CHAPTER I
INTRODUCTION

In 1980, Chartrand raised the following question: "If a graph G possesses a spanning tree having m leaves and another having n leaves, where $m < n$, does G possess a spanning tree having k leaves for every k between m and n ?". This question was answered affirmatively and it led to a host of lots of papers studying the interpolation property of graph parameters over the set of all spanning trees of a given graph.

In this thesis, we study the interpolation property for six graph parameters, namely the clique number, the independence number, the vertex covering number, the chromatic number, the matching number, and the edge covering number over the class of all non-isomorphic graphs of order n and size m; $\mathcal{G}(m, n)$, and the class of all non-isomorphic connected graphs of order n and size m; $\mathcal{CG}(m, n)$, respectively. We also find the minimum and maximum value for the graph parameter f, $f \in \{\omega, \alpha_0, \beta_0, \chi, \alpha_1, \beta_1\}$. With this property and those values, we know all possible values of those graph parameters of a given graph.

In this chapter, we give the notation and terminology used throughout this thesis in Section 1.1. We define some basic concepts, a graph transformation, the switching transformation, the $\Sigma(d)$-graph, and state a property of the $\Sigma(d)$-graph in Section 1.2. And also give the definition of a jumping transformation, the $\mathcal{G}(m,n)$-graph, the $\mathcal{CG}(m,n)$-graph which will use in our thesis in Section 1.2. In Section 1.3, we give the definition of graph parameters.

1.1 Notation and Terminology

A graph G with n *vertices* and m *edges* consists of a *vertex set* $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an *edge set* $E(G) = \{e_1, e_2, \ldots, e_m\}$, where each edge consists of two (possibly equal) vertices called its *endpoints*. We write uv for an edge $e = \{u, v\}$ and we say that the edge e *joins* the vertices u and v. If $uv \in E(G)$, then u and v are *adjacent*. If vertex v belongs to edge e, then v and e are *incident*.

We use $|S|$ to denote the cardinality of a set S. A graph $G = (V, E)$ has a vertex set V and edge set E. The order of G is the cardinality of V and size of G is the cardinality of E.

A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges that have the same pair of endpoints. A *simple graph* is a graph having no loops or multiple edges. A graph is *finite* if its vertex set and edge set are finite.

All graphs considered in this thesis are finite and simple. In most part, our graph theoretic notations and terminology can be found in the textbook of West [38].

A graph in which every pair of distinct vertices is joined by an edge is called a *complete graph* or *clique*. A complete graph on n vertices is denoted by K_n.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we write $H \subseteq G$. When $V(H) = V(G)$, we say that H is a *spanning subgraph* of G. When $E(H)$ consists of all edges of G whose endpoints belong to $V(H)$, we say that H is an *induced subgraph* of G and write $H = G[V(H)]$. A graph G is H-$free$ if H is not an induced subgraph of G.

The *complement* $\overline{G} = (V, \overline{E})$ of a graph $G = (V, E)$ of order n has the same vertex set as G and its edge set is the set complement of E, $\overline{E}$, such that $\overline{E} = \{e \in$

$E(K_n) : e \notin E(G)\}$ that is an edge uv is in $\overline{G}$ if and only if uv is not an edge in G. Figure 1.1 illustrates a graph G and its complement.

Figure 1.1: A graph G and its complement $\overline{G}$

A graph $G = (V, E)$ is said to be *r-partite* (where r is a positive integer) if its vertex set can be partitioned into r subsets as $V_1, V_2, \ldots, V_r$, called partite sets or parts, such that G$[V_i]$ contains no edge for all $i = 1, 2, \ldots, r$. A maximum part of r-partite graph is a part which has a maximum number of vertices. A 2-partite graph is usually referred as a *bipartite graph.* A *complete bipartite graph* is a bipartite graph with partition $\{X, Y\}$ in which each vertex of X is joined to each vertex of Y. A complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$.

A sequence $v_1e_1v_2 \ldots v_{n-1}e_{n-1}v_n$ with elements alternately from the vertex set V and the edge set E of a graph G, such that $e_i = v_iv_{i+1}$ for $1 \leq i \leq n-1$ is called a $v_1 - v_n$ *walk* in G. The walk is said to be *closed* if $v_1 = v_n$. A closed walk in which no vertex (and edge) is repeated is called a *cycle.* A cycle of order n is denoted by C_n: C_3 is often called a *triangle.* A $v_1 - v_n$ walk in which no vertex is repeated is called a $v_1 - v_n$ *path.* A path of order n is denoted by P_n.

Two distinct vertices u and v of a graph G are said to be connected if there is a $u - v$ walk in G. A graph is said to be *connected* if for every two of its vertices are connected; otherwise it is *disconnected.* A maximal connected subgraph of G is called a *component* of G. For a positive integer p and a graph G, pG is denoted for the union of p copies of G.

A graph G is called *acyclic* if it has no cycle. A *tree* is an acyclic connected graph. A spanning subgraph H of a connected graph G such that H is a tree is called a spanning tree of G.

Two graphs G and H are disjoint if $V(G) \cap V(H) = \varnothing$. Any two disjoint graphs G and H we define $G \cup H$, their union, by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Figure 1.2 illustrates a graph $G = G_1 \cup G_2$.

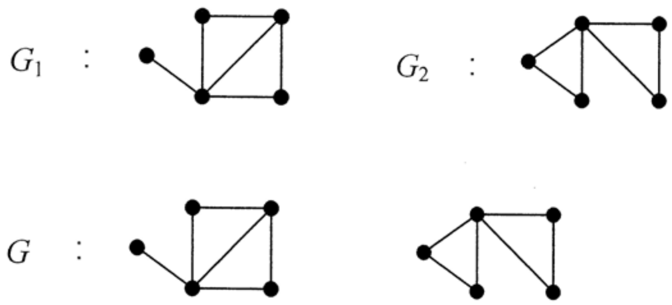


Figure 1.2: A graph $G = G_1 \cup G_2$

Let $G = (V, E)$ be a graph and $S \subseteq V$. We define a graph $G - S$ to be the graph obtained from G by deleting all vertices in S and all edges incident with them. If $S = \{v\}$ we simply write $G - v$ for $G - \{v\}$. If $F \subseteq E$, we can analogously define a graph $G - F$ as the graph obtained from G by removing all edges in F. Also we simply write $G - e$ for $G - \{e\}$. Furthermore, if $e \notin E$, we write $G + e$ for the graph obtained from G by adding the new edge e.

Two graphs $G = (V, E)$ and $H = (U, F)$ are *isomorphic* if and only if there is a bijection $\phi : V \to U$ such that for every pair u, v in $V(G)$ we have $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in F$. The function ϕ is called an *isomorphism* from G onto H and ϕ^{-1} is also an isomorphism from H onto G. We use the notation $G \cong H$

to denote that G is isomorphic to H. Figure 1.3 illustrates that graphs G_1 and G_2 are isomorphic by $\phi(v_i) = u_i$; $i = 1, 2, 3, 4$.

Figure 1.3: The graphs G_1 and G_2 are isomorphic

The *degree* of a vertex v in a graph G, written by $d_G(v)$ or $d(v)$, is the number of edges of G which are incident with v that is $d(v) = |\{e \in E : e = uv$ for some $u \in V\}|$. The *maximum degree* of a graph G is denoted by $\Delta(G)$ and the *minimum degree* of a graph G is denoted by $\delta(G)$. A graph G is *regular* if $\Delta(G) = \delta(G)$; all vertices have the same degree. The graph G is an *r-regular graph* if $\Delta(G) = \delta(G) = r$.

1.2 Basic concepts

In this section, we introduce the fundamental concepts of graph theory, the switching transformation, and the $\Sigma(d)$-graph. We also define a jumping transformation, the $\mathcal{G}(m, n)$-graph, and the $\mathcal{CG}(m, n)$-graph which are used throughout this thesis.

Theorem 1.1. If G is a graph of size m, then

$$\sum_{v \in V(G)} d(v) = 2m.$$

□

Let G be a graph of order n and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. The sequence $(d(v_1), d(v_2), \ldots, d(v_n))$ is called a *degree sequence* of G, where $d(v_i)$ is the degree of vertex v_i. An r-regular graph G has degree sequence $d = r^n = (r, \ldots, r)$. A sequence $d = (d_1, d_2, \ldots, d_n)$ of non-negative integers is a *graphic degree sequence* if it is a degree sequence of some graph G. In this case, G is called a *realization* of d.

An algorithm for determining whether or not a given sequence of non-negative integers is graphic was independently obtained by Havel[18] and Hakimi[13]. We state their results in the following theorem.

Theorem 1.2. Let $d = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of non-negative integers and denote the sequence

$$d' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n).$$

Then d is graphic if an only if d' is graphic.

□

Let $\mathcal{J}$ be a class of graphs or subgraph of a graph. If there is a fixed type of transformation of an element of $\mathcal{J}$ into another, then this will provide a definition of a graph transformation on $\mathcal{J}$. Equivalently, a graph transformation may be defined as a relation ρ from $\mathcal{J}$ to $\mathcal{J}$. In other words $\rho \subseteq \mathcal{J} \times \mathcal{J}$. By the definition, we can see that there are infinitely many such graph transformations. Let ρ be a graph transformation on $\mathcal{J}$. We can define *the ρ-graph* having $\mathcal{J}$ as its vertex set and for $G, H \in \mathcal{J}$, there is a directed edge from G to H if and only if H can be obtained from G by a ρ-transformation. If ρ is symmetric, we have an undirected graph and if not it is a directed graph.

Thus the ρ-graph arises from graphs in at least two different ways:

1. from a class of graphs, or
2. from a class of subgraphs of a fixed graph.

Let G be a graph, ab and cd are independent edges in G, such that ac and bd are not edges in G. Define $G^{\sigma(a,b;c,d)}$ or G^{σ} to be the graph obtained from G by deleting the edges ab and cd and replacing the edges ac and bd (See the below figure).

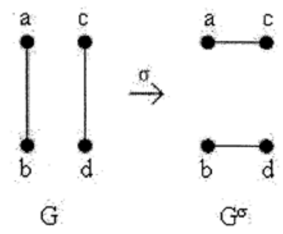


Figure 1.4: A switching transformation

The transformation $\sigma(a, b; c, d)$ is called a *switching transformation.*

The following theorem has been shown by Havel[17] and Hakimi[12].

Theorem 1.3. Let $d = (d_1, d_2, \ldots, d_n)$ be a graphic degree sequence. If G and H are any two realizations of d, then H can be obtained from G by a finite sequence of switchings.

□

Let d be a graphic degree sequence. Let $\mathcal{R}(d)$ be the set of realizations of d. We can see that the graph obtained from G by a switching has the same degree sequence as G, it is reasonable to define $\Sigma(d)$ as a relation on $\mathcal{R}(d)$ as $(G, H) \in \mathcal{R}(d)$ if $G \not\cong H$ and there is a switching σ on G such that $H = G^{\sigma}$. As a consequence of Theorem 1.3, Eggleton and Holton[8] defined *the* $\Sigma(d)$*-graph* of realization of d whose vertices are the graphs with degree sequence d; two vertices are adjacent in the $\Sigma(d)$-graph if one can be obtained from the other by a switching and they obtained the following theorem.

Theorem 1.4. The $\Sigma(d)$-graph is connected.

□

A graph transformation which we use in this thesis is called *the jumping transformation*, we state as follows:

For positive integers m and n with $0 \leq m \leq \binom{n}{2}$. Let $G \in \mathcal{G}(m,n)$ with $e \in E(G), f \in E(\overline{G})$. We define $G^{t(e,f)}$ or $G^t = (V(G^t), E(G^t))$ as the graph with $V(G^t) = V(G)$ and $E(G^t) = E(G - e + f)$. Then $t(e,f)$ is said to be a *jumping transformation* on G. We can see that the graph obtained from G by a jumping transformation has the same order and size as G. Figure 1.5 illustrates the jumping transformation t.

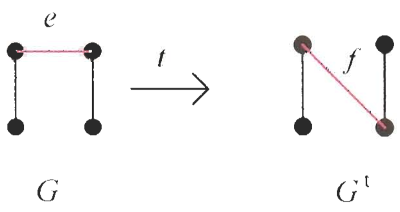


Figure 1.5: A jumping transformation

We define *the* $\mathcal{G}(m,n)$*-graph* as a graph having $\mathcal{G}(m,n)$ as its vertex set (two graphs in distinct vertices are non-isomorphic), two vertices are adjacent if one can be obtained from the other by a jumping transformation. It is clear that the $\mathcal{CG}(m,n)$-graph is the subgraph of the $\mathcal{G}(m,n)$-graph induced by $\mathcal{CG}(m,n)$. Let $\mathcal{G}(m,n;\delta \geq 1)$ be the subclass of $\mathcal{G}(m,n)$ which contains all graphs in $\mathcal{G}(m,n)$ with minimum degree at least 1. The $\mathcal{G}(m,n;\delta \geq 1)$-graph is also defined as the subgraph of the $\mathcal{G}(m,n)$-graph induced by $\mathcal{G}(m,n;\delta \geq 1)$. Note that for any graph G, $t(e,f)$ is well defined on G if and only if $t(f,e)$ is well defined on $G^{t(e,f)}$. Thus the $\mathcal{G}(m,n)$-graph, the $\mathcal{CG}(m,n)$-graph, and the $\mathcal{G}(m,n;\delta \geq 1)$-graph are simple.

It was proved recently by Punnim that the $\mathcal{G}(m,n)$-graph and the $\mathcal{CG}(m,n)$-graph are connected. The following results was shown by Punnim [28].

Theorem 1.5. Let $G, H \in \mathcal{G}(m,n)$. Then $G = H$ or there is a finite sequence of jumping transformations $t(e_1, f_1), t(e_2, f_2), \ldots, t(e_r, f_r)$ for some integer $1 \leq r \leq \binom{n}{2}$ such that $H = G^{t(e_1,f_1)t(e_2,f_2)\ldots t(e_r,f_r)}$.

□

As a consequence of Theorem 1.5, he obtains the following corollary.

Corollary 1.6. The $\mathcal{G}(m,n)$-graph is connected.

□

Similar result in the $\mathcal{CG}(m,n)$-graph and the $\mathcal{G}(m,n;\delta \geq 1)$-graph can also be obtained.

Theorem 1.7. Let $G, H \in \mathcal{CG}(m,n)$. Then $G = H$ or there is a finite sequence of jumping transformations $t(e_1, f_1), t(e_2, f_2), \ldots, t(e_r, f_r)$ such that for all $i = 1, 2, \ldots, r$, $G^{t(e_1,f_1)t(e_2,f_2)\ldots t(e_i,f_i)} \in \mathcal{CG}(m,n)$ and $H = G^{t(e_1,f_1)t(e_2,f_2)\ldots t(e_r,f_r)}$.

□

As a consequence of Theorem 1.7, he obtains the following corollary.

Corollary 1.8. The $\mathcal{CG}(m,n)$-graph is connected.

□

Theorem 1.9. Let $G, H \in \mathcal{G}(m,n;\delta \geq 1)$. Then $G = H$ or there is a finite sequence of jumping transformations $t(e_1, f_1), t(e_2, f_2), \ldots, t(e_r, f_r)$ such that for all $i = 1, 2, \ldots, r$, $G^{t(e_1,f_1)t(e_2,f_2)\ldots t(e_i,f_i)} \in \mathcal{G}(m,n;\delta \geq 1)$ and $H = G^{t(e_1,f_1)t(e_2,f_2)\ldots t(e_r,f_r)}$.

Proof. Note that $\mathcal{G}(m,n;\delta \geq 1) \neq \varnothing$ if and only if $m \geq \lceil \frac{n}{2} \rceil$. If $m = \lceil \frac{n}{2} \rceil$, then $\mathcal{G}(m,n;\delta \geq 1)$ contains only one graph namely, $\frac{n}{2}K_2$ when n is even and $\frac{n-3}{2}K_2 \cup P_3$ when n is odd. We define $G \to H$, if there is a finite sequence of jumping transformations $t(e_1,f_1), t(e_2,f_2), \ldots, t(e_r,f_r)$ such that for all $i = 1,2,\ldots,r$, $G^{t(e_1,f_1)t(e_2,f_2)\ldots t(e_i,f_i)} \in \mathcal{G}(m,n;\delta \geq 1)$ and $H = G^{t(e_1,f_1)t(e_2,f_2)\ldots t(e_r,f_r)}$. We consider two cases.

Case 1. When $\lceil \frac{n}{2} \rceil < m \leq n-2$ and $G \in \mathcal{G}(m,n;\delta \geq 1)$. If $m = n-p$, then G contains at least p components and G contains exactly p components if and only if G is a union of p disjoint trees. If G is not a union of p disjoint trees, then there exists $F \in \mathcal{G}(m,n;\delta \geq 1)$ where F is a union of p disjoint trees such that $G \to F$. Let $T \in \mathcal{G}(n-1,n;\delta \geq 1)$ be a tree. By Corollary 1.7, we have $T \to P_n$. Observe that if $n_1, n_2 \geq 3$, then $P_{n_1} \cup P_{n_2} \to P_{n_1+n_2-2} \cup P_2$. Thus for each $G \in \mathcal{G}(m,n,;\delta \geq 1)$ and $\lceil \frac{n}{2} \rceil < m = n-p \leq n-2$, there exists $F \in \mathcal{G}(m,n;\delta \geq 1)$ where $F = P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_p}$ such that $G \to F$ and $n_1 \geq 2, n_2 = n_3 = \ldots = n_p = 2$. Thus $G \to H$.

Case 2. If $m \geq n-1$ and $G \in \mathcal{G}(m,n;\delta \geq 1)$. If G is disconnected, then there exists a connected graph $G' \in \mathcal{G}(m,n;\delta \geq 1)$ such that $G \to G'$. Thus for $G, H \in \mathcal{G}(m,n;\delta \geq 1)$ there exists $G', H' \in \mathcal{CG}(m,n)$ such that $G \to G'$ and $H \to H'$. Since $G', H' \in \mathcal{CG}(m,n)$, we get a finite sequence of jumping transformations which transforms G' to H'. Therefore, $G \to H$.

□

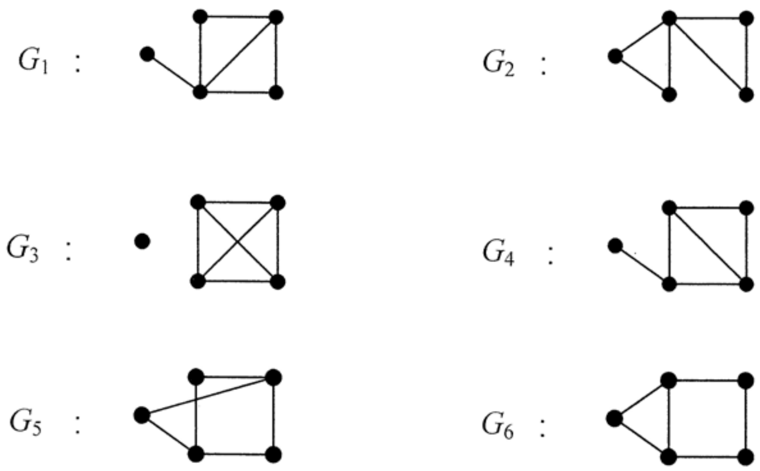
As a consequence of Theorem 1.9, we obtain the following corollary.

Corollary 1.10. The $\mathcal{G}(m,n;\delta \geq 1)$-graph is connected.

□

The following examples show that the $\mathcal{G}(6,5)$-graph, the $\mathcal{CG}(5,5)$-graph, and the $\mathcal{G}(4,5;\delta \geq 1)$-graph are connected.

Example 1.11. Consider six non-isomorphic graphs in $\mathcal{G}(6,5)$.



Let $G_1, G_2, G_3, G_4, G_5, G_6$ represent each set of non-isomorphic graphs where $G_i \ncong G_j$, $i \neq j$; $i,j = 1,2,\ldots,6$. We see that the $\mathcal{G}(6,5)$-graph is connected as shown below.

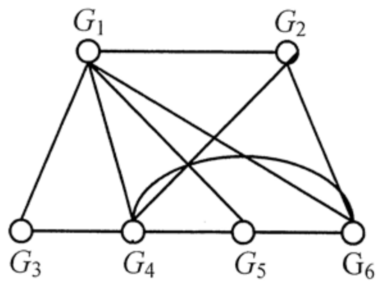
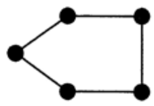
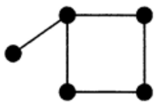
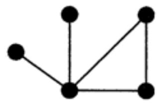


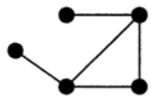
Figure 1.6: The $\mathcal{G}(6,5)$-graph

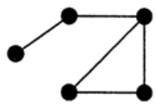
Example 1.12. Consider five non-isomorphic connected graphs in $\mathcal{CG}(5,5)$.

G_1 : 

G_2 : 

G_3 : 

G_4 : 

G_5 : 

Let G_1, G_2, G_3, G_4, G_5 represent each set of non-isomorphic connected graphs where $G_i \ncong G_j$, $i \neq j$; $i, j = 1, 2, 3, 4, 5$. We see that the $\mathcal{CG}(5,5)$-graph is connected as shown below.

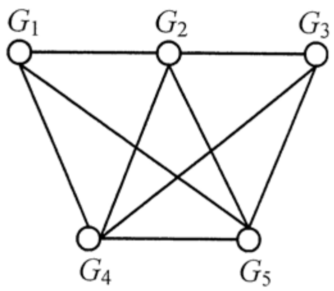
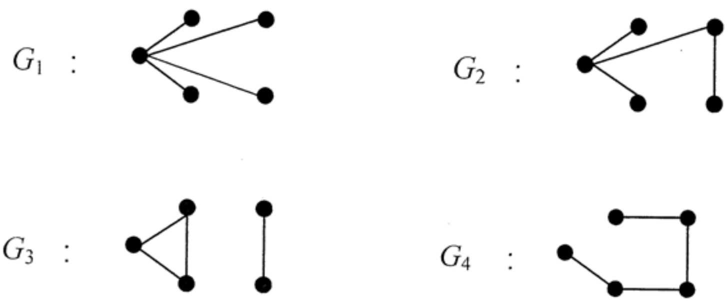


Figure 1.7: The $\mathcal{CG}(5,5)$-graph

Example 1.13. Consider four non-isomorphic graphs in $\mathcal{G}(4,5;\delta \geq 1)$.



Let G_1, G_2, G_3, G_4 represent each set of non-isomorphic graphs whose minimum degree at least 1 where $G_i \ncong G_j$, $i \neq j$; $i, j = 1, 2, 3, 4$. We see that the $\mathcal{G}(4,5;\delta \geq 1)$-graph is connected as shown below.

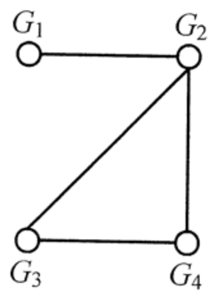


Figure 1.8: The $\mathcal{G}(4,5;\delta \geq 1)$-graph

1.3 Graph Parameters

In this section, We introduce the concepts of graph parameter and interpolation graph parameter. So we give definition of six graph parameters as follows:

Let $\mathcal{G}$ be the set of all graphs and $\mathbb{Z}$ be the set of positive integers, a function $f : \mathcal{G} \rightarrow \mathbb{Z}$ is called a *graph parameter* if f takes the same value on a class of isomorphic graphs. That is if $G \cong H$, then $f(G) = f(H)$. If f is a graph parameter and $\mathcal{J} \subseteq \mathcal{G}$, f is called an *interpolation graph parameter* over $\mathcal{J}$ if there exist integers a and b such that

$$\{f(G) : G \in \mathcal{J}\} = [a, b] = \{k \in Z : a \leq k \leq b\}.$$

A complete subgraph of a graph G is called a *clique* of G. The maximum order of cliques of G is called the *clique number* of G denoted by $\omega(G)$. We can see that if F is a subgraph of G, then $\omega(F) \leq \omega(G)$ and $\omega(G-e) \geq \omega(G)-1$ where $e \in E(G)$.

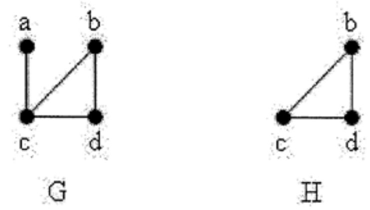


Figure 1.9: H is a maximum complete subgraph of G

Examples 1.14 and 1.15 illustrate the change of $\omega(G)$ of a graph, if some edges are deleted and some edges are added.

Example 1.14. Consider a graph $G \in \mathcal{G}(9, 6)$.

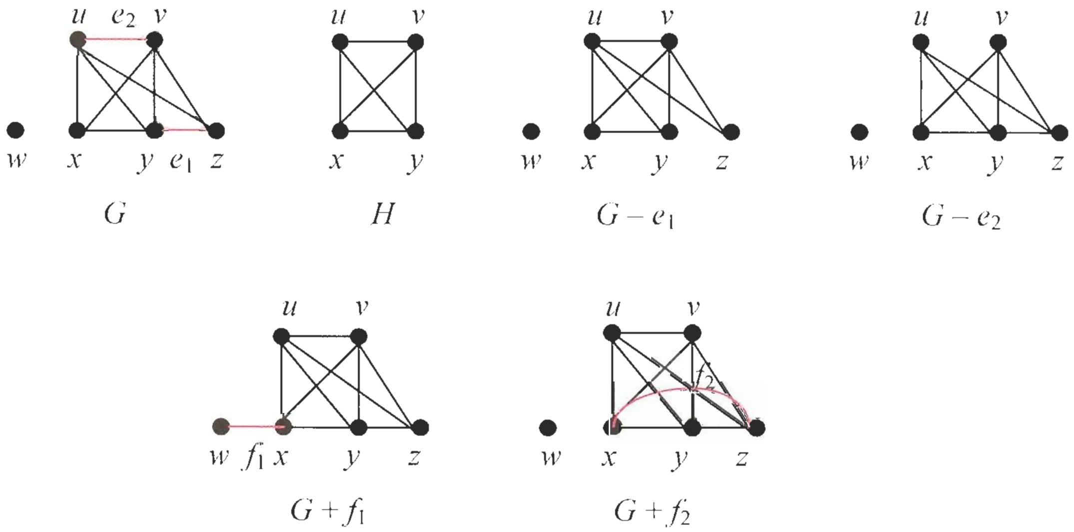


Figure 1.10: The change of $\omega(G)$ of a graph

In Figure 1.10, since H is a maximum complete subgraph of G with $|V(H)| = 4$, $\omega(G) = 4$. We see that if we delete the edge e_1 from G, then $\omega(G - e_1) = 4$ and if we delete the edge e_2 from G, then $\omega(G - e_2) = 3$. We see that if we add the edge f_1 to G, then $\omega(G + f_1) = 4$ and if we add the edge f_2 to G, then $\omega(G + f_2) = 5$.

Example 1.15. Consider a connected graph $G \in \mathcal{CG}(7,6)$.

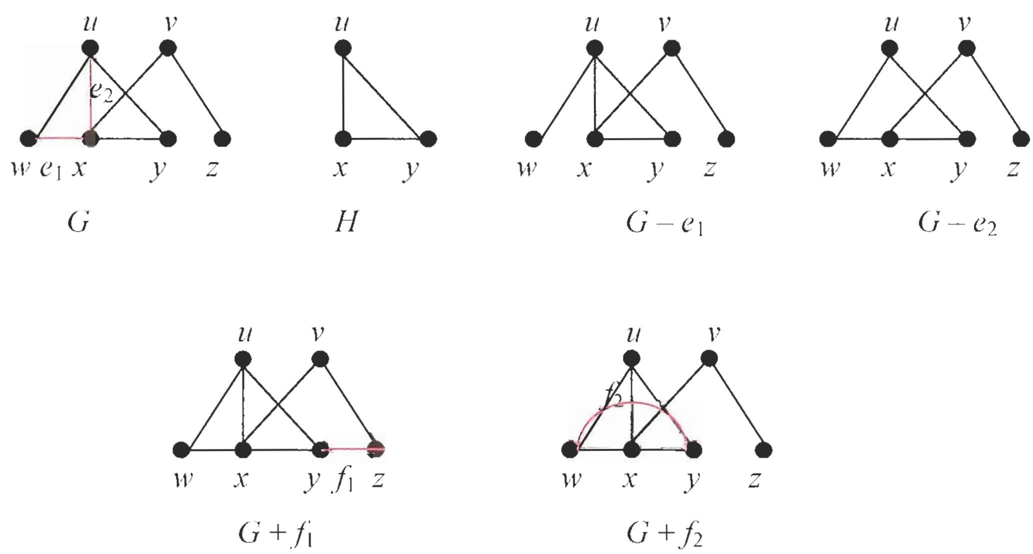


Figure 1.11: The change of $\omega(G)$ of a connected graph

In Figure 1.11, since H is a maximum complete subgraph of G with $|V(H)| = 3$, $\omega(G) = 3$. We see that if we delete the edge e_1 from G, then $\omega(G - e_1) = 3$ and if we delete the edge e_2 from G, then $\omega(G - e_2) = 2$. We see that if we add the edge f_1 to G, then $\omega(G+f_1) = 3$ and if we add the edge f_2 to G, then $\omega(G+f_2) = 4$.

A subset S of the vertex set V of a graph G is an *independent set* of G if the induced subgraph $G[S]$ of G has no edge. An independent set of G containing maximum number of vertices is called a maximum independent set of G. The number of vertices of a maximum independent set of G is called the *independence number* of G denoted by $\alpha_0(G)$. It is clear that $\alpha_0(G) = \omega(\overline{G})$.

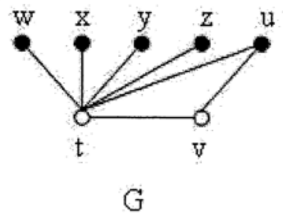


Figure 1.12: A maximum independent set of G

In Figure 1.12, the set $\{w, x, y, z, u\}$ is a maximum independent set of G, indicated by black vertices and so $\alpha_0(G) = 5$.

A vertex of a graph $G = (V, E)$ is said to *cover* the edges incident with it. A *vertex cover* of a graph G is a set of vertices covering all the edges of G. The minimum cardinality of a vertex cover of a graph G is called the *vertex covering number* of G denoted by $\beta_0(G)$.

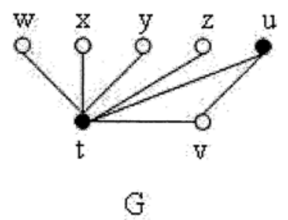


Figure 1.13: A minimum vertex cover of G

In Figure 1.13, the set $\{u, t\}$ is a minimum vertex cover of G, indicated by black vertices and so $\beta_0(G) = 2$.

A *k-coloring* of a graph $G = (V, E)$ is a partition of its vertex set V as $V_1 \cup V_2 \cup \ldots \cup V_k$ such that no two vertices in V_i $(1 \leq i \leq k)$ are adjacent. The V_i's are called the *color classes.* A function $f : V \rightarrow \{1, 2, \ldots, k\}$ such that

$f(v) = i$ for each $v \in V_i (1 \le i \le k)$ is called a *color function*. If G has a *k-coloring*, it is said to be *k-colorable* and the minimum integer k for which G is k-colorable called the *chromatic number* of G and is denoted by $\chi(G)$. If $\chi(G) = k$, we say that G is *k-chromatic*. Figure 1.14 illustrates $\chi(G) = 3$. We can see that if H is a subgraph of G, then $\chi(H) \le \chi(G)$ and $\chi(G-e) \ge \chi(G)-1$ where $e \in E(G)$.

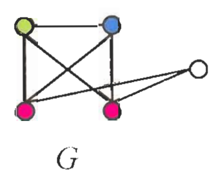


Figure 1.14: A graph G is 3-chromatic

Examples 1.16 and 1.17 illustrate the change of $\chi(G)$ of a graph, if some edges are deleted and some edges are added.

Example 1.16. Consider a graph $G \in \mathcal{G}(9, 6)$.

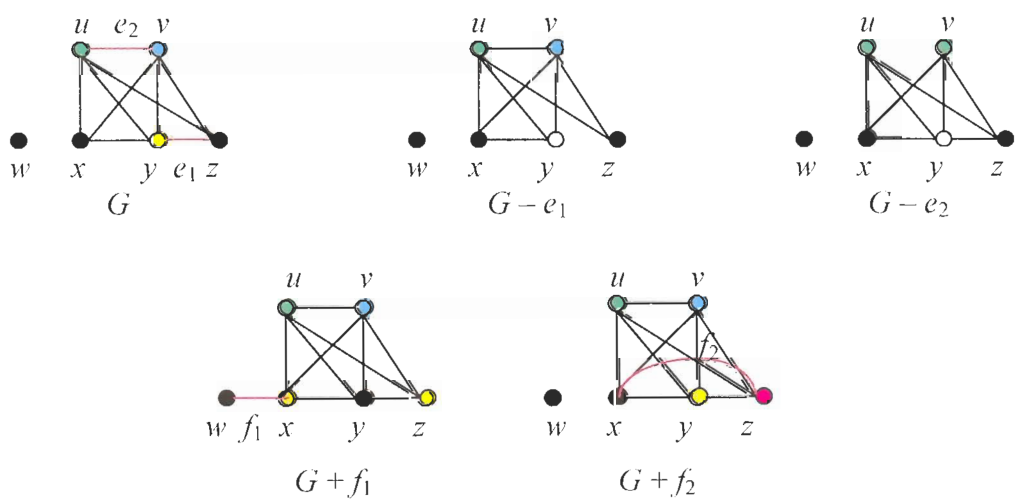


Figure 1.15: The change of $\chi(G)$ of a graph

In Figure 1.15, we have $\chi(G) = 4$. We see that if we delete the edge e_1 from G, then $\chi(G - e_1) = 4$ and if we delete the edge e_2 from G, then $\chi(G - e_2) = 3$. We see that if we add the edge f_1 to G, then $\chi(G + f_1) = 4$ and if we add the edge f_2 to G, then $\chi(G + f_2) = 5$.

Example 1.17. Consider a connected graph $G \in \mathcal{CG}(7, 6)$.

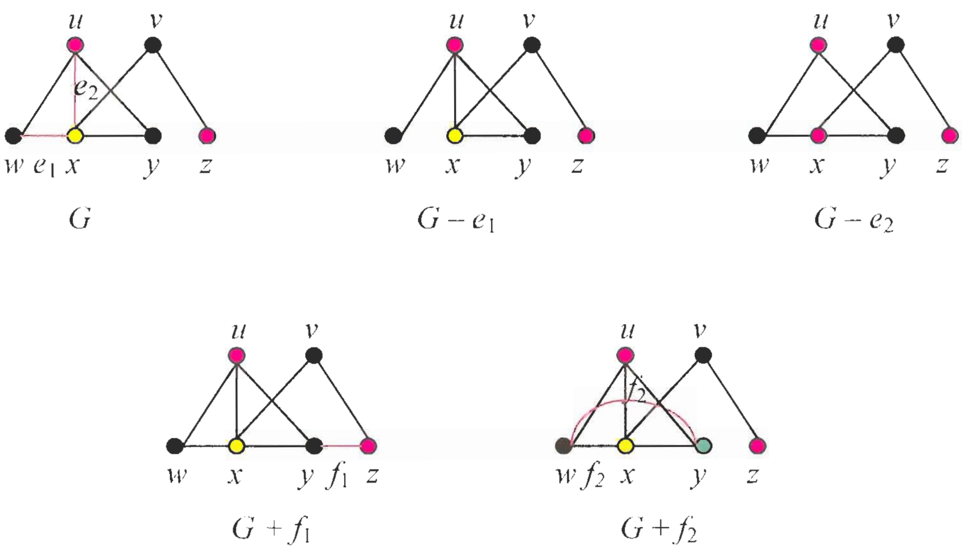


Figure 1.16: The change of $\chi(G)$ of a connected graph

In Figure 1.16, we have $\chi(G) = 3$. We see that if we delete the edge e_1 from G, then $\chi(G - e_1) = 3$ and if we delete the edge e_2 from G, then $\chi(G - e_2) = 2$. We see that if we add the edge f_1 to G, then $\chi(G + f_1) = 3$ and if we add the edge f_2 to G, then $\chi(G + f_2) = 4$.

A subset M of the edge set E of a graph $G = (V, E)$ is an *independent edge set* or *matching* in G if no two distinct edges in M have a common vertex. A matching M is *maximum* in G if there is no matching M' of G with $|M'| > |M|$.

The cardinality of a maximum matching of G is called the *matching number* of G, denoted by $\alpha_1(G)$.

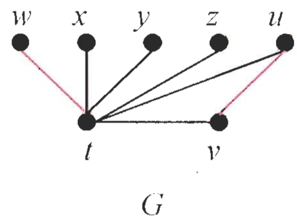


Figure 1.17: A maximum matching of G

In Figure 1.17, the set $\{wt, uv\}$ is a maximum matching of G, indicated by red edges and so $\alpha_1(G) = 2$.

Examples 1.18 and 1.19 illustrate the change of $\alpha_1(G)$ of a graph, if some edges are deleted and some edges are added.

Example 1.18. Consider a graph $G \in \mathcal{G}(3, 6)$.

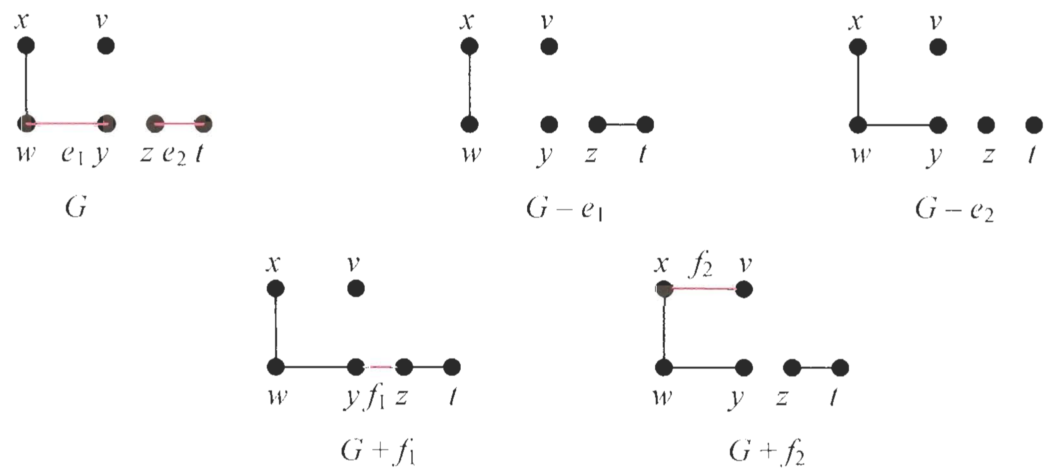


Figure 1.18: The change of $\alpha_1(G)$ of a graph

In Figure 1.18, the set $H = \{yw, zt\}$ is a maximum matching of G, $\alpha_1(G) = 2$. We see that if we delete the edge e_1 from G, then $\alpha_1(G - e_1) = 2$ and if we delete the edge e_2 from G, then $\alpha_1(G - e_2) = 1$. We see that if we add the edge f_1 to G, then $\alpha_1(G + f_1) = 2$ and if we add the edge f_2 to G, then $\alpha_1(G + f_2) = 3$.

Example 1.19. Consider a connected graph $G \in \mathcal{CG}(5, 6)$.

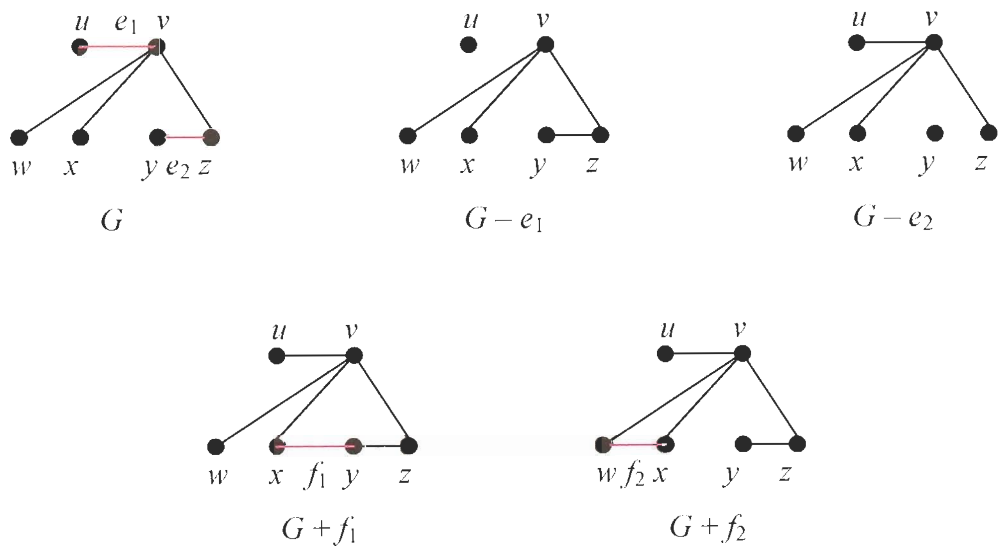
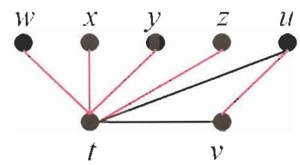


Figure 1.19: The change of $\alpha_1(G)$ of a connected graph

In Figure 1.19, the set $H = \{uv, yz\}$ is a maximum matching of G, $\alpha_1(G) = 2$. We see that if we delete the edge e_1 from G, then $\alpha_1(G - e_1) = 2$ and if we delete the edge e_2 from G, then $\alpha_1(G - e_2) = 1$. We see that if we add the edge f_1 to G, then $\alpha_1(G + f_1) = 2$ and if we add the edge f_2 to G, then $\alpha_1(G + f_2) = 3$.

An edge of a graph $G = (V, E)$ is said to *cover* the two vertices incident with it. An *edge cover* of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of G is called the *edge covering number* of G denoted by $\beta_1(G)$.



G

Figure 1.20: A minimum edge cover of G

In Figure 1.20, the set $\{wt, xt, yt, zt, uv\}$ is a minimum edge cover of G, indicated by red edges and so $\beta_1(G) = 5$.

In Chapter 2, we review significant related results on interpolation theorems on graph parameters. We analyze results of interpolation property for six graph parameters ω, α_0, β_0, χ, α_1, and β_1 over $\mathcal{G}(m,n)$ and $\mathcal{CG}(m,n)$ in Chapter 3. We also find the minimum and maximum value for all those graph parameters in Chapter 4.