = \{K_n\}$.

$\square$

Theorem 2.37. [35] For any $r \geq 3$, $n = r + j$ and $1 \leq j \leq r + 1$. Then

1. $\min(\alpha_0, r^n) = 1$ if and only if n = r + 1,
2. $\min(\alpha_0, r^n) = 2$ for all even integers n and $2 \leq j \leq r$,
3. $\min(\alpha_0, r^n) = 2$ for all odd integers $n, 3 \leq j \leq r + 1$ and $n \geq f(j)$,
4. $\min(\alpha_0, r^n) = 3$ for all odd integers $n, 3 \leq j \leq r + 1$ and $n < f(j)$,

where $f(j) = \frac{5}{2}(j - 1)$ if $j \equiv 3$ (mod4) and $f(j) = 1 + \frac{5}{2}(j - 1)$ if $j \equiv 1$ (mod4).

$\square$

Theorem 2.38. [35] For $n \geq 2r + 2$ and even integer $r \geq 4$, with $n = (r + 1)q + t$, for some $q \geq 2$, and $0 \leq t \leq r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,
2. $\min(\alpha_0, r^n) = q + 1$ if $1 \leq t \leq r - 1$,
3. $\min(\alpha_0, r^n) = q + 2$ if $t = r$.

$\square$

Theorem 2.39. [35] For any even integer $n \geq 2r+2$ and any odd integer $r \geq 3$, with $n = (r+1)q + t$, for some $q \geq 2$, and $0 \leq t < r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,
2. $\min(\alpha_0, r^n) = q + 1$ if $0 < t < r$.

□

Theorem 2.40. [35] For any integer $r \geq 3$ and any integer s such that $3 \leq s < r$, let q and t be integers satisfying $n = r + s = sq + t$; $q \geq 2$ and $0 \leq t < s$. Then $\max(\alpha_0, r^n) = s$.

□

Theorem 2.41. [35] Let $d = r^n$ be a graphic degree sequence with $n \geq 2r$. Then $\max(\alpha_0, r^n) = \lfloor \frac{n}{2} \rfloor$.

□

In [35], Samanmoo also found the minimum and maximum value of the independence number of connected regular graphs as follows:

Theorem 2.42. [35] For $r \geq 3$, $n \geq 2r+2$ and a graphic degree sequence r^n. $\max(\alpha_0, r^n) = \lfloor \frac{n}{2} \rfloor$ if $n \geq 2r$.

□

Theorem 2.43. [35] For $n \geq 2r+2$ and even integer $r \geq 4$, write $n = rq + t$ where $0 \leq t < r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,
2. $\min(\alpha_0, r^n) = q + 1$ if $1 \leq t < r$.

□

Theorem 2.44. [35] For even integer $n \geq 2r+2$ and odd integer $r \geq 3$, write $n = rq + t$ where $0 \leq t < r$. Then

1. $\min(\alpha_0, r^n) = q$ if $t = 0$,

2. $\min(\alpha_0, r^n) = q + 1$ if $1 \leq t < r$.

$\square$

Theorem 2.45. [35] $\min(\alpha_0, r^n) = \frac{n}{r}$, for all integer $r \geq 3$.

$\square$

Punnim showed in [26] the following results that the chromatic number is an interpolation graph parameter over $\mathcal{R}(d)$ and also found its minimum and maximum value.

Theorem 2.46. [26] Let G be a graph and σ be a switching on G. Then

$$|\chi(G) - \chi(G^\sigma)| \leq 1$$

$\square$

Theorem 2.47. [26] If $r \geq 2$ and $n \geq 2r$, then

$$\min(\chi, r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

$\square$

Theorem 2.48. [26] If $r \geq 2$, then

$$\min(\chi, r^{r+1}) = \max(\chi, r^{r+1}) = r + 1, \quad \text{and}$$
$$\min(\chi, r^{r+2}) = \max(\chi, r^{r+2}) = \frac{r+2}{2}.$$

$\square$

Theorem 2.49. [26] For any $r \geq 4$ and odd integer s such that $3 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } 1 \leq t \leq s-2, \\ q+2 & \text{if } t = s-1. \end{cases}$$

□

Theorem 2.50. [26] For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$. Then

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q+1 & \text{if } t \geq 2. \end{cases}$$

□

Theorem 2.51. [26] Let $r \geq 2$. Then

1. $\max(\chi, r^{2r}) = r$,
2. $\max(\chi, r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2 \\ r & \text{if } r \geq 4, \end{cases}$
3. $\max(\chi, r^n) = r + 1$ for $n \geq 2r + 2$.

□

Theorem 2.52. [26] For any r and s such that $3 \leq s \leq r - 1$, we have

1. $\max(\chi, r^{r+s}) \geq \frac{r+s}{2}$ if $r + s$ is even, and
2. $\max(\chi, r^{r+s}) \geq \frac{r+s-1}{2}$ if $r + s$ is odd.

□

Punnim showed in [29] the following results that the matching number is an interpolation graph parameter over $\mathcal{R}(d)$.

Theorem 2.53. [29] Let G be a graph and σ be a switching on G. Then

$$|\alpha_1(G) - \alpha_1(G^\sigma)| \leq 1$$

□

In [34], Punnim investigated the values of $\min(\alpha_1, r^n)$ and $\max(\alpha_1, r^n)$ for all r and n. It is easy to see that $\min(\alpha_1, 0^n) = \max(\alpha_1, 0^n) = 0$ and $\min(\alpha_1, 1^{2n}) = \max(\alpha_1, 1^{2n}) = n$. Because of this fact, we will consider $r \geq 2$ and $n \geq r + 1$.

Theorem 2.54. [34] For $r \geq 2, n \geq r + 1$ and $nr \equiv 0 \pmod 2$, there exists an r-regular hamiltonian graph of order n. In particular, $\max(\alpha_1, r^n) = \lfloor \frac{n}{2} \rfloor$.

□

Let $F(r, d)$ be the minimum order of an r-regular graph G with $(\alpha_1(G) = \frac{1}{2}(|V(G)| - d)$. It is clear that $|V(G)| \equiv d \pmod 2$.

Theorem 2.55. [34] Let r be an even integer, $r \geq 2$. Then $F(r, d) = d(r + 1)$.

□

Theorem 2.56. [34] Let r be an even integer, $r \geq 2$. If $n = (r+1)d+e, 0 \leq e \leq r$, then $\min(\alpha_1, r^n) = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor$.

□

Theorem 2.57. [34] For an odd integer $r \geq 3$. Then

1. $F(r, 2q) = (r + 2)(1 + 2q) + 1$, for $q = 1, 2, \ldots, \frac{r-1}{2}$,
2. if $q = \frac{r-1}{2}s + t, 0 \leq t < \frac{r-1}{2}$, then $F(r, 2q) = sF(r, r - 1) + F(r, 2t)$, where $F(r, 0) = 0$.

□

Corollary 2.58. [34] Let r be an odd integer, $r \geq 3$.
If $F(r, 2q) \leq n < F(r, 2(q+1))$, then $\min(\alpha_1, r^n) = \frac{1}{2}(n - 2q)$.

□

Punnim showed in [30] the following results that $I(G)$ is an interpolation graph parameter over $\mathcal{R}(d)$ and and also found its maximum values.

Theorem 2.59. [30] Let G be a graph and σ be a switching on G. Then

$$|I(G) - I(G^\sigma)| \leq 1$$

□

Theorem 2.60. [30]

$$\max(I, r^n) = \begin{cases} n - r + 1 & \text{if } \ r + 1 \leq n \leq 2r - 1, \\ \lfloor \frac{nr-2}{2(r-1)} \rfloor & \text{if } \ n \geq 2r. \end{cases}$$

□

Theorem 2.61. [27] Let $d = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then

1. $\min(I, d) = 2$ if and only if $d_1 = d_2 = d_3 = \ldots = d_n$ and $n = d_1 + 1$ and
2. if d does not have a complete graph as its realization, then $\min(I, d) = 3$ if and only if $\overline{d}$ has a union of stars as its realization.

□

The values of $\min(I, r^n)$, for all r and n, were obtained by Punnim in [25] in terms of the graph parameter ϕ as stated in the following theorems. Note that $\min(I, r^n) = \max(\phi, r^n)$.

Theorem 2.62. [25] For $r \geq 3$, and $n = r + j, 1 \leq j \leq r + 1$

1. $\min(I, r^n) = 2$, if and only if $n = r + 1$,

2. $\min(I, r^n) = 3$, if and only if $n = r + 2$,

3. $\min(I, r^n) = 4$, for all even integers $n, r + 3 \leq n$,

4. $\min(I, r^n) = 4$, for all odd integers $n, r + 3 \leq n$ and $n \geq f(j)$,

5. $\min(I, r^n) = 5$, for all odd integers $n, r + 3 \leq n$ and $n < f(j)$,

where $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 \pmod 4$, and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 \pmod 4$.

□

Theorem 2.63. [25] For $n \geq 2r + 2$ and $r \geq 3$, write $n = (r+1)q + t, q \geq 2$ and $0 \leq t \leq r$.

1. $\min(I, r^n) = 2q$ if $t = 0$,

2. $\min(I, r^n) = 2q + 1$ if $t = 1$,

3. $\min(I, r^n) = 2q + 2$ if $2 \leq t \leq r - 1$,

4. $\min(I, r^n) = 2q + 3$ if $t = r$.

□

Let Δ be a nonnegative integer and n be a positive integer such that $n \geq \Delta + 1$. Let $\mathbb{G}(\Delta, n)$ be the class of all graphs of order n and of maximum degree Δ. *The* (Δ, n)*-graph* is a graph having $\mathbb{G}(\Delta, n)$ as its vertex set and two such graphs being adjacent if one can be obtained from the other by either adding or deleting an edge.

Punnim showed in [27] the following results that $I(G)$ is an interpolation graph parameter over $\mathbb{G}(\Delta, n)$ and also found its minimum values.

Theorem 2.64. [27] *The* (Δ, n)*-graph* is connected.

□

Theorem 2.65. [27] If G_1 and G_2 are adjacent in $\mathbb{G}(\Delta, n)$, then

$$|I(G_1) - I(G_2)| \leq 1$$

□

Theorem 2.66. [27] Let $n = (\Delta+1)q + t, 0 \leq t \leq \Delta$. Then

1. $\min(I, \mathbb{G}(\Delta, n)) = 2q$ if $t = 0$,
2. $\min(I, \mathbb{G}(\Delta, n)) = 2q + 1$ if $t = 1$, and
3. $\min(I, \mathbb{G}(\Delta, n)) = 2q + 2$ if $2 \leq t \leq \Delta$.

□

For a graph G, the minimum number of vertices whose removal eliminates all cycles in a graph G is the *decycling number* of G, and is denoted by $\phi(G)$. It is easy to see that for a graph G of order n, $\phi(G) + I(G) = n$. Thus the interpolation result for ϕ over $\mathcal{R}(d)$ is easily obtained.

Theorem 2.67. [25] Let G be a graph and σ be a switching on G. Then

$$|\phi(G) - \phi(G^\sigma)| \leq 1$$

□

In [25], Punnim showed the minimum and maximum value of the decycling number of regular graphs.

Theorem 2.68. [25]

$$\min(\phi, r^n) = \begin{cases} r - 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \lceil \frac{nr-2n+2}{2(r-1)} \rceil & \text{if } n \geq 2r. \end{cases}$$

□

Theorem 2.69. [25] For $r \geq 3$, and $r+1 \leq n \leq 2r+1$,

1. $\min(\phi, r^n) = n-2$, if and only if $n = r+1$,
2. $\min(\phi, r^n) = n-3$, if and only if $n = r+2$,
3. $\min(\phi, r^n) = n-4$, for all even integers $n, r+3 \leq n$,
4. $\min(\phi, r^n) = n-4$, for all odd integers $n, r+3 \leq n$ and $n \geq f(j)$,
5. $\min(\phi, r^n) = n-5$, for all odd integers $n, r+3 \leq n$ and $n < f(j)$,

where $f(j) = \frac{5}{2}(j-1)$ if $j \equiv 3 \pmod 4$, and $f(j) = 1 + \frac{5}{2}(j-1)$ if $j \equiv 1 \pmod 4$.

□

Theorem 2.70. [25] For $n \geq 2r+2$ and $r \geq 3$, write $n = (r+1)q + t, q \geq 2$ and $0 \leq t \leq r$.

1. $\max(\phi, r^n) = n - 2q$ if $t = 0$,
2. $\max(\phi, r^n) = n - 2q - 1$ if $t = 1$,
3. $\max(\phi, r^n) = n - 2q - 2$ if $2 \leq t \leq r-1$,
4. $\max(\phi, r^n) = n - 2q - 3$ if $t = r$.

□

Theorem 2.71. [25] Let $\mathrm{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then

1. $\max(\phi, d) = n-2$ if and only if $\mathcal{R}(d) = \{K_n\}$ and
2. if $K_n \notin \mathcal{R}(d)$ then $\max(\phi, d) = n-3$ if and only if there exists a union of stars as a realization of $\overline{d}$, where $\overline{d} = (n-d_n, n-d_{n-1}, \ldots, n-d_1)$.

□

Punnim also showed in [25] the following results that the decycling number is an interpolation graph parameter over $\mathbb{G}(\Delta, n)$ and also found its maximum values.

Theorem 2.72. [25] If G_1 and G_2 are adjacent in $\mathbb{G}(\Delta, n)$, then

$$|\phi(G_1) - \phi(G_2)| \leq 1$$

□

Theorem 2.73. [25] Let $n = (\Delta + 1)q + t, 0 \leq t \leq \Delta$. Then

1. $\max(\phi, \mathbb{G}(\Delta, n)) = n - 2q, \quad$ if $t = 0$,
2. $\max(\phi, \mathbb{G}(\Delta, n)) = n - 2q - 1,$ if $t = 1$, and
3. $\max(\phi, \mathbb{G}(\Delta, n)) = n - 2q - 2,$ if $2 \leq t \leq \Delta$.

□

Moreover, in [25] Punnim showed the minimum and maximum value of the decycling number of cubic graphs.

Theorem 2.74. [25] For any integer $n \geq 2$, we have

$$\min(\phi, 3^{2n}) = \left\lceil \frac{n+1}{2} \right\rceil, \text{ and}$$
$$\max(\phi, 3^{2n}) = \begin{cases} n & \text{if } n \text{ is even,} \\ n - 1 & \text{if } n \text{ is odd.} \end{cases}$$

□