CHAPTER 1 INTRODUCTION

1.1 Rational

Influenza is caused by Influenza virus. It is inflection in respiratory the virus may spread in to the lungs. The patient from this virus will be high flu head age and very muscle pain. Normally it is found in childhood but death rate is found in elderly or congenital disease for example heart disease, lung disease, liver disease, kidney disease etc. The vaccination is the best way to protect. It can reduce the in a rate of hospital bed pause working and suspending in teenager. This disease easily infect by respiratory contact. The way to spread is cough. The virus transmission on eye and mouse. The incubation period is a bout 1-4 day. The patient will be tired, anorexia, nausea, severe pain, limb pain around eye, high temperature. The fever and queasy will gone in two day. The contact period is 1 day before indicate symptom or the five days after performed symptom. In child may be spread disease in 6 day be form and spread along 10 days [1]. Current approach for the prevention and control of influenza epidemics is annual vaccination with a trivalent inactivated influenza vaccine [2]. And numerical methods developed for the solution of the model equations.

1.2 Literature Review

Moghadas.et.al[3] described quantitative analysis of a mathematical model for the transmission dynamics of a childhood disease in the presence of a preventive vaccine. The global stability analysis is proved by a Lyapunov function. It is shown that the disease can be eradicated from the population it the vaccination coverage level exceed a certain threshold value.

The implicit finite difference schemes are developed. The numerical schemes showed that that they have the identical stability properties to the continuous model. Numerical results with different initial condition, parameters and time step sizes converge to the disease free and endemic equilibrium points [4].

The non-standard numerical scheme for a $SIRS$ model for RSV transmission is developed and numerical results is compared to the well-known explicit methods [5].

Samsuzzoha.et.al [6] formulate a H1N1 influenza model. The numerical result of three initial population are shown. The effect of disease transmission coefficient is a constant. The reproduction number is illustrated. Numerical experimental shows that system supports the existence and damped oscillation depend on initial population, the disease transmission rate and diffusion.

Samsuzzoha [7] developed the vaccinated diffusive compartmental epidemic model to explore the impact of vaccination as well as diffusion on the transmission dynamics of influenza by using the operator splitting method.

The impact of vaccination as well as diffusion on a vaccinated diffusive epidemic model is investigated. Sensitivity analysis of the reproduction number is described. The model is analyzed to develop the criteria for control influenza epidemic [8].

1.3 Objective

To simulate the dynamics of influenza epidemic model with vaccination and diffusion by standard finite difference method and non- standard finite difference method.

1.4 Scope

Construct the numerical method using standard and non-standard finite-difference method for influenza epidemic model. And analyze the stability of standard finite-difference method.

CHAPTER 2 THEORIES

In this chapter, the background in mathematical are described in the following sections.

2.1 Finite Difference Scheme

Given a function *f(x)* shown in Figure. 1, we can approximate its derivative, slope or tangent at **P** by the slope of the arcs **PB**, **PA**, or **AB**, for obtaining the forward difference, backward-difference, and central-difference formulas respectively [9].

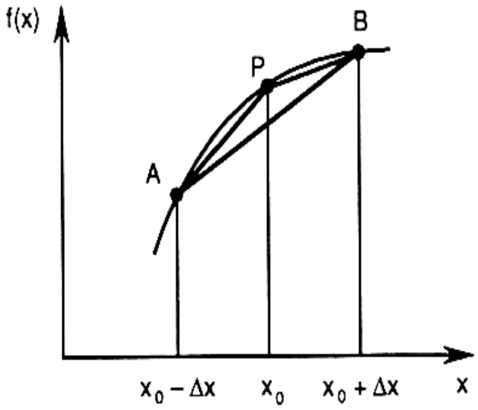

Figure 2.1 Estimates for the derivative of $f(x)$ at P by using forward, backward, and central differences.

The approach used for obtaining above finite difference equations is Taylor's series:

$$f(x_0+\Delta x)=f(x_0)+\Delta x f'(x_0)+\frac{1}{2!}(\Delta x)^2 f''(x_0)+\frac{1}{3!}(\Delta x)^3 f'''(x_0)+O(\Delta x)^4, \qquad (2.1)$$

and

$$f(x_0-\Delta x)=f(x_0)-\Delta x f'(x_0)+\frac{1}{2!}(\Delta x)^2 f''(x_0)-\frac{1}{3!}(\Delta x)^3 f'''(x_0)+O(\Delta x)^4, \qquad (2.2)$$

where $O(\Delta x)^4$ is the error introduced by truncating the series.
To subtract (2.1.1) by (2.1.2), we can obtain
$f(x_0+\Delta x)-f(x_0-\Delta x)=2\Delta x f'(x_0)+O(\Delta x)^3,$
Which could be – written as

$f'(x_0)\cong\dfrac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}+O(\Delta x)^2$, i.e. the central-difference formula. Note that the $O(\Delta x)^2$ means the truncation error is the order of $O(\Delta x)^2$ for the central - difference. The forward-difference and backward-difference formulas could be obtained by re-arranging (2.1) and (2.2) respectively, and we have

$f'(x_0) \cong \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + O(\Delta x),$ for forward difference,

and

$f'(x_0) \cong \frac{f(x_0) - f(x_0-\Delta x)}{\Delta x} + O(\Delta x),$ for backward difference. We can find the truncation errors of these two formulas are of order Δx.

Upon adding (2.1) and (2.2),

$$f(x_0+\Delta x) - f(x_0-\Delta x) = 2f(x_0) + (\Delta x)^2 f''(x_0) + O(\Delta x)^4,$$

and we have

$$f''(x_0) \cong \frac{f(x_0+\Delta x) - 2f(x_0) + f(x_0-\Delta x)}{(\Delta x)^2} + O(\Delta x)^2,$$

Higher order finite difference approximations can be obtained by taking more terms in Taylor series expansion.

Table 2.1 Finite Difference Approximations for Φ_x and Φ_{xx}, where FD = Forward Difference, BD = Backward Difference, and CD = Central Difference

Derivative	Finite Difference Approximation	Type	Error
Φ_x	$\frac{\Phi_{i+1}-\Phi_i}{\Delta x}$	FD	$O(\Delta x)$
	$\frac{\Phi_i-\Phi_{i+1}}{\Delta x}$	BD	$O(\Delta x)$
	$\frac{\Phi_{i+1}-\Phi_{i-1}}{\Delta x}$	CD	$O(\Delta x)^2$
Φ_{xx}	$\frac{\Phi_{i+2}-2\Phi_{i+1}+\Phi_i}{(\Delta x)^2}$	FD	$O(\Delta x)^2$
	$\frac{\Phi_i-2\Phi_{i-1}+\Phi_{i-2}}{(\Delta x)^2}$	BD	$O(\Delta x)^2$
	$\frac{\Phi_{i+1}-2\Phi_i+\Phi_{i-1}}{(\Delta x)^2}$	CD	$O(\Delta x)^2$

To apply the difference method to find the solution of a function $\Phi(x,t)$, we divide the solution region in $x-t$ plane into equal rectangles or meshes of sides Δx and Δt. The derivatives of Φ at the $(i, j)^{th}$ node are shown in the table, where

$x = i\Delta x$

$t = j\Delta t$

Thus we have

$$\begin{aligned}
\Phi_x\Big|_{i,j} &\approx \frac{\Phi(i+1,j)-\Phi(i-1,j)}{2\Delta x},\\
\Phi_t\Big|_{i,j} &\approx \frac{\Phi(i,j+1)-\Phi(i,j-1)}{2\Delta t},\\
\Phi_{xx}\Big|_{i,j} &\approx \frac{\Phi(i+1,j)-2\Phi(i,j)+\Phi(i-1,j)}{(\Delta x)^2},\\
\Phi_{tt}\Big|_{i,j} &\approx \frac{\Phi(i,j+1)-2\Phi(i,j)+\Phi(i,j-1)}{(\Delta t)^2},
\end{aligned} \tag{2.3}$$

2.2 The Von Nuemann Method

The concept of stability is concerned with boundedness of the solution of the finite-difference equations and is examined by finding conditions under which the difference between the theoretical and numerical solution of the difference equation given at the mesh point $(mh,\ nl)$ by

$$Z_m^n = U_m^n - \tilde{U}_m^n$$

remains bounded as n increases, for fixed h and E. The following methods are used in this thesis for examining the stability of finite-difference schemes in Chapters 3. The von Nuemann method is the most widely-used method for determining the stability (or instability) of a finite difference approximation. Here, a harmonic decomposition is made of the error Z at the grid points on a given time level, leading to the error function

$$E(x) = \sum_j A_j e^{i\beta_j x} \tag{2.4}$$

Where $|\beta_j|$ is the frequency of the error, *j* are arbitrary in general and *i* is the complex number $\sqrt{-1}$. It is necessary to consider only the single term $e^{i\beta x}$ where β is any real number. For convenience, suppose that the time level being considered corresponds to $t=0$. To investigate the error propagation as t increases, it is necessary to find a solution
of the finite-difference equation which reduces to $e^{i\beta x}$ when $t=0$. Let such a solution be

$$E(t,x) \approx e^{\alpha t} e^{i\beta x} \tag{2.5}$$

The original error component $e^{i\beta x}$ will not grow with time if

$$\left|e^{\alpha l}\right| \le 1 \tag{2.6}$$

for all α. This is von Neumann's criterion for stability, sometimes called the von Neumann necessary condition for stability.

CHAPTER 3 METHODOLOGY

3.1 Introduction

The *SVEIRS* Influenza epidemic model with vaccination consists of five non-linear partial differential equations (PDEs). The disease may spread faster in some part than in others and it is necessary to allow the variables to depend on space as well as time. It would thus seem natural to extend the model by including diffusional effects, allowing for investigation of the spatial spread of Influenza epidemic model. In this chapter a reaction-diffusion equation will be studied and extended to incorporate diffusion in one-space dimension to enable the geographic spread of the disease in a population first-order in time and second-order in spaces based on finite difference method are constructed to obtained the numerical solution of the proposal model.

This model for influenza is based on the standard *SEIRS* model. The population is divided into five subgroups: susceptible, *S*, vaccinated, *V*, exposed, *E*, infective, *I* and recovered, *R*. The total population size is denoted by $N = S + V + E + I + R.$ The model without diffusion term is represented by the following system of ordinary differential equations:

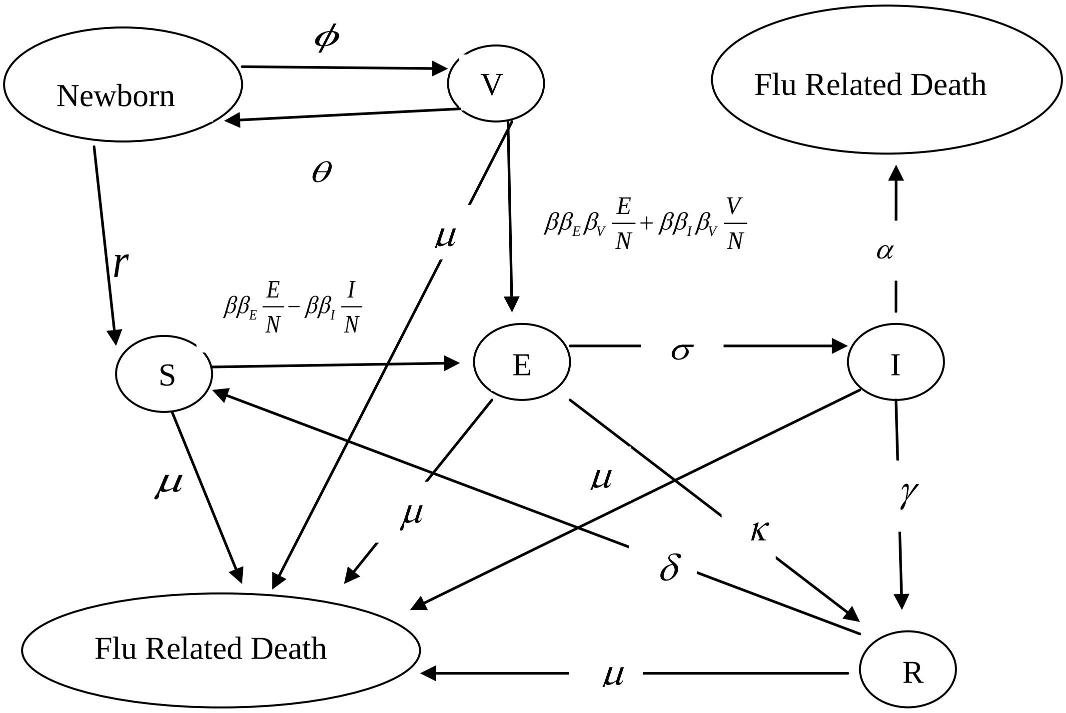

Figure 3.1 The flow diagram of the *SVEIRS* model.

$$\begin{aligned}
\frac{dS}{dt} &= -\beta\beta_E \frac{ES}{N} - \beta\beta_I \frac{IS}{N} - \phi S - \mu S + \delta R + \theta V + rN, \\
\frac{dV}{dt} &= -\beta\beta_E\beta_V \frac{EV}{N} - \beta\beta_I\beta_V \frac{IV}{N} - \mu V - \theta V + \phi S, \\
\frac{dE}{dt} &= \beta\beta_E \frac{ES}{N} + \beta\beta_I \frac{IS}{N} + \beta\beta_E\beta_V \frac{EV}{N} + \beta\beta_I\beta_V \frac{IV}{N} - (\mu + \kappa + \sigma)E, \\
\frac{dI}{dt} &= \sigma E - (\mu + \alpha + \gamma)I, \\
\frac{dR}{dt} &= \kappa E + \gamma I - \mu R - \delta R
\end{aligned} \tag{3.1}$$

and $N = S + V + E + I + R$, $\frac{dN}{dt} = rN - \mu N - \alpha I$.

A transfer diagram for this model between these epidemic classes is shown in Figure 3.1 under the following assumptions:

The susceptible population is increased by new born and those who have loss of immunity due to earlier infection and vaccination. The susceptible population is reduced through vaccination (moving to class *V*), infection (moving to class *E*) and natural death.

The vaccinated population is increased with the vaccination of susceptible. The vaccinated populations have susceptibility to infection reduced by a factor $(1-\beta_V)$ with $0 \le \beta_V \le 1$. The vaccinated class is reduced by the infection (moving to class *E*), natural death and waning of vaccine-based immunity.

The exposed population is increased by infected individuals who remain susceptible even after being vaccinated. The exposed population is also increased by infected, who are not vaccinated yet. The exposed population is reduced by the recovery (moving to class *R*), natural death and onset of infection (moving to class *I*). The population during the exposed stage has a low level of infectivity. Technically, this compartment represents mildly infectious stage.

The population of infective class is increased by a fraction of exposed individuals becoming infective. The population of infective individuals is reduced by natural death, disease related death and recovery from the disease.

The recovered population gets increased with the recovered individual from exposed and infective classes. The recovered population is reduced by natural death and loss of immunity.

In order to analyze in terms of the proportions of susceptible, vaccinated, exposed, infectious and recovered individuals, let $s = \frac{S}{N}$, $v = \frac{V}{N}$, $e = \frac{E}{N}$, $i = \frac{I}{N}$ and $r_1 = \frac{R}{N}$ denote the fraction of the classes *S*; *V*; *E*; *I* and *R* in the population, respectively. After some calculations and replacing *S* by *s*; *V* by *v*; *E* by *e*; *I* by *i* and *R* by r_1, Equation. (3.1) can be written as

$$\begin{aligned}
\frac{dS}{dt} &= -\beta_1 ES - \beta_2 IS + \alpha IS - \phi S - rS + \delta R + \theta V + r \\
\frac{dV}{dt} &= -\beta_3 EV - \beta_4 IV + \alpha IV - rV - \theta V + \phi S \\
\frac{dE}{dt} &= \beta_1 ES + \beta_2 IS + \beta_3 EV + \beta_4 IV + \alpha IE - w_1 E \\
\frac{dI}{dt} &= \sigma E - w_2 I + \alpha I^2 \\
\frac{dR}{dt} &= \kappa E + \gamma I - rR - \delta R + \alpha IR \\
S + V &+ E + I + R = 1
\end{aligned} \tag{3.2}$$

where $\beta_1 = \beta\beta_E, \beta_2 = \beta\beta_I, \beta_3 = \beta\beta_E\beta_V, \beta_4 = \beta\beta_I\beta_V, w_1 = r + \kappa + \sigma$ and $w_2 = r + \alpha + \gamma$.

The Influenza epidemic model with vaccination and diffusion by Samsuzzoha et al [9],

$$\begin{aligned}
\frac{\partial S}{\partial t} &= -\beta_1 ES - \beta_2 IS + \alpha IS - \phi S - rS + \delta R + \theta V + r + d_1 \frac{\partial^2 S}{\partial x^2} \\
\frac{\partial V}{\partial t} &= -\beta_3 EV - \beta_4 IV + \alpha IV - rV - \theta V + \phi S + d_2 \frac{\partial^2 V}{\partial x^2} \\
\frac{\partial E}{\partial t} &= \beta_1 ES + \beta_2 IS + \beta_3 EV + \beta_4 IV + \alpha IE - w_1 E + d_3 \frac{\partial^2 E}{\partial x^2} \\
\frac{\partial I}{\partial t} &= \sigma E - w_2 I + \alpha I^2 + d_4 \frac{\partial^2 I}{\partial x^2} \\
\frac{\partial R}{\partial t} &= \kappa E + \gamma I - rR - \delta R + \alpha IR + d_5 \frac{\partial^2 R}{\partial x^2} \\
S + V &+ E + I + R = 1
\end{aligned} \tag{3.3}$$

where $\beta_1 = \beta\beta_E, \beta_2 = \beta\beta_I, \beta_3 = \beta\beta_E\beta_V, \beta_4 = \beta\beta_I\beta_V, w_1 = r + \kappa + \sigma$ and $w_2 = r + \alpha + \gamma$.

The initial conditions of (3.3) are of the form,

$$S(x,0) = S_0(x), V(x,0) = V_0(x), E(x,0) = E_0(x), \text{ for } -L \le x \le L,$$
$$I(x,0) = I_0(x), R(x,0) = R_0(x), \tag{3.4}$$

and the boundary conditions are chosen to be

$$\frac{\partial S(\pm 2,t)}{\partial x} = \frac{\partial V(\pm 2,t)}{\partial x} = \frac{\partial E(\pm 2,t)}{\partial x} = \frac{\partial I(\pm 2,t)}{\partial x} = \frac{\partial R(\pm 2,t)}{\partial x} = 0, \; t > 0, \tag{3.5}$$

when $L = 2$.

Table 3.1 Model parameters [7]

Parametes	**Description**	**Value**
β	Contact rate	$0.514\ \text{day}^{-1}$
β_E	Ability to cause infection by exposed individuals	0.250
β_I	Ability to cause infection by infectious individuals	1.000
β_v	Ability to cause infection by vaccination individuals	$0.1,\ 0.2\,\text{day}^{-1}$
σ	Rate of latency	$0.5\ \text{day}^{-1}$
γ	Rate of clinically ill	$0.2\ \text{day}^{-1}$
δ	Rate of duration of immunity loss	$1/365\ \text{day}^{-1}$
μ	Natural mortality rate	$5.5\times10^{-8}\ \text{day}^{-1}$
r	Birth rate	$7.140\times10^{-5}\ \text{day}^{-1}$
κ	Recovery rate of latents	$1.857\times10^{-4}\ \text{day}^{-1}$
α	Flu induced mortality rate	$9.3\times10^{-6}\ \text{day}^{-1}$
θ	Rate of susceptible	$1/365\ \text{day}^{-1}$
ϕ	Rate of vaccination	Variable
d_1	Diffusivity constants of susceptible population	0.05
d_2	Diffusivity constants of vaccinated population	0.05
d_3	Diffusivity constants of exposed population	0.025
d_4	Diffusivity constants of infectious population	0.001
d_5	Diffusivity constants of recovered population	0.0

3.2 Discretization and Notations

A solution of the system of partial equation (3.3) - (3.5) may be computed by finite-difference methods by discretizing the space interval $[-L, L]$ into M sub-intervals each of width $h > 0,$ and the time interval $t \ge 0$ is discretized in steps each of length $\ell > 0.$ The open rectangle $\Omega = [-L \le x \le L] \times [t > 0]$ and its boundary $\partial\Omega,$

consisting of the lines $x=-L,\ x=L$ and $t=0$, are covered by a rectangular grid having coordinates of the form $(x,t)=(x_m,t_n)=(-L+mh,nl)$ where $x_m=-L+mh,\ t_n=n\ell(n=0,1,2,..)$. The solutions of (3.3) - (3.5) at the typical mesh point (x_m,t_n) are $S(x_m,t_n),V(x_m,t_n)\ E(x_m,t_n),I(x_m,t_n),R(x_m,t_n)$ denoted by $S_m^n,V_m^n,E_m^n,I_m^n,$ and R_m^n respectively. The solutions of numerical approximations at the same mesh point will be denoted by $\tilde{S}_m^n,\tilde{V}_m^n,\tilde{E}_m^n,\tilde{I}_m^n,$ and $\tilde{R}_m^n$ respectively.

3.3 Standard Finite-Difference Methods

Finite-difference methods are constructed by approximating all time derivative in (3.3) by the first-order forward-difference replacement

$$\frac{\partial u}{\partial t}=\frac{u(x,t+\ell)-u(x,t)}{\ell}+O(\ell)\quad \text{as } \ell\to 0, \tag{3.6}$$

and the space derivative by the second-order approximant

$$\frac{\partial^2 u}{\partial x^2}=\frac{u(x-h,t+\ell)-2u(x,t+\ell)+u(x+h,t+\ell)}{h^2}+O(h^2)\ \text{ as }\ h\to 0. \tag{3.7}$$

In which $u\left(x,t\right)=S\left(x,t\right)$ or $V\left(x,t\right)$ or $E\left(x,t\right)$ or $I\left(x,t\right)$ or $R(x,t)$.

Using (3.6) and (3.7) making appropriate approximations for the right hand-side functions of the model (3.1), we obtain

$$\begin{aligned}\frac{\tilde{S}_m^{n+1}-\tilde{S}_m^n}{\ell}&=\left(\beta_1\tilde{E}_m^n+\beta_2\tilde{I}_m^n+\alpha\tilde{I}_m^n-\phi-r\right)\tilde{S}_m^n+\delta\tilde{R}_m^n+\theta\tilde{V}_m^n+r\\&\quad+d_1\left[\frac{\tilde{S}_{m-1}^{n+1}-2\tilde{S}_m^{n+1}+\tilde{S}_{m+1}^{n+1}}{h^2}\right],\end{aligned} \tag{3.8}$$

$$\begin{aligned}\frac{\tilde{V}_m^{n+1}-\tilde{V}_m^n}{\ell}&=\left(-\beta_3\tilde{E}_m^n-\beta_4\tilde{I}_m^n+\alpha\tilde{I}_m^n-r-\theta\right)\tilde{V}_m^n+\phi\tilde{S}_m^n\\&\quad+d_2\left[\frac{\tilde{V}_{m-1}^{n+1}-2\tilde{V}_m^{n+1}+V_{m+1}^{n+1}}{h^2}\right],\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{\tilde{E}_m^{n+1}-\tilde{E}_m^n}{\ell}&=\left(\beta_1\tilde{S}_m^n+\beta_3\tilde{V}_m^n+\alpha\tilde{I}_m^n-w_1\right)\tilde{E}_m^n+\beta_2\tilde{I}_m^n\tilde{S}_m^n+\beta_4\tilde{I}_m^n\tilde{V}_m^n\\&\quad+d_3\left[\frac{\tilde{E}_{m-1}^{n+1}-2\tilde{E}_m^{n+1}+\tilde{E}_{m+1}^{n+1}}{h^2}\right],\end{aligned} \tag{3.10}$$

$$\frac{\tilde{I}_m^{n+1}-\tilde{I}_m^{n}}{\ell}=\left(-w_2+\alpha\tilde{I}_m^{n}\right)\tilde{I}_m^{n}+\sigma\tilde{E}_m^{n}+d_4\left[\frac{\tilde{I}_{m-1}^{n+1}-2\tilde{I}_m^{n+1}+\tilde{I}_{m+1}^{n+1}}{h^2}\right], \tag{3.11}$$

$$\frac{\tilde{R}_m^{n+1}-\tilde{R}_m^{n}}{\ell}=\left(-r-\delta+\alpha\tilde{I}_m^{n}\right)\tilde{R}_m^{n}+\kappa\tilde{E}_m^{n}+\gamma\tilde{I}_m^{n}$$
$$d_5\left[\frac{\tilde{R}_{m-1}^{n+1}-2\tilde{R}_m^{n+1}+\tilde{R}_{m+1}^{n+1}}{h^2}\right], \tag{3.12}$$

After rearranging, the scheme becomes

$$\begin{aligned}&-d_1p\tilde{S}_{m-1}^{n+1}+\left(1+2d_1p\right)\tilde{S}_m^{n+1}-d_1p\tilde{S}_{m-1}^{n+1}\\&=\left[1-(\tilde{E}_m^{n}+\beta_2\tilde{I}_m^{n}+\alpha\tilde{I}_m^{n}-\phi-r)\ell\right]\tilde{S}_m^{n}+r\ell+\delta\ell\tilde{R}_m^{n}+\theta\ell\tilde{V}_m^{n}\ .\end{aligned} \tag{3.13}$$

$$\begin{aligned}&-d_2p\tilde{V}_{m-1}^{n+1}+\left(1+2d_2p\right)\tilde{V}_m^{n+1}-d_2pv_{m-1}^{n+1}\\&=\left[1+\left(-\beta_2\beta\tilde{E}_m^{n}-\beta_3\tilde{I}_m^{n}+\alpha\tilde{I}_m^{n}-r-\theta\right)\ell\right]\tilde{V}_m^{n}+\phi\ell\tilde{S}_m^{n}.\end{aligned} \tag{3.14}$$

$$\begin{aligned}&-d_3p\tilde{E}_{m-1}^{n+1}+\left(1+2d_3p\right)\tilde{E}_m^{n+1}-d_3p\tilde{E}_{m-1}^{n+1}\\&=\left[1+\left(\beta_1\tilde{S}_m^{n}+\beta_3\tilde{V}_m^{n}+\alpha\tilde{I}_m^{n}-w_1\right)\ell\right]\tilde{E}_m^{n}+\beta_2\ell\tilde{I}_m^{n}\tilde{S}_m^{n}+\beta_4\ell\tilde{I}_m^{n}\tilde{V}_m^{n}.\end{aligned} \tag{3.15}$$

$$\begin{aligned}&-d_4p\tilde{I}_{m-1}^{n+1}+\left(1+2d_4p\right)\tilde{I}_m^{n+1}-d_4p\tilde{I}_{m-1}^{n+1}\\&=\left[1+\left(-w_2+\alpha\tilde{I}_m^{n}\right)\ell\right]\tilde{I}_m^{n}+\sigma\ell\tilde{E}_m^{n}.\end{aligned} \tag{3.16}$$

$$\begin{aligned}&-d_5p\tilde{R}_{m-1}^{n+1}+\left(1+2d_5p\right)\tilde{R}_m^{n+1}-d_5p\tilde{R}_{m-1}^{n+1}\\&=\left[1+\left(-r-\delta+\alpha\tilde{I}_m^{n}\right)\ell\right]\tilde{R}_m^{n}+\kappa\ell\tilde{E}_m^{n}+\gamma\ell\tilde{I}_m^{n}.\end{aligned} \tag{3.17}$$

where $p=\ell/h^2$ and m=0,1,2,...,M, $n=0,1,2,\ldots$

3.3.1 The Local Truncation Error

The local truncation error of the numerical scheme which associated (3.8) – (3.12) respectively by

$\mathscr{L}_S\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right]$;

$\mathscr{L}_V\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right]$;

$\mathscr{L}_E\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right]$;

$\mathscr{L}_I\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right]$;

$\mathscr{L}_R\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right]$;

Thus, it is easy to see that

$$\begin{aligned}\mathscr{L}_S =\ & \frac{S(x,t+\ell)-S(x,t)}{\ell}\\ & -r+(\beta_1 E(x,t)+\beta_2 I(x,t)+\alpha I(x,t)-\phi-r)S(x,t)-\delta R(x,t)\\ & -\theta V(x,t)-d_1\left[\frac{S(x-h,t+\ell)-2S(x,t+\ell)+S(x+h,t+\ell)}{h^2}\right]\\ & -S_t+r-(\beta_1 E(x,t)+\beta_2 I(x,t)+\alpha I(x,t)-\phi-r)S(x,t)\\ & +\delta R(x,t)+\theta V(x,t)+d_1 S_{xx}.\end{aligned} \tag{3.18}$$

$$\begin{aligned}\mathscr{L}_V =\ & \frac{V(x,t+\ell)-V(x,t)}{\ell}\\ & -\left(-\beta_2\beta E(x,t)-\beta_3 I(x,t)+\alpha I(x,t)-r-\theta\right)V(x,t)-\phi S(x,t)\\ & -d_2\left[\frac{V(x-h,t+\ell)-2V(x,t+\ell)+V(x+h,t+\ell)}{h^2}\right]\\ & -V_t+(-\beta_2\beta E(x,t)-\beta_3 I(x,t)+\alpha I(x,t)-r-\theta)V(x,t)+\phi S(x,t)+d_2 V_{xx}.\end{aligned} \tag{3.19}$$

$$\begin{aligned}\mathscr{L}_E =\ & \frac{E(x,t+\ell)-E(x,t)}{\ell}\\ & -\left(\beta_1 E(x,t)+\beta_3 V(x,t)+\alpha I(x,t)-w_1\right)E(x,t)-\beta_2 I(x,t)S(x,t)\\ & -\beta_4 I(x,t)V(x,t)-d_3\left[\frac{E(x-h,t+\ell)-2E(x,t+\ell)+E(x+h,t+\ell)}{h^2}\right]\\ & -E_t+(\beta_1 S(x,t)+\beta_3 V(x,t)+\alpha I(x,t)-w_1)E+\beta_2 I(x,t)S(x,t)\\ & +\beta_4 I(x,t)V(x,t)+d_3 E_{xx}.\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathscr{L}_I =\ & \frac{I(x,t+\ell)-I(x,t)}{\ell}\\ & -(-w_2+\alpha I(x,t))I(x,t)-\sigma E(x,t)\\ & -d_4\left[\frac{I(x-h,t+\ell)-2I(x,t+\ell)+I(x+h,t+\ell)}{h^2}\right]\\ & -I_t+(-w_2+\alpha I(x,t))I(x,t)+\sigma E(x,t)+d_4 I_{tt}.\end{aligned} \tag{3.21}$$

$$\begin{aligned}\mathscr{L}_R =\ & \frac{R(x,t+\ell)-R(x,t)}{\ell}\\ & -\left(-r-\delta+\alpha I(x,t)\right)R(x,t)-\kappa E(x,t)-\gamma I(x,t)\\ & -d_5\left[\frac{R(x-h,t+\ell)-2R(x,t+\ell)+R(x+h,t+\ell)}{h^2}\right]\\ & -R_t+(-r-\delta+\alpha I(x,t))R(x,t)+\kappa E(x,t)+\gamma I(x,t)+d_5 R_{xx}.\end{aligned} \tag{3.22}$$

Expanding and $S(x\pm h,t),\ V(x\pm h,t),\ E(x\pm h,t),\ I(x\pm h,t),\ R(x\pm h,t)$ and $S(x,t+\ell),\ V(x,t+\ell),$ $E(x,t+\ell),\ I(x,t+\ell),\ R(x,t+\ell)$ in (3.18)-(3.22) as Taylor series about (x, t) leads to

$$\mathscr{L}_S = -\frac{1}{2}d_1h^2\frac{\partial^4 S}{\partial x^4}+l\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2}-d_1\frac{\partial^3 S}{\partial x^2\partial t}\right]+\ldots, \tag{3.23}$$

$$\mathscr{L}_V = -\frac{1}{2}d_2h^2\frac{\partial^4 S}{\partial x^4}+l\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2}-d_2\frac{\partial^3 S}{\partial x^2\partial t}\right]+\ldots, \tag{3.24}$$

$$\mathscr{L}_E = -\frac{1}{2}d_3h^2\frac{\partial^4 S}{\partial x^4}+l\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2}-d_3\frac{\partial^3 S}{\partial x^2\partial t}\right]+\ldots, \tag{3.25}$$

$$\mathscr{L}_I = -\frac{1}{2}d_4h^2\frac{\partial^4 S}{\partial x^4}+l\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2}-d_4\frac{\partial^3 S}{\partial x^2\partial t}\right]+\ldots, \tag{3.26}$$

$$\mathscr{L}_R = -\frac{1}{2}d_5h^2\frac{\partial^4 S}{\partial x^4}+l\left[\frac{1}{2}\frac{\partial^2 S}{\partial t^2}-d_5\frac{\partial^3 S}{\partial x^2\partial t}\right]+\ldots, \tag{3.27}$$

Equation (3.23)-(3.27) verify that the method are (3.13)-(3.17) $O(h^2+\ell)$ as $h,\ell\to 0,$

3.3.2 Stability Analyses

The von Neumann or Fourier series method of analyzing stability will be used to gain some insight into the stability of the standard finite-difference method. These methods seek the condition under which small errors of the forms

$$Z_{S,m}^n = \tilde{S}_m^n - S_m^n = e^{\psi_1 nl}e^{i\tau_1(-L+mh)} \tag{3.28}$$

$$Z_{V,m}^n = \tilde{V}_m^n - V_m^n = e^{\psi_2 nl}e^{i\tau_2(-L+mh)} \tag{3.29}$$

$$Z_{E,m}^n = \tilde{E}_m^n - E_m^n = e^{\psi_3 nl}e^{i\tau_3(-L+mh)} \tag{3.30}$$

$$Z_{I,m}^n = \tilde{I}_m^n - I_m^n = e^{\psi_4 nl}e^{i\tau_4(-L+mh)} \tag{3.31}$$

$$Z_{R,m}^n = \tilde{R}_m^n - R_m^n = e^{\psi_5 nl}e^{i\tau_5(-L+mh)} \tag{3.32}$$

where ψ_j and τ_j which $j=1,2,\ldots,5$ are real, $i=\sqrt{-1}$ where $S_m^n, V_m^n, E_m^n, I_m^n$ and R_m^n are perturbed numerical solutions, necessary conditions for the errors not to grow as $n\to\infty$ are $\left|e^{\omega_1 l}\right|\le 1,\ \left|e^{\omega_2 l}\right|\le 1,\ \left|e^{\omega_3 l}\right|\le 1,\ \left|e^{\omega_4 l}\right|\le 1$ and $\left|e^{\omega_5 l}\right|\le 1$

Substituting Z_S, Z_V, Z_E, Z_I, and Z_R in (3.13)-(3.17) lead to

$$\xi_S = \frac{\left(1-(\beta_1\tilde{E}_m^n+\beta_2\tilde{I}_m^n+\alpha\tilde{I}_m^n-\phi-r)\ell\right)}{(1+4d_1p)}, \tag{3.33}$$

$$\xi_V = \frac{\left(1+(-\beta_3\tilde{E}_m^n-\beta_4\tilde{I}_m^n+\alpha\tilde{I}_m^n-r-\theta)\ell\right)}{(1+4d_2p)}, \tag{3.34}$$

$$\xi_E = \frac{\left(1+(\beta_1\tilde{S}_m^n+\beta_3\tilde{V}_m^n+\alpha\tilde{I}_m^n-w_1)\ell\right)}{(1+4d_3p)}, \tag{3.35}$$

$$\xi_I = \frac{\left(1+(-w_2+\alpha\tilde{I}_m^n)\ell\right)}{(1+4d_4p)}, \tag{3.36}$$

$$\xi_R = \frac{\left(1+(-r-\delta+\alpha\tilde{I}_m^n)\ell\right)}{(1+4d_5p)}, \tag{3.37}$$

which $p=\frac{\ell}{h^2}$,

where $\xi_S = e^{\psi_1\ell}$, $\xi_V = e^{\psi_2\ell}$, $\xi_E = e^{\psi_3\ell}$, $\xi_I = e^{\psi_4\ell}$, $\xi_R = e^{\psi_5\ell}$, $\tilde{S}_m^n$, $\tilde{V}_m^n$, $\tilde{E}_m^n$, $\tilde{I}_m^n$ and $\tilde{R}_m^n$ is a constant, that is, the stability restrictions are

The stability condition for S

$$p \ge \frac{\left(-2+(\beta_1\tilde{E}_m^n+\beta_2\tilde{I}_m^n+\alpha\tilde{I}_m^n-\phi-r)\ell\right)}{4d_1},$$

and

$$p \ge \frac{-(\beta_1\tilde{E}_m^n+\beta_2\tilde{I}_m^n+\alpha\tilde{I}_m^n-\phi-r)\ell}{4d_1}. \tag{3.38}$$

The stability condition for V

$$p \ge \frac{\left(-2-(-\beta_3\tilde{E}_m^n-\beta_4\tilde{I}_m^n+\alpha\tilde{I}_m^n-r-\theta)\ell\right)}{4d_2}\ ,$$

and

$$p \ge \frac{(-\beta_3\tilde{E}_m^n-\beta_4\tilde{I}_m^n+\alpha\tilde{I}_m^n-r-\theta)\ell}{4d_2}. \tag{3.39}$$

The stability condition for E

$$p \ge \frac{\left(-2-(\beta_1\tilde{S}_m^n+\beta_3\tilde{V}_m^n+\alpha\tilde{I}_m^n-w_1)\ell\right)}{4d_3},$$

and

$$p \ge \frac{(\beta_1\tilde{S}_m^n+\beta_3\tilde{V}_m^n+\alpha\tilde{I}_m^n-w_1)\ell}{4d_3}. \tag{3.40}$$

The stability condition for I

$$p \ge \frac{\left(-2-(-w_2+\alpha\tilde{I}_m^n)\ell\right)}{4d_4} \quad \text{and} \quad p \ge \frac{(-w_2+\alpha\tilde{I}_m^n)\ell}{4d_4}. \tag{3.41}$$

The stability condition for R

$$p \ge \frac{\left(-2-(-r-\delta+\alpha\tilde{I}_m^n)\ell\right)}{4d_5} \quad \text{and} \quad p \ge \frac{(-r-\delta+\alpha\tilde{I}_m^n)\ell}{4d_5}. \tag{3.42}$$

The stability intervals of the method are given by the intersection on p.

3.3.3 Implementations

The partial derivative in the boundary conditions (3.5), is approximated by the second order, central-difference replacement

$$\frac{\partial u}{\partial x} = \frac{u(x+h,t) - u(x-h,t)}{2h} + O(h^2)$$

Apply the replacement to (3.5) on $x = \pm L$

$$\tilde{S}_1^n = \tilde{S}_{-1}^n \quad \text{and} \quad \tilde{S}_{M+1}^n = \tilde{S}_{M-1}^n \tag{3.43}$$

$$\tilde{V}_1^n = \tilde{V}_{-1}^n \quad \text{and} \quad \tilde{V}_{M+1}^n = \tilde{V}_{M-1}^n \tag{3.44}$$

$$\tilde{E}_1^n = \tilde{E}_{-1}^n \quad \text{and} \quad \tilde{E}_{M+1}^n = \tilde{E}_{M-1}^n \tag{3.45}$$

$$\tilde{I}_1^n = \tilde{I}_{-1}^n \quad \text{and} \quad \tilde{I}_{M+1}^n = \tilde{I}_{M-1}^n \tag{3.46}$$

$$\tilde{R}_1^n = \tilde{R}_{-1}^n \quad \text{and} \quad \tilde{R}_{M+1}^n = \tilde{R}_{M-1}^n \tag{3.47}$$

Let

$\tilde{\mathbf{S}}^{n+1} = \left[\tilde{S}_0^{n+1}, \tilde{S}_1^{n+1}, \ldots, \tilde{S}_M^{n+1}\right]^T$, $\tilde{\mathbf{V}}^{n+1} = \left[\tilde{V}_0^{n+1}, \tilde{V}_1^{n+1}, \ldots, \tilde{V}_M^{n+1}\right]^T$, $\tilde{\mathbf{E}}^{n+1} = \left[\tilde{E}_0^{n+1}, \tilde{E}_1^{n+1}, \ldots, \tilde{E}_M^{n+1}\right]^T$,

$\tilde{\mathbf{I}}^{n+1} = \left[\tilde{I}_0^{n+1}, \tilde{I}_1^{n+1}, \ldots, \tilde{I}_M^{n+1}\right]^T$, $\tilde{\mathbf{R}}^{n+1} = \left[\tilde{R}_0^{n+1}, \tilde{R}_1^{n+1}, \ldots, \tilde{R}_M^{n+1}\right]^T$,

where T denotes transpose of the vector. The modifications to the formulae of the families of method (3.13) - (3.17), and their implication, are as follows.

Taking $m = 0$, M in (3.13)-(3.17) and using (3.43)-(3.47) gives

$m = 0$,

$$\begin{aligned}&(1+2d_1 p)\tilde{S}_0^{n+1} - 2d_1 p\tilde{S}_1^{n+1} \\ &= \left[1 - (\beta_1 \tilde{E}_0^n + \beta_2 \tilde{I}_0^n + \alpha \tilde{I}_0^n - \phi - r)\ell\right]\tilde{S}_0^n + r\ell + \delta\ell\tilde{R}_0^n + \theta\ell\tilde{V}_0^n.\end{aligned} \tag{3.48}$$

$$\begin{aligned}&(1+2d_2 p)\tilde{V}_0^{n+1} - 2d_2 p\tilde{V}_1^{n+1} \\ &= \left[1 + \left(-\beta_2\beta\tilde{E}_0^n - \beta_3\tilde{I}_0^n + \alpha\tilde{I}_0^n - r - \theta\right)\ell\right]\tilde{V}_0^n + \phi\ell\tilde{S}_0^n.\end{aligned} \tag{3.49}$$

$$\begin{aligned}&(1+2d_3 p)\tilde{E}_0^{n+1} - 2d_3 p\tilde{E}_1^{n+1} \\ &= \left[1 + \left(\beta_1\tilde{S}_0^n + \beta_3\tilde{V}_0^n + \alpha\tilde{I}_0^n - w_1\right)\ell\right]\tilde{E}_0^n + \beta_2\ell\tilde{I}_0^n\tilde{S}_0^n + \beta_4\ell\tilde{I}_0^n\tilde{V}_0^n.\end{aligned} \tag{3.50}$$

$$\begin{aligned}&(1+2d_4 p)\tilde{I}_0^{n+1} - 2d_4 p\tilde{I}_1^{n+1} \\ &= \left[1 + \left(-w_2 + \alpha\tilde{I}_0^n\right)\ell\right]\tilde{I}_0^n + \sigma\ell\tilde{E}_0^n.\end{aligned} \tag{3.51}$$

$$\begin{aligned}&(1+2d_5 p)\tilde{R}_0^{n+1} - 2d_5 p\tilde{R}_1^{n+1} \\ &= \left[1 + \left(-r - \delta + \alpha\tilde{I}_0^n\right)\ell\right]\tilde{R}_0^n + \kappa\ell\tilde{E}_0^n + \gamma\ell\tilde{I}_0^n.\end{aligned} \tag{3.52}$$

$m = M$,

$$-2d_1 p\tilde{S}_{M-1}^{n+1} + \left(1+2d_1 p\right)\tilde{S}_M^{n+1}$$
$$= \left[1-(\beta_1 \tilde{E}_M^n + \beta_2 \tilde{I}_M^n + \alpha \tilde{I}_M^n - \phi - r)\ell\right]\tilde{S}_M^n + r\ell + \delta\ell\tilde{R}_M^n + \theta\ell\tilde{V}_M^n. \quad (3.53)$$

$$-2d_2 p\tilde{V}_{M-1}^{n+1} + \left(1+2d_2 p\right)\tilde{V}_M^{n+1}$$
$$= \left[1+\left(-\beta_2\beta\tilde{E}_M^n - \beta_3 \tilde{I}_M^n + \alpha \tilde{I}_M^n - r - \theta\right)\ell\right]\tilde{V}_M^n + \phi\ell\tilde{S}_M^n. \quad (3.54)$$

$$-2d_3 p\tilde{E}_{M-1}^{n+1} + \left(1+2d_3 p\right)\tilde{E}_M^{n+1}$$
$$= \left[1+\left(\beta_1 \tilde{S}_M^n + \beta_3 \tilde{V}_M^n + \alpha \tilde{I}_M^n - w_1\right)\ell\right]\tilde{E}_M^n + \beta_2\ell\tilde{I}_M^n\tilde{S}_M^n + \beta_4\ell\tilde{I}_M^n\tilde{V}_M^n. \quad (3.55)$$

$$-2d_4 p\tilde{I}_{M-1}^{n+1} + \left(1+2d_4 p\right)\tilde{I}_M^{n+1}$$
$$= \left[1+\left(-w_2 + \alpha \tilde{I}_M^n\right)\ell\right]\tilde{I}_M^n + \sigma\ell\tilde{E}_M^n. \quad (3.56)$$

$$-2d_5 p\tilde{R}_{M-1}^{n+1} + \left(1+2d_5 p\right)\tilde{R}_M^{n+1}$$
$$= \left[1+\left(-r - \delta + \alpha \tilde{I}_M^n\right)\ell\right]\tilde{R}_M^n + \kappa\ell\tilde{E}_M^n + \gamma\ell\tilde{I}_M^n. \quad (3.57)$$

For $m = 1,2,\ldots,M$-1,

$$-d_1 p\tilde{S}_{m-1}^{n+1} + \left(1+2d_1 p\right)\tilde{S}_m^{n+1} - d_1 p\tilde{S}_{m+1}^{n+1}$$
$$= \left[1-(\beta_1 \tilde{E}_m^n + \beta_2 \tilde{I}_m^n + \alpha \tilde{I}_m^n - \phi - r)\ell\right]\tilde{S}_m^n + r\ell + \delta\ell\tilde{R}_m^n + \theta\ell\tilde{V}_m^n. \quad (3.58)$$

$$-d_2 p\tilde{V}_{m-1}^{n+1} + \left(1+2d_2 p\right)\tilde{V}_m^{n+1} - d_2 p\tilde{V}_{m+1}^{n+1}$$
$$= \left[1+\left(-\beta_2\beta\tilde{E}_m^n - \beta_3 \tilde{I}_m^n + \alpha \tilde{I}_m^n - r - \theta\right)\ell\right]\tilde{V}_m^n + \phi\ell\tilde{S}_m^n. \quad (3.59)$$

$$-d_3 p\tilde{E}_{m-1}^{n+1} + \left(1+2d_3 p\right)\tilde{E}_m^{n+1} - d_3 p\tilde{E}_{m+1}^{n+1}$$
$$= \left[1+\left(\beta_1 \tilde{S}_m^n + \beta_3 \tilde{V}_m^n + \alpha \tilde{I}_m^n - w_1\right)\ell\right]\tilde{E}_m^n + \beta_2\ell\tilde{I}_m^n\tilde{S}_m^n + \beta_4\ell\tilde{I}_m^n\tilde{V}_m^n. \quad (3.60)$$

$$-d_4 p\tilde{I}_{m-1}^{n+1} + \left(1+2d_4 p\right)\tilde{I}_m^{n+1} - d_4 p\tilde{I}_{m+1}^{n+1}$$
$$= \left[1+\left(-w_2 + \alpha \tilde{I}_m^n\right)\ell\right]\tilde{I}_m^n + \sigma\ell\tilde{E}_m^n. \quad (3.61)$$

$$-d_5 p\tilde{R}_{m-1}^{n+1} + \left(1+2d_5 p\right)\tilde{R}_m^{n+1} - d_5 p\tilde{R}_{m+1}^{n+1}$$
$$= \left[1+\left(-r - \delta + \alpha \tilde{I}_m^n\right)\ell\right]\tilde{R}_m^n + \kappa\ell\tilde{E}_m^n + \gamma\ell\tilde{I}_m^n. \quad (3.62)$$

Thus, the solution vector $\tilde{\mathbf{S}},\tilde{\mathbf{V}},\tilde{\mathbf{E}},\tilde{\mathbf{I}}$ and $\tilde{\mathbf{R}}$ may be obtained using the following parallel algorithm:

Processor I $\quad A_1\tilde{\mathbf{S}}^{n+1} = Q_1\tilde{\mathbf{S}}^{n} + H_1$ (3.63)

Processor II $\quad A_2\tilde{\mathbf{V}}^{n+1} = Q_2\tilde{\mathbf{V}}^{n} + H_2$ (3.64)

Processor III $\quad A_3\tilde{\mathbf{E}}^{n+1} = Q_3\tilde{\mathbf{E}}^{n} + H_3$ (3.65)

Processor IV $\quad A_4\tilde{\mathbf{I}}^{n+1} = Q_4\tilde{\mathbf{I}}^{n} + H_4$ (3.66)

Processor V $\quad A_5\tilde{\mathbf{R}}^{n+1} = Q_5\tilde{\mathbf{R}}^{n} + H_5$ (3.67)

The matrix A_1 is a constant tridiagonal matrix of order M+1 given by

$$A_1 = \begin{bmatrix} 1+2d_1 p & -2pd_1 & 0 & \cdots & 0 \\ -pd_1 & 1+2pd_1 & -pd_1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -pd_1 & 1+2pd_1 & -pd_1 \\ 0 & \cdots & 0 & -2pd_1 & 1+2pd_1 \end{bmatrix}, \tag{3.68}$$

The diagonal matrix Q_1 of order M+1 given by

$$Q_1 = \begin{bmatrix} f_0 & 0 & 0 & \cdots & 0 \\ 0 & f_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & f_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & f_M \end{bmatrix}, \tag{3.69}$$

with $\quad f_i = 1-(\beta_1\tilde{E}_i^n + \beta_2\tilde{I}_i^n + \alpha\tilde{I}_i^n - \phi - r)\ell,\ i = 0,1,2,\ldots,M.$

The column matrix H_1 of order M+1 given by

$$H_1 = \begin{bmatrix} r\ell + \delta\ell\tilde{R}_0^n + \theta\ell\tilde{V}_0^n \\ r\ell + \delta\ell\tilde{R}_1^n + \theta\ell\tilde{V}_1^n \\ \vdots \\ r\ell + \delta\ell\tilde{R}_{M-1}^n + \theta\ell\tilde{V}_{M-1}^n \\ r\ell + \delta\ell\tilde{R}_M^n + \theta\ell\tilde{V}_M^n \end{bmatrix}, \tag{3.70}$$

similary, the remain A_i, Q_i and H_i, $i = 2,3,...,5$ are given by

$$A_2 = \begin{bmatrix} 1+2d_2 p & -2pd_2 & 0 & \cdots & 0 \\ -pd_2 & 1+2pd_2 & -pd_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -pd_2 & 1+2pd_2 & -pd_2 \\ 0 & \cdots & 0 & -2pd_2 & 1+2pd_2 \end{bmatrix}, \quad (3.71)$$

$$Q_2 = \begin{bmatrix} j_0 & 0 & 0 & \cdots & 0 \\ 0 & j_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & j_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & j_M \end{bmatrix}, \quad (3.72)$$

with $j_i = 1+\left(-\beta_3\tilde{E}_i^n - \beta_4\tilde{I}_i^n + \alpha\tilde{I}_i^n - r - \theta\right)\ell,\ i = 0,1,2,...,M.$

The matrix H_2, A_3 and Q_3 are given by

$$H_2 = \begin{bmatrix} \phi\ell\tilde{S}_0^n \\ \phi\ell\tilde{S}_1^n \\ \vdots \\ \phi\ell\tilde{S}_{M-1}^n \\ \phi\ell\tilde{S}_M^n \end{bmatrix}, \quad (3.73)$$

$$A_3 = \begin{bmatrix} 1+2d_3 p & -2pd_3 & 0 & \cdots & 0 \\ -pd_3 & 1+2pd_3 & -pd_3 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -pd_3 & 1+2pd_3 & -pd_3 \\ 0 & \cdots & 0 & -2pd_3 & 1+2pd_3 \end{bmatrix}, \quad (3.74)$$

$$Q_3 = \begin{bmatrix} z_0 & 0 & 0 & \cdots & 0 \\ 0 & z_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & z_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & z_M \end{bmatrix}, \tag{3.75}$$

with $z_i = 1+\left(\beta_1\tilde{S}_i^n + \beta_3\tilde{V}_i^n + \alpha\tilde{I}_i^n - w_1\right)\ell,\ i = 0,1,2,\ldots,M.$

The remain matrices are shown as the following

$$H_3 = \begin{bmatrix} \beta_2\tilde{I}_0^n\tilde{S}_0^n + \beta_4\tilde{I}_0^n\tilde{V}_0^n \\ \beta_2\tilde{I}_1^n\tilde{S}_1^n + \beta_4\tilde{I}_1^n\tilde{V}_1^n \\ \vdots \\ \beta_2\tilde{I}_{M-1}^n\tilde{S}_{M-1}^n + \beta_4\tilde{I}_{M-1}^n\tilde{V}_{M-1}^n \\ \beta_2\tilde{I}_M^n\tilde{S}_M^n + \beta_4\tilde{I}_M^n\tilde{V}_M^n \end{bmatrix}, \tag{3.76}$$

$$A_4 = \begin{bmatrix} 1+2d_4p & -2pd_4 & 0 & \cdots & 0 \\ -pd_4 & 1+2pd_4 & -pd_4 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -pd_4 & 1+2pd_4 & -pd_4 \\ 0 & \cdots & 0 & -2pd_4 & 1+2pd_4 \end{bmatrix}, \tag{3.77}$$

$$Q_4 = \begin{bmatrix} b_0 & 0 & 0 & \cdots & 0 \\ 0 & b_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & b_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & b_M \end{bmatrix}, \tag{3.78}$$

with $b_i = 1+\left(-w_2 + \alpha\tilde{I}_i^n\right)\ell,\ i = 0,1,2,\ldots,M.$

$$H_4 = \begin{bmatrix} \sigma\ell\tilde{E}_0^n \\ \sigma\ell\tilde{E}_1^n \\ \vdots \\ \sigma\ell\tilde{E}_{M-1}^n \\ \sigma\ell\tilde{E}_M^n \end{bmatrix}, \tag{3.79}$$

$$A_5 = \begin{bmatrix} 1+2d_5 p & -2p & 0 & \cdots & 0 \\ -pd_5 & 1+2pd_5 & -pd_5 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -pd_5 & 1+2pd_5 & -pd_5 \\ 0 & \cdots & 0 & -2pd_5 & 1+2pd_5 \end{bmatrix}, \tag{3.80}$$

$$Q_5 = \begin{bmatrix} n_0 & 0 & 0 & \cdots & 0 \\ 0 & n_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & n_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & n_M \end{bmatrix}, \tag{3.81}$$

With $n_i = 1 + \left(-r + \delta + \alpha\tilde{I}_i^n\right)\ell,\ i = 0,1,2,\ldots,M.$

$$H_5 = \begin{bmatrix} \kappa\ell\tilde{E}_0^n + \gamma\ell\tilde{I}_0^n \\ \kappa\ell\tilde{E}_1^n + \gamma\ell\tilde{I}_1^n \\ \vdots \\ \kappa\ell\tilde{E}_{M-1}^n + \gamma\ell\tilde{I}_{M-1}^n \\ \kappa\ell\tilde{E}_M^n + \gamma\ell\tilde{I}_M^n \end{bmatrix}, \tag{3.82}$$

The linear algebraic systems given by (3.63)-(3.67) can be solved by using parallel computation, The vector $\tilde{\mathbf{S}}, \tilde{\mathbf{V}}, \tilde{\mathbf{E}}, \tilde{\mathbf{I}}$ and $\tilde{\mathbf{R}}$ can be obtained simultaneously and thus the time taken to solve PDEs will be reduced significant.

3.4 Nonstandard Finite-Difference Method

Numerical method for *S*

In this section, the non-standard finite difference is studied. The approximation of the time derivative is obtained by the first-order forward-difference

$$\frac{\partial S}{\partial t} \approx \frac{S(x,t+\ell)-S(x,t)}{\ell},$$

and the space derivative is obtained by the second-order approximant

$$\frac{\partial^2 S}{\partial x^2} \approx \frac{1}{2h^2}\{S(x-h,t+\ell)-2S(x,t+\ell)+S(x+h,t+\ell) \\ +S(x-h,t)-2S(x,t)+S(x+h,t)\},$$

The result of approximating becomes,

$$\frac{\tilde{S}_m^{n+1}-\tilde{S}_m^n}{\ell} = (-\beta_1\tilde{E}_m^n-\beta_2\tilde{I}_m^n+\alpha\tilde{I}_m^n-\phi-r)\tilde{S}_m^{n+1}+\delta\tilde{R}_m^{n+1}+\theta\tilde{V}_m^{n+1}+r \\ +d_1\left[\frac{\tilde{S}_{m-1}^{n+1}-2\tilde{S}_m^{n+1}+\tilde{S}_{m+1}^{n+1}+\tilde{S}_{m-1}^n-2\tilde{S}_m^n+\tilde{S}_{m+1}^n}{2h^2}\right]. \tag{3.83}$$

and

$$\frac{\tilde{S}_m^{n+1}-\tilde{S}_m^n}{\ell} = (-\beta_1\tilde{E}_m^{n+1}-\beta_2\tilde{I}_m^{n+1}+\alpha\tilde{I}_m^{n+1}-\phi-r)\tilde{S}_m^n+\delta\tilde{R}_m^n+\theta\tilde{V}_m^n+r \\ +d_1\left[\frac{\tilde{S}_{m-1}^{n+1}-2\tilde{S}_m^{n+1}+\tilde{S}_{m+1}^{n+1}+\tilde{S}_{m-1}^n-2\tilde{S}_m^n+\tilde{S}_{m+1}^n}{2h^2}\right]. \tag{3.84}$$

The average of equation (3.83) and (3.84) becomes

$$\frac{\tilde{S}_m^{n+1}-\tilde{S}_m^n}{\ell} = \frac{1}{2}(-\beta_1\tilde{E}_m^n-\beta_2\tilde{I}_m^n+\alpha\tilde{I}_m^n-\phi-r)\tilde{S}_m^{n+1}+\frac{1}{2}\delta\tilde{R}_m^{n+1}+\frac{1}{2}\theta\tilde{V}_m^{n+1} \\ +\frac{1}{2}\delta\tilde{R}_m^n+\frac{1}{2}\theta\tilde{V}_m^n+r+\frac{1}{2}(-\beta_1\tilde{E}_m^{n+1}-\beta_2\tilde{I}_m^{n+1}+\alpha\tilde{I}_m^{n+1}-\phi-r)\tilde{S}_m^n \\ +d_1\left[\frac{\tilde{S}_{m-1}^{n+1}-2\tilde{S}_m^{n+1}+\tilde{S}_{m+1}^{n+1}+\tilde{S}_{m-1}^n-2\tilde{S}_m^n+\tilde{S}_{m+1}^n}{2h^2}\right]. \tag{3.85}$$

Equation (3.85) is rearranged. This provides

$$\begin{aligned}
&\frac{-P_1}{2}\tilde{S}_{m-1}^{n+1} + (1+\frac{1}{2}(\beta_1\ell\tilde{E}_m^n + \beta_2\ell\tilde{I}_m^n - \alpha\ell\tilde{I}_m^n + \phi\ell + r\ell + P_1))\tilde{S}_m^{n+1} - \frac{P_1}{2}\tilde{S}_{m+1}^{n+1} \\
&-\frac{1}{2}\theta\tilde{V}_m^{n+1} - \frac{1}{2}\beta_1\ell\tilde{S}_m^n\tilde{E}_m^{n+1} + \left(-\frac{1}{2}\alpha\ell\tilde{S}_m^n + \frac{1}{2}\beta_2\ell\tilde{S}_m^n\right)\tilde{I}_m^{n+1} - \frac{1}{2}\delta\ell\tilde{R}_m^{n+1} \\
&= \frac{P_1}{2}\tilde{S}_{m-1}^n + \left(1 - \phi\frac{\ell}{2} - r\frac{\ell}{2} - P_1\right)\tilde{S}_m^n + \frac{P_1}{2}\tilde{S}_{m+1}^n + \frac{\ell}{2}\theta\tilde{V}_m^n + \frac{\ell}{2}\delta\tilde{R}_m^n + \ell r \ .
\end{aligned} \tag{3.86}$$

Similarly, the nonstandard finite-difference for V may be approximate the time derivative in the second equation of (3.3) by the first-order forward-difference replacement

$$\frac{\partial V}{\partial t} \approx \frac{V(x,t+\ell) - V(x,t)}{\ell},$$

And the space derivative by the second-order approximant

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} \approx \frac{1}{2h^2}\{&V(x-h,t+\ell) - 2V(x,t+\ell) + V(x+h,t+\ell) \\
&+V(x-h,t) - 2V(x,t) + V(x+h,t)\},
\end{aligned}$$

The result of approximating becomes,

$$\begin{aligned}
\frac{\tilde{V}_m^{n+1} - \tilde{V}_m^n}{\ell} &= \phi\tilde{S}_m^{n+1} + \left(-\beta_2\beta\tilde{E}_m^n - \beta_3\tilde{I}_m^n + \alpha\tilde{I}_m^n - r - \theta\right)\tilde{V}_m^{n+1} \\
&+ d_2\left[\frac{\tilde{V}_{m-1}^{n+1} - 2\tilde{V}_m^{n+1} + \tilde{V}_{m+1}^{n+1} + \tilde{V}_{m-1}^n - 2\tilde{V}_m^n + \tilde{V}_{m+1}^n}{2h^2}\right].
\end{aligned} \tag{3.87}$$

and

$$\begin{aligned}
\frac{\tilde{V}_m^{n+1} - \tilde{V}_m^n}{\ell} &= \phi\tilde{S}_m^n + \left(-\beta_2\beta\tilde{E}_m^{n+1} - \beta_3\tilde{I}_m^{n+1} + \alpha\tilde{I}_m^{n+1} - r - \theta\right)\tilde{V}_m^n \\
&+ d_2\left[\frac{\tilde{V}_{m-1}^{n+1} - 2\tilde{V}_m^{n+1} + \tilde{V}_{m+1}^{n+1} + \tilde{V}_{m-1}^n - 2\tilde{V}_m^n + \tilde{V}_{m+1}^n}{2h^2}\right].
\end{aligned} \tag{3.88}$$

The average of equation (3.87) and (3.88) becomes

$$-\frac{\ell}{2}\phi\tilde{S}_m^{n+1}-\frac{p_2}{2}\tilde{V}_{m-1}^{n+1}+\left(1+\beta_3\frac{\ell}{2}\tilde{E}_m^n+\beta_4\frac{\ell}{2}\tilde{I}_m^n-\alpha\frac{\ell}{2}\tilde{I}_m^n+r\frac{\ell}{2}+\theta\frac{\ell}{2}+P_2\right)\tilde{V}_m^{n+1} \tag{3.89}$$
$$-\frac{P_2}{2}\tilde{V}_{m+1}^{n+1}+\beta_3\frac{\ell}{2}\tilde{V}_m^n\tilde{E}_m^{n+1}+\left(\beta_4\frac{\ell}{2}\tilde{V}_m^n-\alpha\frac{\ell}{2}\tilde{V}_m^n\right)\tilde{I}_m^{n+1}$$
$$=\frac{\ell}{2}\phi\tilde{S}_m^n+\frac{p_2}{2}\tilde{V}_{m-1}^n+\left(1-r\frac{\ell}{2}-\theta\frac{\ell}{2}-P_2\right)\tilde{V}_m^n+\frac{p_2}{2}\tilde{V}_{m+1}^n$$

The nonstandard finite-difference for E may be approximate the time derivative in the third equation of (3.3) by the first-order forward-difference replacement

$$\frac{\partial E}{\partial t}\approx\frac{E(x,t+\ell)-E(x,t)}{\ell},$$

And the space derivative by the second-order approximant

$$\frac{\partial^2 E}{\partial x^2}\approx\frac{1}{2h^2}\{E(x-h,t+\ell)-2E(x,t+\ell)+E(x+h,t+\ell)$$
$$+E(x-h,t)-2E(x,t)+E(x+h,t)\},$$

The result of approximating become,

$$\frac{\tilde{E}_m^{n+1}-\tilde{E}_m^n}{\ell}=\left(\beta_1\tilde{S}_m^n+\beta_3\tilde{V}_m^n+\alpha\tilde{I}_m^n-w_1\right)\tilde{E}_m^{n+1}+\beta_2\tilde{I}_m^n\tilde{S}_m^{n+1}+\beta_4\tilde{I}_m^n\tilde{V}_m^{n+1} \tag{3.90}$$
$$+d_3\left[\frac{\tilde{E}_{m-1}^{n+1}-2\tilde{E}_m^{n+1}+\tilde{E}_{m+1}^{n+1}+\tilde{E}_{m-1}^n-2\tilde{E}_m^n+\tilde{E}_{m+1}^n}{2h^2}\right].$$

and

$$\frac{\tilde{E}_m^{n+1}-\tilde{E}_m^n}{\ell}=\left(\beta_1\tilde{S}_m^{n+1}+\beta_3\tilde{V}_m^{n+1}+\alpha\tilde{I}_m^{n+1}-w_1\right)\tilde{E}_m^n+\beta_2\tilde{I}_m^{n+1}\tilde{S}_m^n+\beta_4\tilde{I}_m^{n+1}\tilde{V}_m^n \tag{3.91}$$
$$+d_3\left[\frac{\tilde{E}_{m-1}^{n+1}-2\tilde{E}_m^{n+1}+\tilde{E}_{m+1}^{n+1}+\tilde{E}_{m-1}^n-2\tilde{E}_m^n+\tilde{E}_{m+1}^n}{2h^2}\right].$$

The average of equation (3.90) and (3.91) becomes

$$-\left(\beta_1\frac{\ell}{2}\tilde{E}_m^n+\beta_2\frac{\ell}{2}\tilde{I}_m^n\right)\tilde{S}_m^{n+1}-\left(\beta_4\frac{\ell}{2}\tilde{I}_m^n+\beta_3\frac{\ell}{2}\tilde{E}_m^n\right)\tilde{V}_m^{n+1}-\frac{P_3}{2}\tilde{E}_{m+1}^{n+1} \tag{3.92}$$
$$+\left(1-\beta_1\frac{\ell}{2}\tilde{S}_m^n-\beta_3\frac{\ell}{2}\tilde{V}_m^n-\alpha\frac{\ell}{2}\tilde{I}_m^n+w_1\frac{\ell}{2}+P_3\right)\tilde{E}_m^{n+1}-\beta_2\tilde{I}_m^n\tilde{S}_m^{n+1}$$
$$+\beta_4\tilde{I}_m^n\tilde{V}_m^{n+1}-\frac{P_3}{2}\tilde{E}_{m-1}^{n+1}-\left(\alpha\frac{\ell}{2}\tilde{E}_m^n+\beta_2\frac{\ell}{2}\tilde{S}_m^n+\beta_4\frac{\ell}{2}\tilde{V}_m^n\right)\tilde{I}_m^{n+1}$$
$$=\frac{P_3}{2}\tilde{E}_{m-1}^n+\left(1-w_1\frac{\ell}{2}-P_3\right)+\frac{P_3}{2}\tilde{E}_{m+1}^n$$

The nonstandard finite-difference for I may be approximates the time derivative in the fourth equation of (3.3) by the first-order forward-difference replacement.

$$\frac{\partial I}{\partial t}\approx\frac{I(x,t+\ell)-I(x,t)}{\ell},$$

and the space derivative by the second-order approximant

$$\frac{\partial^2 I}{\partial x^2}\approx\frac{1}{2h^2}\{I(x-h,t+\ell)-2I(x,t+\ell)+I(x+h,t+\ell)$$
$$+I(x-h,t)-2I(x,t)+I(x+h,t)\},$$

The result of approximating becomes,

$$\frac{\tilde{I}_m^{n+1}-\tilde{I}_m^n}{\ell}=\left(-w_2+\alpha\tilde{I}_m^n\right)\tilde{I}_m^{n+1}+\sigma\tilde{E}_m^{n+1} \tag{3.93}$$
$$+d_4\left[\frac{\tilde{I}_{m-1}^{n+1}-2\tilde{I}_m^{n+1}+\tilde{I}_{m+1}^{n+1}+\tilde{I}_{m-1}^{n+1}-2\tilde{I}_m^{n+1}+\tilde{I}_{m+1}^{n+1}}{2h^2}\right].$$

and

$$\frac{\tilde{I}_m^{n+1}-\tilde{I}_m^n}{\ell}=\left(-w_2+\alpha\tilde{I}_m^{n+1}\right)\tilde{I}_m^n+\sigma\tilde{E}_m^n \tag{3.94}$$
$$+d_4\left[\frac{\tilde{I}_{m-1}^{n+1}-2\tilde{I}_m^{n+1}+\tilde{I}_{m+1}^{n+1}+\tilde{I}_{m-1}^{n+1}-2\tilde{I}_m^{n+1}+\tilde{I}_{m+1}^{n+1}}{2h^2}\right].$$

The average of equation (3.93) and (3.94) becomes

$$-\sigma\frac{\ell}{2}\tilde{E}_m^{n+1}-\frac{P_4}{2}\tilde{I}_{m-1}^{n+1}+\left(1+w_2\frac{\ell}{2}-\alpha\ell\tilde{I}_m^{n+1}+P_4\right)\tilde{I}_m^{n+1}-\frac{P_4}{2}\tilde{I}_{m+1}^{n+1} \tag{3.95}$$
$$=\sigma\frac{\ell}{2}\tilde{E}_m^{n}+\frac{P_4}{2}\tilde{I}_{m-1}^{n}+\left(1-w_2\frac{\ell}{2}-P_4\right)\tilde{I}_m^{n}+\frac{P_4}{2}\tilde{I}_{m+1}^{n}$$

The nonstandard finite-difference for R may be approximates the time derivative in the fifth equation of (3.3) by the first-order forward-difference replacement.

$$\frac{\partial R}{\partial t}\approx\frac{R(x,t+\ell)-R(x,t)}{\ell},$$

and the space derivative by the second-order approximant

$$\frac{\partial^2 R}{\partial x^2}\approx\frac{1}{2h^2}\{R(x-h,t+\ell)-2R(x,t+\ell)+R(x+h,t+\ell)$$
$$+R(x-h,t)-2R(x,t)+R(x+h,t)\},$$

Approximating in last equation in (3.3) as follows

$$\frac{\tilde{R}_m^{n+1}-\tilde{R}_m^{n}}{\ell}=\left(-r-\delta+\alpha\tilde{I}_m^{n}\right)\tilde{R}_m^{n+1}+\kappa\tilde{E}_m^{n+1}+\gamma\tilde{I}_m^{n+1} \tag{3.96}$$
$$+d_5\left[\frac{\tilde{R}_{m-1}^{n+1}-2\tilde{R}_m^{n+1}+\tilde{R}_{m+1}^{n+1}+\tilde{R}_{m-1}^{n+1}-2\tilde{R}_m^{n+1}+\tilde{R}_{m+1}^{n+1}}{2h^2}\right].$$

and

$$\frac{\tilde{R}_m^{n+1}-\tilde{R}_m^{n}}{\ell}=\left(-r-\delta+\alpha\tilde{I}_m^{n+1}\right)\tilde{R}_m^{n}+\kappa\tilde{E}_m^{n}+\gamma\tilde{I}_m^{n} \tag{3.97}$$
$$+d_5\left[\frac{\tilde{R}_{m-1}^{n+1}-2\tilde{R}_m^{n+1}+\tilde{R}_{m+1}^{n+1}+\tilde{R}_{m-1}^{n+1}-2\tilde{R}_m^{n+1}+\tilde{R}_{m+1}^{n+1}}{2h^2}\right].$$

The result of approximating becomes,

$$-\kappa\frac{\ell}{2}\tilde{E}_m^n-\left(\gamma\frac{\ell}{2}+\alpha\frac{\ell}{2}\tilde{R}_m^n\right)\tilde{I}_m^{n+1}-\frac{P_5}{2}\tilde{R}_{m-1}^{n+1}+\left(1+r\frac{\ell}{2}+\delta\frac{\ell}{2}-\alpha\frac{\ell}{2}\tilde{I}_m^n+P_5\right)\tilde{R}_m^{n+1} \tag{3.98}$$

$$-\frac{P_5}{2}\tilde{R}_{m+1}^{n+1} = \kappa\frac{\ell}{2}\tilde{E}_m^n+\gamma\frac{\ell}{2}\tilde{I}_m^n+\frac{P_5}{2}\tilde{R}_{m-1}^n+\left(1-r\frac{\ell}{2}-\delta\frac{\ell}{2}-P_5\right)\tilde{R}_m^n+\frac{P_5}{2}\tilde{R}_{m+1}^n$$

3.4.1 The Local Truncation Error

The local truncation error of the numerical scheme which associated (3.86), (3.89), (3.93), (3.96) and (3.98), are given by

$$\mathcal{L}_S\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right];$$
$$\mathcal{L}_V\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right];$$
$$\mathcal{L}_E\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right];$$
$$\mathcal{L}_I\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right];$$
$$\mathcal{L}_R\left[S(x,t),V(x,t),E(x,t),I(x,t),R(x,t);h,\ell\right];$$

Thus, it is easy to see that

$$\begin{aligned}\mathcal{L}_S = {} & \frac{S(x,t+\ell)-S(x,t)}{\ell}+\frac{1}{2}(-\beta_1E(x,t)-\beta_2I(x,t)+\alpha I(x,t)-\phi-r)S(x,t+\ell) \\ & -\frac{1}{2}\theta V(x,t+\ell)-\frac{1}{2}\delta R(x,t+\ell)+\beta_1\frac{1}{2}S(x,t)E(x,t+\ell) \\ & +\frac{1}{2}\big(\beta_2S(x,t)-\alpha S(x,t)\big)I(x,t+\ell)+\frac{1}{2}\theta S(x,t)+\frac{1}{2}rS(x,t)-\frac{1}{2}\theta V(x,t)-\frac{1}{2}\delta R(x,t)-r \\ & -d_1\left[\frac{S(x-h,t+\ell)-2S(x,t+\ell)+S(x+h,t+\ell)+S(x-h,t)-2S(x,t)+S(x+h,t)}{2h^2}\right] \\ & -S_t+r-(\beta_1E(x,t)+\beta_2I(x,t)+\alpha I(x,t)-\phi-r)S(x,t)+\delta R(x,t)+\theta V(x,t)+d_1S_{xx}\ .\end{aligned} \tag{3.99}$$

$$\begin{aligned}\mathcal{L}_V = {} & \frac{V(x,t+\ell)-V(x,t)}{\ell}-\frac{1}{2}\big(-\beta_3E(x,t)-\beta_4I(x,t)+\alpha I(x,t)-r-\theta\big)V(x,t+\ell) \\ & -\frac{\theta}{2}S(x,t+\ell)-\frac{\theta}{2}S(x,t)+\frac{1}{2}\big(\beta_3E(x,t+\ell)+\beta_4I(x,t)\big)I(x,t+\ell) \\ & -\frac{\alpha}{2}V(x,t)I(x,t+\ell)+\frac{1}{2}\big(r+\theta\big)V(x,t) \\ & -d_2\left[\frac{V(x-h,t+\ell)-2V(x,t+\ell)+V(x+h,t+\ell)+V(x-h,t)-2V(x,t)+V(x+h,t)}{2h^2}\right] \\ & -V_t+(-\beta_3E(x,t)-\beta_4I(x,t)+\alpha I(x,t)-r-\theta)V(x,t)+\phi S(x,t)+d_2V_{xx}.\end{aligned} \tag{3.100}$$

$$\mathscr{L}_E = \frac{E(x,t+\ell)-E(x,t)}{\ell} - \frac{1}{2}\left(\beta_1 S(x,t)+\beta_3 V(x,t)+\alpha I(x,t)-w_1\right)E(x,t+\ell) \tag{3.101}$$

$$-\frac{1}{2}\left(\beta_2 I(x,t)+\beta_1 E(x,t)\right)S(x,t+\ell)-\frac{1}{2}\left(\beta_4 I(x,t)+\beta_3 E(x,t)\right)V(x,t+\ell)$$

$$-\frac{1}{2}\left(\alpha E(x,t)+\beta_2 S(x,t)\right)I(x,t+\ell)+\frac{1}{2}w_1 E(x,t)-\frac{1}{2}\beta_4 I(x,t)V(x,t+\ell)$$

$$-d_3\left[\frac{E(x-h,t+\ell)-2E(x,t+\ell)+E(x+h,t+\ell)+E(x-h,t)-2E(x,t)+E(x+h,t)}{2h^2}\right]$$

$$-E_t+(\beta_1 S(x,t)+\beta_3 V(x,t)+\alpha I(x,t)-w_1)E+\beta_2 I(x,t)S(x,t)+\beta_4 I(x,t)V(x,t)+d_3 E_{xx}.$$

$$\mathscr{L}_I = \frac{I(x,t+\ell)-I(x,t)}{\ell} - \frac{1}{2}\sigma E(x,t+\ell)-\frac{1}{2}\sigma E(x,t) \tag{3.102}$$

$$+(\frac{1}{2}w_2-\alpha I(x,t))I(x,t+\ell)+\frac{1}{2}w_2 I(x,t)$$

$$-d_4\left[\frac{I(x-h,t+\ell)-2I(x,t+\ell)+I(x+h,t+\ell)+I(x-h,t)-2I(x,t)+I(x+h,t)}{2h^2}\right]$$

$$-I_t+(-w_2+\alpha I(x,t))I(x,t)+\sigma E(x,t)+d_4 I_{tt}.$$

$$\mathscr{L}_R = \frac{R(x,t+\ell)-R(x,t)}{\ell} - \frac{1}{2}\left(-r-\delta+\alpha I(x,t)\right)R(x,t+\ell)-\kappa\frac{1}{2}E(x,t+\ell) \tag{3.103}$$

$$-\kappa\frac{1}{2}E(x,t)-\gamma\frac{1}{2}I(x,t+\ell)-\gamma\frac{1}{2}I(x,t)-\frac{1}{2}\left(-r-\delta+\alpha I(x,t+\ell)\right)R(x,t)$$

$$-d_5\left[\frac{R(x-h,t+\ell)-2R(x,t+\ell)+R(x+h,t+\ell)+R(x-h,t)-2R(x,t)+R(x+h,t)}{2h^2}\right]$$

$$-R_t+(-r-\delta+\alpha I(x,t))R(x,t)+\kappa E(x,t)+\gamma I(x,t)+d_5 R_{xx}\ .$$

Expanding $S(x\pm h,t)$, $V(x\pm h,t)$, $E(x\pm h,t)$, $I(x\pm h,t)$, $R(x\pm h,t)$ and $S(x,t+\ell)$, $V(x,t+\ell)$, $E(x,t+\ell)$, $I(x,t+\ell)$, $R(x,t+\ell)$ in (3.18)-(3.22) by Taylor series about the point $(x\ ,t)$ leads to

$$\mathscr{L}_S = -\frac{1}{12}d_1h^2S_{xxxx} + \left\{\frac{1}{6}S_{ttt} + \frac{1}{4}(\beta_1E + \beta_2I - \alpha I + \phi + r)S_{tt} - \frac{1}{4}\theta V_{tt}\right. \tag{3.104}$$

$$\left. -\frac{1}{4}\delta R_{tt} + \beta_1\frac{1}{4}SE_{tt} + \frac{1}{4}(\beta_2S - \alpha S)I_{tt} - d_1\frac{1}{4}S_{xxtt}\right\}\ell^2 + \dots,$$

$$\mathscr{L}_V = -d_2\frac{h^2}{12}V_{xxxx} + \left\{\frac{1}{6}V_{ttt} - \frac{1}{4}(-\beta_3E - \beta_4I + \alpha I - r - \theta)V_{tt} - \frac{\theta}{4}S_{tt}\right. \tag{3.105}$$

$$\left. +\frac{1}{4}(\beta_3VE_{tt} + \beta_4VI_{tt}) - \frac{\alpha}{2}VI_{tt} - d_2\frac{1}{2}V_{xxtt}\right\}\ell^2 + \dots,$$

$$\mathscr{L}_E = -d_3\frac{h^2}{12}E_{xxxx} + \left\{\frac{1}{6}E_{tt} - \frac{1}{4}(\beta_1S + \beta_3V + \alpha I - w_1)E_t - \frac{1}{4}(\beta_2I + \beta_1E)S_t\right. \tag{3.106}$$

$$\left. -\frac{1}{4}(\beta_2I + \beta_1E)S_t - \frac{1}{2}(\beta_4I + \beta_3E)V_t - \frac{1}{2}(\alpha E + \beta_2S)I_t - \frac{1}{2}\beta_4IV_t\right\}\ell^2 + \dots,$$

$$\mathscr{L}_I = -d_4\frac{h^2}{12}I_{xxxx} + \left\{\frac{1}{6}I_{ttt} - \frac{1}{4}\sigma E_{tt} + (\frac{1}{4}w_2 - \alpha I)I_{tt} - d_4\frac{1}{4}I_{xxtt}\right\}\ell^2 + \dots, \tag{3.107}$$

$$\mathscr{L}_R = -d_5\frac{h^2}{2}R_{xxxx} + \left\{\frac{1}{6}R_{ttt} - \frac{1}{4}(-r - \delta + \alpha I)R_{tt} - \kappa\frac{1}{4}E_{tt} - \gamma\frac{1}{4}I_{tt}\right. \tag{3.108}$$

$$\left. -\frac{1}{4}\alpha RI_{tt} - d_5\frac{1}{4}R_{xxtt}\right\}\ell^2 + \dots,$$

Equations (3.86), (3.89), (3.93), (3.96), (3.98) verify that the methods the order of methods $O(h^2 + \ell^2)$ as $h, \ell \to 0$,

3.4.2 Implementations

The boundary conditions (3.5) are applied by the replacement (3.43) - (3.47), We obtain Taking $m = 0$, M in (3.13)-(3.17) and using (3.43)-(3.47) gives

$m = 0$

$$\begin{aligned}
&(1+\frac{1}{2}(\beta_1\ell\tilde{E}_0^n+\beta_2\ell\tilde{I}_0^n-\alpha\ell\tilde{I}_0^n+\phi\ell+r\ell+P_1))\tilde{S}_0^{n+1}-P_1\tilde{S}_1^{n+1}-\frac{1}{2}\theta\tilde{V}_0^{n+1}\\
&-\frac{1}{2}\beta_1\ell\tilde{S}_0^n\tilde{E}_0^{n+1}+\left(-\frac{1}{2}\alpha\ell\tilde{S}_0^n+\frac{1}{2}\beta_2\ell\tilde{S}_0^n\right)\tilde{I}_0^{n+1}-\frac{1}{2}\delta\ell\tilde{R}_0^{n+1}\\
&=\left(1-\phi\frac{\ell}{2}-r\frac{\ell}{2}-P_1\right)\tilde{S}_0^n+P_1\tilde{S}_1^n+\frac{\ell}{2}\theta\tilde{V}_0^n+\frac{\ell}{2}\delta\tilde{R}_0^n+\ell r\ .
\end{aligned}\tag{3.109}$$

$$\begin{aligned}
&-\frac{\ell}{2}\phi\tilde{S}_0^{n+1}+\left(1+\beta_3\frac{\ell}{2}\tilde{E}_0^n+\beta_4\frac{\ell}{2}\tilde{I}_0^n-\alpha\frac{\ell}{2}\tilde{I}_0^n+r\frac{\ell}{2}+\theta\frac{\ell}{2}+P_2\right)\tilde{V}_0^{n+1}\\
&-P_2\tilde{V}_1^{n+1}+\beta_3\frac{\ell}{2}\tilde{V}_0^n\tilde{E}_0^{n+1}+\left(\beta_4\frac{\ell}{2}\tilde{V}_0^n-\alpha\frac{\ell}{2}\tilde{V}_0^n\right)\tilde{I}_0^{n+1}\\
&=\ \frac{\ell}{2}\phi\tilde{S}_0^n+\left(1-r\frac{\ell}{2}-\theta\frac{\ell}{2}-P_2\right)\tilde{V}_0^n+P_2\tilde{V}_1^n+\left(\beta_1\frac{\ell}{2}\tilde{E}_0^n+\beta_2\frac{\ell}{2}\tilde{I}_0^n\right)\tilde{S}_0^{n+1}\\
&-\left(\beta_4\frac{\ell}{2}\tilde{I}_0^n+\beta_3\frac{\ell}{2}\tilde{E}_0^n\right)\tilde{V}_0^{n+1}.
\end{aligned}\tag{3.110}$$

$$\begin{aligned}
&\left(1-\beta_1\frac{\ell}{2}\tilde{S}_0^n-\beta_3\frac{\ell}{2}\tilde{V}_0^n-\alpha\frac{\ell}{2}\tilde{I}_0^n+w_1\frac{\ell}{2}+P_3\right)\tilde{E}_0^{n+1}-P_3\tilde{E}_1^{n+1}-\beta_2\tilde{I}_0^n\tilde{S}_0^{n+1}\\
&+\beta_4\tilde{I}_0^n\tilde{V}_0^{n+1}-\left(\alpha\frac{\ell}{2}\tilde{E}_0^n+\beta_2\frac{\ell}{2}\tilde{S}_0^n+\beta_4\frac{\ell}{2}\tilde{V}_0^n\right)\tilde{I}_0^{n+1}\\
&=\ \left(1-w_1\frac{\ell}{2}-P_3\right)\tilde{E}_0^n+P_3\tilde{E}_1^n.
\end{aligned}\tag{3.111}$$

$$\begin{aligned}
&-\sigma\frac{\ell}{2}\tilde{E}_0^{n+1}+\left(1+w_2\frac{\ell}{2}-\alpha\ell\tilde{I}_0^{n+1}+P_4\right)\tilde{I}_0^{n+1}-P_4\tilde{I}_1^{n+1}\\
&=\sigma\frac{\ell}{2}\tilde{E}_0^n+\left(1-w_2\frac{\ell}{2}-P_4\right)\tilde{I}_0^n+P_4\tilde{I}_1^n.
\end{aligned}\tag{3.112}$$

$$\begin{aligned}
&-\kappa\frac{\ell}{2}\tilde{E}_0^n-\left(\gamma\frac{\ell}{2}+\alpha\frac{\ell}{2}\tilde{R}_0^n\right)\tilde{I}_0^{n+1}+\left(1+r\frac{\ell}{2}+\delta\frac{\ell}{2}-\alpha\frac{\ell}{2}\tilde{I}_0^n+P_5\right)\tilde{R}_0^{n+1}\\
&-P_5\tilde{R}_1^{n+1}\\
&=\kappa\frac{\ell}{2}\tilde{E}_0^n+\gamma\frac{\ell}{2}\tilde{I}_0^n+\left(1-r\frac{\ell}{2}-\delta\frac{\ell}{2}-P_5\right)\tilde{R}_0^n+P_5\tilde{R}_1^n.
\end{aligned}\tag{3.113}$$

$m = M$,

$$-P_1\,\tilde{S}_{M-1}^{n+1} + \left(1+\frac{1}{2}(\beta_1\ell\tilde{E}_M^n + \beta_2\ell\tilde{I}_M^n - \alpha\ell\tilde{I}_M^n + \phi\ell + r\ell + P_1\,)\right)\tilde{S}_M^{n+1} - \frac{1}{2}\theta\tilde{V}_M^{n+1}$$
$$-\frac{1}{2}\beta_1\ell\tilde{S}_M^n\,\tilde{E}_M^{n+1} + \left(-\frac{1}{2}\alpha\ell\tilde{S}_M^n + \frac{1}{2}\beta_2\ell\tilde{S}_0^n\right)\tilde{I}_M^{n+1} - \frac{1}{2}\delta\ell\tilde{R}_M^{n+1}$$
$$= P_1\,\tilde{S}_{M-1}^n + \left(1-\phi\frac{\ell}{2} - r\frac{\ell}{2} - P_1\right)\tilde{S}_M^n + \frac{\ell}{2}\theta\tilde{V}_M^n + \frac{\ell}{2}\delta\tilde{R}_M^n + \ell r\ . \tag{3.114}$$

$$-\frac{\ell}{2}\phi\tilde{S}_M^{n+1} - P_2\,\tilde{V}_{M-1}^{n+1} + \left(1+\beta_3\frac{\ell}{2}\tilde{E}_M^n + \beta_4\frac{\ell}{2}\tilde{I}_M^n - \alpha\frac{\ell}{2}\tilde{I}_M^n + r\frac{\ell}{2} + \theta\frac{\ell}{2} + P_2\right)\tilde{V}_M^{n+1}$$
$$+\beta_3\frac{\ell}{2}\tilde{V}_M^n\,\tilde{E}_M^{n+1} + \left(\beta_4\frac{\ell}{2}\tilde{V}_M^n - \alpha\frac{\ell}{2}\tilde{V}_M^n\right)\tilde{I}_M^{n+1}$$
$$= \frac{\ell}{2}\phi\tilde{S}_M^n + P_2\,\tilde{V}_{M-1}^n + \left(1-r\frac{\ell}{2} - \theta\frac{\ell}{2} - P_2\right)\tilde{V}_M^n + \left(\beta_1\frac{\ell}{2}\tilde{E}_M^n + \beta_2\frac{\ell}{2}\tilde{I}_M^n\right)\tilde{S}_M^{n+1}$$
$$-\left(\beta_4\frac{\ell}{2}\tilde{I}_M^n + \beta_3\frac{\ell}{2}\tilde{E}_M^n\right)\tilde{V}_M^{n+1}. \tag{3.115}$$

$$-P_3\,\tilde{E}_{M-1}^{n+1} + \left(1-\beta_1\frac{\ell}{2}\tilde{S}_M^n - \beta_3\frac{\ell}{2}\tilde{V}_M^n - \alpha\frac{\ell}{2}\tilde{I}_M^n + w_1\frac{\ell}{2} + P_3\right)\tilde{E}_M^{n+1} - \beta_2\tilde{I}_M^n\,\tilde{S}_M^{n+1}$$
$$+\beta_4\tilde{I}_M^n\tilde{V}_M^{n+1} - \left(\alpha\frac{\ell}{2}\tilde{E}_M^n + \beta_2\frac{\ell}{2}\tilde{S}_M^n + \beta_4\frac{\ell}{2}\tilde{V}_M^n\right)\tilde{I}_M^{n+1}$$
$$= P_3\,\tilde{E}_{M-1}^n + \left(1-w_1\frac{\ell}{2} - P_3\right)\tilde{E}_M^n\,. \tag{3.116}$$

$$-\sigma\frac{\ell}{2}\tilde{E}_M^{n+1} - P_4\,\tilde{I}_{M-1}^{n+1} + \left(1+w_2\frac{\ell}{2} - \alpha\ell\tilde{I}_M^{n+1} + P_4\right)\tilde{I}_M^{n+1}$$
$$= \sigma\frac{\ell}{2}\tilde{E}_M^n + P_4\,\tilde{I}_{M-1}^n + \left(1-w_2\frac{\ell}{2} - P_4\right)\tilde{I}_M^n\,. \tag{3.117}$$

$$-\kappa\frac{\ell}{2}\tilde{E}_M^n - \left(\gamma\frac{\ell}{2} + \alpha\frac{\ell}{2}\tilde{R}_M^n\right)\tilde{I}_M^{n+1} - P_5\,\tilde{R}_{M-1}^{n+1} + \left(1+r\frac{\ell}{2} + \delta\frac{\ell}{2} - \alpha\frac{\ell}{2}\tilde{I}_M^n + P_5\right)\tilde{R}_M^{n+1}$$
$$= \kappa\frac{\ell}{2}\tilde{E}_M^n + \gamma\frac{\ell}{2}\tilde{I}_M^n + P_5\,\tilde{R}_{M-1}^n + \left(1-r\frac{\ell}{2} - \delta\frac{\ell}{2} - P_5\right)\tilde{R}_M^n\,. \tag{3.118}$$

For m = 1, 2,…, M-1,

$$\frac{-P_1}{2}\tilde{S}_{m-1}^{n+1}+(1+\frac{1}{2}(\beta_1\ell\tilde{E}_m^n+\beta_2\ell\tilde{I}_m^n-\alpha\ell\tilde{I}_m^n+\phi\ell+r\ell+P_1))\tilde{S}_m^{n+1}-\frac{P_1}{2}\tilde{S}_{m+1}^{n+1} \tag{3.119}$$

$$-\frac{1}{2}\theta\tilde{V}_m^{n+1}-\frac{1}{2}\beta_1\ell\tilde{S}_m^n\tilde{E}_m^{n+1}+\left(-\frac{1}{2}\alpha\ell\tilde{S}_m^n+\frac{1}{2}\beta_2\ell\tilde{S}_m^n\right)\tilde{I}_m^{n+1}-\frac{1}{2}\delta\ell\tilde{R}_m^{n+1}$$

$$=\frac{P_1}{2}\tilde{S}_{m-1}^n+\left(1-\phi\frac{\ell}{2}-r\frac{\ell}{2}-P_1\right)\tilde{S}_m^n+\frac{P_1}{2}\tilde{S}_{m+1}^n+\frac{\ell}{2}\theta\tilde{V}_m^n+\frac{\ell}{2}\delta\tilde{R}_m^n+\ell r\ .$$

$$-\frac{\ell}{2}\phi\tilde{S}_m^{n+1}-\frac{p_2}{2}\tilde{V}_{m-1}^{n+1}+\left(1+\beta_3\frac{\ell}{2}\tilde{E}_m^n+\beta_4\frac{\ell}{2}\tilde{I}_m^n-\alpha\frac{\ell}{2}\tilde{I}_m^n+r\frac{\ell}{2}+\theta\frac{\ell}{2}+P_2\right)\tilde{V}_m^{n+1} \tag{3.120}$$

$$-\frac{P_2}{2}\tilde{V}_{m+1}^{n+1}+\beta_3\frac{\ell}{2}\tilde{V}_m^n\tilde{E}_m^{n+1}+\left(\beta_4\frac{\ell}{2}\tilde{V}_m^n-\alpha\frac{\ell}{2}\tilde{V}_m^n\right)\tilde{I}_m^{n+1}=\ \frac{\ell}{2}\phi\tilde{S}_m^n+\frac{p_2}{2}\tilde{V}_{m-1}^n$$

$$+\left(1-r\frac{\ell}{2}-\theta\frac{\ell}{2}-P_2\right)\tilde{V}_m^n+\frac{p_2}{2}\tilde{V}_{m+1}^n-\left(\beta_1\frac{\ell}{2}\tilde{E}_m^n+\beta_2\frac{\ell}{2}\tilde{I}_m^n\right)\tilde{S}_m^{n+1}.$$

$$-\left(\beta_4\frac{\ell}{2}\tilde{I}_m^n+\beta_3\frac{\ell}{2}\tilde{E}_m^n\right)\tilde{V}_m^{n+1}-\frac{P_3}{2}\tilde{E}_{m-1}^{n+1} \tag{3.121}$$

$$+\left(1-\beta_1\frac{\ell}{2}\tilde{S}_m^n-\beta_3\frac{\ell}{2}\tilde{V}_m^n-\alpha\frac{\ell}{2}\tilde{I}_m^n+w_1\frac{\ell}{2}+P_3\right)\tilde{E}_m^{n+1}$$

$$-\beta_2\tilde{I}_m^n\tilde{S}_m^{n+1}-\frac{P_3}{2}\tilde{E}_{m+1}^{n+1}+\beta_4\tilde{I}_m^n\tilde{V}_m^{n+1}-\left(\alpha\frac{\ell}{2}\tilde{E}_m^n+\beta_2\frac{\ell}{2}\tilde{S}_m^n+\beta_4\frac{\ell}{2}\tilde{V}_m^n\right)\tilde{I}_m^{n+1}$$

$$=\ \frac{P_3}{2}\tilde{E}_{m-1}^n+\left(1-w_1\frac{\ell}{2}-P_3\right)+\frac{P_3}{2}\tilde{E}_{m+1}^n.$$

$$-\sigma\frac{\ell}{2}\tilde{E}_m^{n+1}-\frac{P_4}{2}\tilde{I}_{m-1}^{n+1}+\left(1+w_2\frac{\ell}{2}-\alpha\ell\tilde{I}_m^{n+1}+P_4\right)\tilde{I}_m^{n+1}-\frac{P_4}{2}\tilde{I}_{m+1}^{n+1} \tag{3.122}$$

$$=\sigma\frac{\ell}{2}\tilde{E}_m^n+\frac{P_4}{2}\tilde{I}_{m-1}^n+\left(1-w_2\frac{\ell}{2}-P_4\right)\tilde{I}_m^n+\frac{P_4}{2}\tilde{I}_{m+1}^n.$$

$$-\kappa\frac{\ell}{2}\tilde{E}_m^n-\left(\gamma\frac{\ell}{2}+\alpha\frac{\ell}{2}\tilde{R}_m^n\right)\tilde{I}_m^{n+1}-\frac{P_5}{2}\tilde{R}_{m-1}^{n+1}+ \tag{3.123}$$

$$\left(1+r\frac{\ell}{2}+\delta\frac{\ell}{2}-\alpha\frac{\ell}{2}\tilde{I}_m^n+P_5\right)\tilde{R}_m^{n+1}-\frac{P_5}{2}\tilde{R}_{m+1}^{n+1}$$

$$=\kappa\frac{\ell}{2}\tilde{E}_m^n+\gamma\frac{\ell}{2}\tilde{I}_m^n+\frac{P_5}{2}\tilde{R}_{m-1}^n+\left(1-r\frac{\ell}{2}-\delta\frac{\ell}{2}-P_5\right)\tilde{R}_m^n+\frac{P_5}{2}\tilde{R}_{m+1}^n.$$

The solution vector $\tilde{\mathbf{S}}, \tilde{\mathbf{V}}, \tilde{\mathbf{E}}, \tilde{\mathbf{I}}$ and $\tilde{\mathbf{R}}$ can be obtained simultaneously by solving a linear algebraic. It may be seen that (3.109)-(3.123) may be written form in the

$$W^n \mathbf{U}^{n+1} = M^n \mathbf{U}^n + \mathbf{b} \tag{3.124}$$

Let

$\tilde{\mathbf{S}}^{n+1} = \left[\tilde{S}_0^{n+1}, \tilde{S}_1^{n+1}, \ldots, \tilde{S}_M^{n+1}\right]^T$, $\tilde{\mathbf{V}}^{n+1} = \left[\tilde{V}_0^{n+1}, \tilde{V}_1^{n+1}, \ldots, \tilde{V}_M^{n+1}\right]^T$, $\tilde{\mathbf{E}}^{n+1} = \left[\tilde{E}_0^{n+1}, \tilde{E}_1^{n+1}, \ldots, \tilde{E}_M^{n+1}\right]^T$,

$\tilde{\mathbf{I}}^{n+1} = \left[\tilde{I}_0^{n+1}, \tilde{I}_1^{n+1}, \ldots, \tilde{I}_M^{n+1}\right]^T$, $\tilde{\mathbf{R}}^{n+1} = \left[\tilde{R}_0^{n+1}, \tilde{R}_1^{n+1}, \ldots, \tilde{R}_M^{n+1}\right]^T$

$$\mathbf{U}^{n+1} = \left[\left(\tilde{\mathbf{S}}^{n+1}\right)^T, \left(\tilde{\mathbf{V}}^{n+1}\right)^T, \left(\tilde{\mathbf{E}}^{n+1}\right)^T, \left(\tilde{\mathbf{I}}^{n+1}\right)^T, \left(\tilde{\mathbf{R}}^{n+1}\right)^T\right]^T,$$

$$\mathbf{U}^{n} = \left[\left(\tilde{\mathbf{S}}^{n}\right)^T, \left(\tilde{\mathbf{V}}^{n}\right)^T, \left(\tilde{\mathbf{E}}^{n}\right)^T, \left(\tilde{\mathbf{I}}^{n}\right)^T, \left(\tilde{\mathbf{R}}^{n}\right)^T\right]^T,$$

and

$$W^n = \begin{bmatrix} A_1^n & A_2 & A_3^n & A_4^n & A_5 \\ B_1 & B_2^n & B_3^n & B_4^n & 0 \\ C_1^n & C_2^n & C_3^n & C_4^n & 0 \\ 0 & 0 & Q_3 & Q_4^n & 0 \\ 0 & 0 & G_3^n & G_4^n & G_5^n \end{bmatrix}, \tag{3.125}$$

$$M^n = \begin{bmatrix} a_1 & a_2 & 0 & 0 & a_5^n \\ b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & q_3 & q_4 & 0 \\ 0 & 0 & g_3 & g_4 & g_5^n \end{bmatrix}, \tag{3.126}$$

Where 0 is the zero matrix of order (M+1) and the vector **b** is a column-vector of order (5M+5) wish given by

$$\mathbf{b} = \Big[\underbrace{\ell r, \ldots, \ell r,}_{M+1 \text{ times}} \; 0, \ldots, 0 \Big]^T , \tag{3.127}$$

The matrices W^n and M^n are both of order (5M+5) and their sub matrices are of order (M+1) These matrices are given by

$$A_1^n = \begin{bmatrix} o_0 & -P_1 & 0 & \cdots & 0 \\ -\frac{P_1}{2} & o_1 & -\frac{P_1}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -\frac{P_1}{2} & o_{M-1} & -\frac{P_1}{2} \\ 0 & \cdots & 0 & -P_1 & o_M \end{bmatrix}, \tag{3.128}$$

with $\text{o}_i = 1 + \frac{1}{2}(\beta_1 \ell \tilde{E}_i^n + \beta_2 \ell \tilde{I}_i^n - \alpha \ell \tilde{I}_i^n + \phi \ell + r\ell + P_1);\; i = 0,1,2,\ldots,M-1,M.$

$$a_1 = \begin{bmatrix} \left(1-\phi\frac{\ell}{2}-r\frac{\ell}{2}-P_1\right) & P_1 & 0 & \cdots & 0 \\ \frac{P_1}{2} & \left(1-\phi\frac{\ell}{2}-r\frac{\ell}{2}-P_1\right) & \frac{P_1}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{P_1}{2} & \left(1-\phi\frac{\ell}{2}-r\frac{\ell}{2}-P_1\right) & \frac{P_1}{2} \\ 0 & \cdots & 0 & P_1 & \left(1-\phi\frac{\ell}{2}-r\frac{\ell}{2}-P_1\right) \end{bmatrix}, \tag{3.129}$$

$$B_2^n = \begin{bmatrix} b_0 & -P_2 & 0 & \cdots & 0 \\ -\frac{P_2}{2} & b_1 & -\frac{P_2}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -\frac{P_2}{2} & b_{M-1} & -\frac{P_2}{2} \\ 0 & \cdots & 0 & -P_2 & b_M \end{bmatrix}, \tag{3.130}$$

with $b_i = 1 + \beta_3 \frac{\ell}{2} \tilde{E}_i^n + \beta_4 \frac{\ell}{2} \tilde{I}_i^n - \alpha \frac{\ell}{2} \tilde{I}_i^n + r\frac{\ell}{2} + \theta \frac{\ell}{2} + P_2 \,; \quad i = 0,1,2,\ldots,M-1,M.$

$$b_2 = \begin{bmatrix} \left(1-\theta\frac{\ell}{2}-r\frac{\ell}{2}-P_2\right) & P_2 & 0 & \cdots & 0 \\ \frac{P_2}{2} & \left(1-\theta\frac{\ell}{2}-r\frac{\ell}{2}-P_2\right) & \frac{P_2}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{P_2}{2} & \left(1-\theta\frac{\ell}{2}-r\frac{\ell}{2}-P_2\right) & \frac{P_2}{2} \\ 0 & \cdots & 0 & P_2 & \left(1-\theta\frac{\ell}{2}-r\frac{\ell}{2}-P_2\right) \end{bmatrix}, \tag{3.131}$$

$$C_3^n = \begin{bmatrix} z_0 & -P_3 & 0 & \cdots & 0 \\ -\frac{P_3}{2} & z_1 & -\frac{P_3}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -\frac{P_3}{2} & z_{M-1} & -\frac{P_3}{2} \\ 0 & \cdots & 0 & -P_3 & z_M \end{bmatrix}, \tag{3.132}$$

with $\mathrm{z}_i = 1-\beta_1\frac{\ell}{2}\tilde{S}_i^n - \beta_3\frac{\ell}{2}\tilde{V}_i^n - \alpha\frac{\ell}{2}\tilde{I}_i^n + w_1\frac{\ell}{2} + P_3\,; \quad i = 0,1,2,\ldots,M-1,M.$

$$c_3 = \begin{bmatrix} \left(1-w_1\frac{\ell}{2}-P_3\right) & P_3 & 0 & \cdots & 0 \\ \frac{P_3}{2} & \left(1-w_1\frac{\ell}{2}-P_3\right) & \frac{P_3}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{P_3}{2} & \left(1-w_1\frac{\ell}{2}-P_3\right) & \frac{P_3}{2} \\ 0 & \cdots & 0 & P_3 & \left(1-w_1\frac{\ell}{2}-P_3\right) \end{bmatrix}, \tag{3.133}$$

$$Q_4^n = \begin{bmatrix} e_0 & -P_4 & 0 & \cdots & 0 \\ -\frac{P_4}{2} & e_1 & -\frac{P_4}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -\frac{P_4}{2} & e_{M-1} & -\frac{P_4}{2} \\ 0 & \cdots & 0 & -P_4 & e_M \end{bmatrix}, \tag{3.134}$$

with $\mathrm{e}_i = 1 + w_2\frac{\ell}{2} - \alpha\ell\tilde{I}_i^{n+1} + P_4\,; \quad i = 0,1,2,\ldots,M-1,M.$

$$q_4 = \begin{bmatrix} \left(1-w_2\frac{\ell}{2}-P_3\right) & P_4 & 0 & \cdots & 0 \\ \frac{P_4}{2} & \left(1-w_2\frac{\ell}{2}-P_4\right) & \frac{P_4}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{P_4}{2} & \left(1-w_2\frac{\ell}{2}-P_4\right) & \frac{P_4}{2} \\ 0 & \cdots & 0 & P_4 & \left(1-w_2\frac{\ell}{2}-P_4\right) \end{bmatrix}, \tag{3.135}$$

$$g_5^n = \begin{bmatrix} j & P_5 & 0 & \cdots & 0 \\ \frac{P_5}{2} & j & \frac{P_5}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{P_5}{2} & j & \frac{P_5}{2} \\ 0 & \cdots & 0 & P_5 & j \end{bmatrix}, \tag{3.136}$$

with $j = 1 - r\frac{\ell}{2} - \delta\frac{\ell}{2} - P_5$

$$G_5^n = \begin{bmatrix} g_0 & -P_5 & 0 & \cdots & 0 \\ -\frac{P_5}{2} & g_1 & -\frac{P_5}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -\frac{P_5}{2} & g_{M-1} & -\frac{P_5}{2} \\ 0 & \cdots & 0 & -P_5 & g_M \end{bmatrix}, \tag{3.137}$$

with $\mathrm{g}_i = 1 + r\frac{\ell}{2} + \delta\frac{\ell}{2} - \alpha\frac{\ell}{2}\tilde{I}_i^n + P_5; \quad i = 0,1,2,\ldots,M-1,M.$

$$A_2^n = \begin{bmatrix} \theta\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \theta\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \theta\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \theta\frac{\ell}{2} \end{bmatrix}, \tag{3.138}$$

$$A_3^n = \begin{bmatrix} \beta_1 \frac{\ell}{2}\tilde{S}_0^n & 0 & 0 & \cdots & 0 \\ 0 & \beta_1 \frac{\ell}{2}\tilde{S}_1^n & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \beta_1 \frac{\ell}{2}\tilde{S}_{M-1}^n & 0 \\ 0 & \cdots & 0 & 0 & \beta_1 \frac{\ell}{2}\tilde{S}_M^n \end{bmatrix}, \tag{3.139}$$

$$A_4^n = \begin{bmatrix} \frac{\ell}{2}\tilde{S}_0^n\left(\beta_2 - \alpha\right) & 0 & 0 & \cdots & 0 \\ 0 & \frac{\ell}{2}\tilde{S}_1^n\left(\beta_2 - \alpha\right) & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \frac{\ell}{2}\tilde{S}_{M-1}^n\left(\beta_2 - \alpha\right) & 0 \\ 0 & \cdots & 0 & 0 & \frac{\ell}{2}\tilde{S}_M^n\left(\beta_2 - \alpha\right) \end{bmatrix}, \tag{3.140}$$

$$A_5 = \begin{bmatrix} -\delta\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & -\delta\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -\delta\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & -\delta\frac{\ell}{2} \end{bmatrix}, \tag{3.141}$$

$$a_2 = \begin{bmatrix} \theta\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \theta\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \theta\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \theta\frac{\ell}{2} \end{bmatrix}, \tag{3.142}$$

$$a_5 = \begin{bmatrix} \delta\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \delta\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \delta\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \delta\frac{\ell}{2} \end{bmatrix}, \quad (3.143)$$

$$B_1 = \begin{bmatrix} \phi\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \phi\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \phi\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \phi\frac{\ell}{2} \end{bmatrix}, \quad (3.144)$$

$$B_3^n = \begin{bmatrix} \beta_3\frac{\ell}{2}\tilde{V}_0^n & 0 & 0 & \cdots & 0 \\ 0 & \beta_3\frac{\ell}{2}\tilde{V}_1^n & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \beta_3\frac{\ell}{2}\tilde{V}_{M-1}^n & 0 \\ 0 & \cdots & 0 & 0 & \beta_3\frac{\ell}{2}\tilde{V}_M^n \end{bmatrix}, \quad (3.145)$$

$$B_4^n = \begin{bmatrix} \frac{\ell}{2}\tilde{V}_0^n(\beta_4-\alpha) & 0 & 0 & \cdots & 0 \\ 0 & \frac{\ell}{2}\tilde{V}_1^n(\beta_4-\alpha) & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \frac{\ell}{2}\tilde{V}_{M-1}^n(\beta_4-\alpha) & 0 \\ 0 & \cdots & 0 & 0 & \frac{\ell}{2}\tilde{V}_M^n(\beta_4-\alpha) \end{bmatrix}, \tag{3.146}$$

$$b_1 = \begin{bmatrix} \phi\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \phi\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \phi\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \phi\frac{\ell}{2} \end{bmatrix}, \tag{3.147}$$

$$C_1^n = \begin{bmatrix} -f_0 & 0 & 0 & \cdots & 0 \\ 0 & -f_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -f_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & -f_M \end{bmatrix}, \tag{3.148}$$

Where $f_i = \beta_1 \frac{\ell}{2}\tilde{E}_i^n + \beta_2 \frac{\ell}{2}\tilde{I}_i^n; \quad i = 0,1,2,\ldots,M-1,M$

$$C_2^n = \begin{bmatrix} -j_0 & 0 & 0 & \cdots & 0 \\ 0 & -j_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -j_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & -j_M \end{bmatrix}, \tag{3.149}$$

with $j_i = \beta_3 \frac{\ell}{2}\tilde{E}_i^n + \beta_4 \frac{\ell}{2}\tilde{I}_i^n; \quad i = 0,1,2,\ldots,M-1,M.$

$$C_4^n = \begin{bmatrix} -k_0 & 0 & 0 & \cdots & 0 \\ 0 & -k_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -k_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & -k_M \end{bmatrix}, \quad (3.150)$$

with $k_i = \alpha \frac{\ell}{2} \tilde{E}_i^n + \beta_2 \frac{\ell}{2} \tilde{S}_i^n + \beta_4 \frac{\ell}{2} \tilde{V}_i^n; \quad i = 0,1,2,\ldots,M-1,M$

$$Q_3 = \begin{bmatrix} -\sigma\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & -\sigma\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -\sigma\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & -\sigma\frac{\ell}{2} \end{bmatrix}, \quad (3.151)$$

$$q_3 = \begin{bmatrix} \sigma\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \sigma\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \sigma\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \sigma\frac{\ell}{2} \end{bmatrix}, \quad (3.152)$$

$$G_3 = \begin{bmatrix} -\kappa\frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & -\kappa\frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -\kappa\frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & -\kappa\frac{\ell}{2} \end{bmatrix}, \quad (3.153)$$

$$G_4^n = \begin{bmatrix} -u_0 & 0 & 0 & \cdots & 0 \\ 0 & -u_1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & -u_{M-1} & 0 \\ 0 & \cdots & 0 & 0 & -u_M \end{bmatrix}, \tag{3.154}$$

with $u_i = \gamma \frac{\ell}{2} + \alpha \frac{\ell}{2} \tilde{R}_i^n; \quad i = 0, 1, 2, \ldots, M-1, M.$

$$g_3 = \begin{bmatrix} \kappa \frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \kappa \frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \kappa \frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \kappa \frac{\ell}{2} \end{bmatrix}, \tag{3.155}$$

$$g_4 = \begin{bmatrix} \gamma \frac{\ell}{2} & 0 & 0 & \cdots & 0 \\ 0 & \gamma \frac{\ell}{2} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \gamma \frac{\ell}{2} & 0 \\ 0 & \cdots & 0 & 0 & \gamma \frac{\ell}{2} \end{bmatrix}, \tag{3.156}$$

The vector $\tilde{\mathbf{S}}, \tilde{\mathbf{V}}, \tilde{\mathbf{E}}, \tilde{\mathbf{I}}$ and $\tilde{\mathbf{R}}$ can be solved by implementation from equation (3.124).

CHAPTER 4 RESULTS AND DISCUSSION

4.1 Parameters and Initial Condition of Numerical Method

To test the standard finite difference method and nonstandard finite difference method in the model (3.3) is solved for by using the set of parameters [7]:

$$\beta = 0.514\ ,\ \beta_E = 0.25, \beta_I = 1\ ,\ \mu = 5.5\times10^{-8}\ ,\ r = 7.14\times10^{-5}$$
$$\kappa = 1.857\times10^{-4}\ ,\ \beta_V = 0.1\ ,\ 0.2\quad \alpha = 9.3\times10^{-6}\ ,\ \sigma = \frac{1}{2}\ ,\ \theta = \frac{1}{365}$$
$$\gamma = \frac{1}{5}\ ,\ \delta = \frac{1}{365}\ ,$$

In this experiment, the interval $-2 \le x \le 2$ may be discretized by using h= 0.2 of the model are applied with initial condition (i). And the interval $-0.6 \le x \le 0.6$ may be discretize by using h= 0.05 of the model are applied with initial condition (ii) which is shown in Figure 4.1 and Figure 4.2

$$S_0(x) = 0.86\exp\left(-(x/1.4)^2\right), \quad -2 \le x \le 2.$$
$$V_0(x) = 0.1\exp\left(-(x/1.4)^2\right), \quad -2 \le x \le 2.$$
$$E_0(x) = 0, \quad -2 \le x \le 2.$$
$$I_0(x) = 0.04\exp\left(-x^2\right), \quad -2 \le x \le 2.$$
$$R_0(x) = 0, \quad -2 \le x \le 2.$$

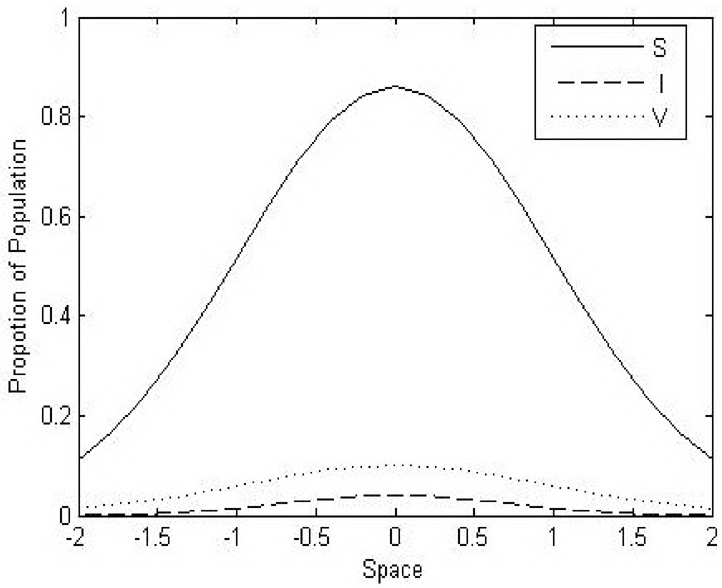

Figure 4.1 Experiment initial condition (i)

$$
\begin{aligned}
&S_0 = 0.86Sech^2(10x), && -2 \le x \le 2.\\
&V_0 = 0.10Sech^2(10x), && -2 \le x \le 2.\\
&E_0 = 0, && -2 \le x \le 2.\\
&I_0 = \begin{cases} 0, & -2 \le x < -0.5,\\ 0.04, & -0.5 \le x \le 0.5,\\ 0, & 0.5 < x \le 2.\end{cases}\\
&R_0 = 0, && -2 \le x \le 2.
\end{aligned}
$$

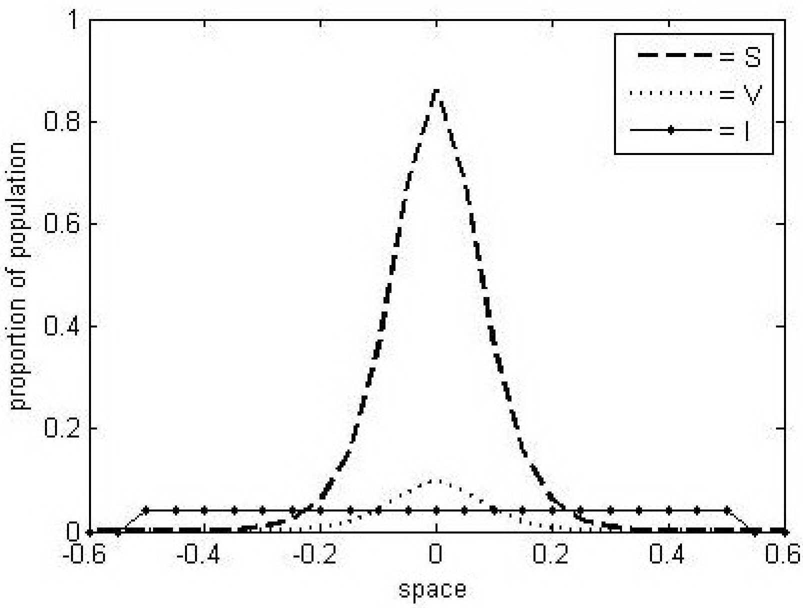

Figure 4.2 Experiment initial condition (ii)

4.2 Numerical Experiments of Standard Finite-Difference Method

Proportion of Susceptible Population (S)

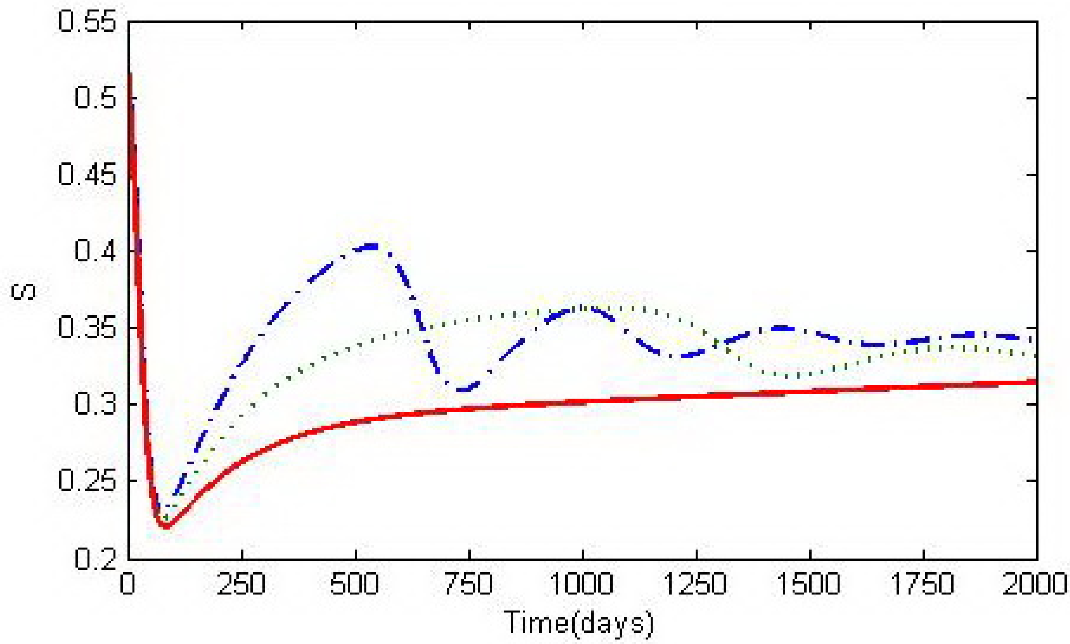

Figure 4.3 Solutions at x =1 ; $\beta_v = 0.1,\ h = 0.2,\ \ell$=0.025, $\phi = 0.001$ (– · –), $\phi = 0.002$ (........), $\phi = 0.003$ (——).

In figure 4.3, with initial condition (i) and for $\phi = 0.001$ the proportion of susceptible population produce pulses until 1500 days. The first pulse for proportion of susceptible population appears at t = 550 days with peak value as 0.41. The next pulse appears with peak value for S as 0.37 at t = 1000 days. After that, pulses appear with small peak value for the proportion of susceptible population till t = 1500 days and gradually decrease to a steady state value. For $\phi = 0.002$ the proportion of susceptible population produce only one pulse at t = 1000 days with peak value as 0.36. For $\phi = 0.003$ the proportion of susceptible population, there are no oscillations produced.

Proportion of Infected Population (I)

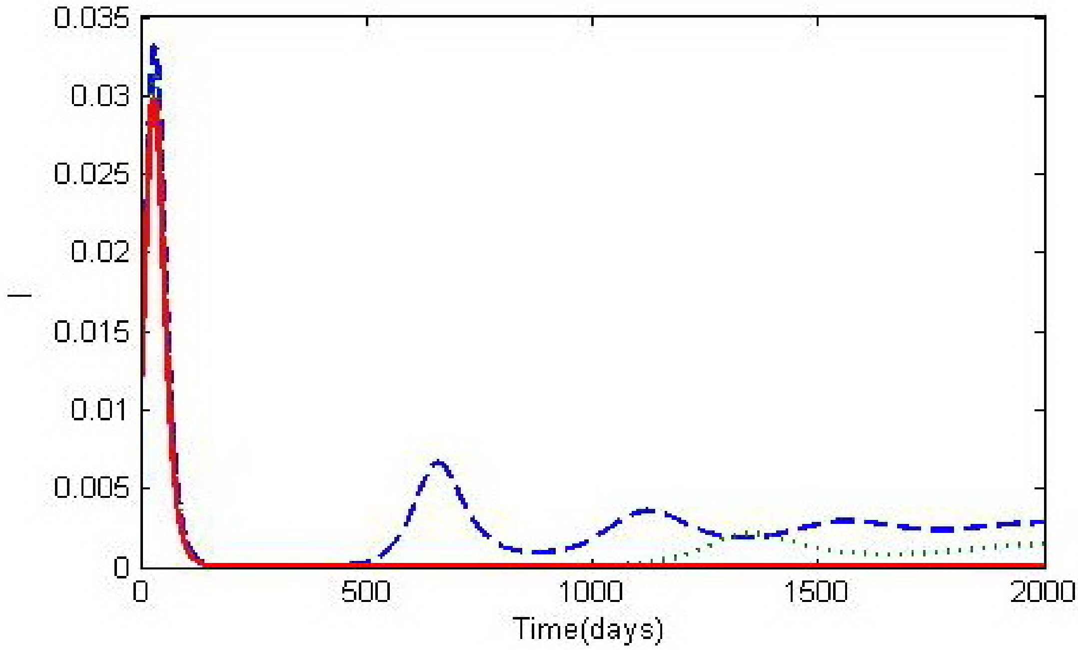

Figure 4.4 Solutions at x =1 ; $\beta_v = 0.1$, $h = 0.2$, ℓ=0.025, $\phi = 0.001$ ( ), $\phi = 0.002$ (........), $\phi = 0.003$ ( ).

In figure 4.4, with initial condition (i) and case $\phi = 0.001$, the proportion of infected population produce pulses until 1500 days. The first pulse for I population appears at t = 35 days with peak value as 0.0335. The next pulse appears with peak value for I as 0.007 at t = 650 days. After that, pulses appear with small peak value for proportion of infected population till t = 1500 days and gradually decrease to a steady state value. Case $\phi = 0.002$, the proportion of infected population produce two pulses at t = 35 and t = 1500 days with peak value as 0.032 and 0.002, respectively. Case $\phi = 0.003$, the proportion of infected population produce only one pulse at t = 40 days with peak value as 0.03.

4.3 Numerical Experiments of Non-Standard Finite-Difference Methods

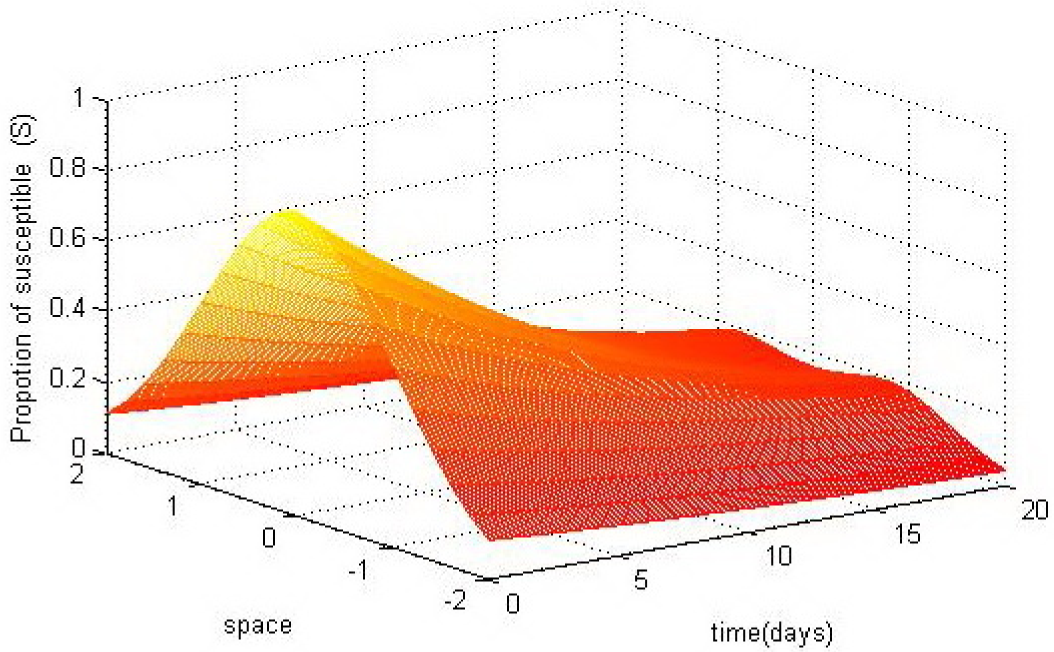

Figure 4.5 Three-dimensional distribution with initial condition (i) of proportion susceptible; $\beta_v = 0.1,\ h = 0.2,\ \ell$=0.1, ϕ=0.05.

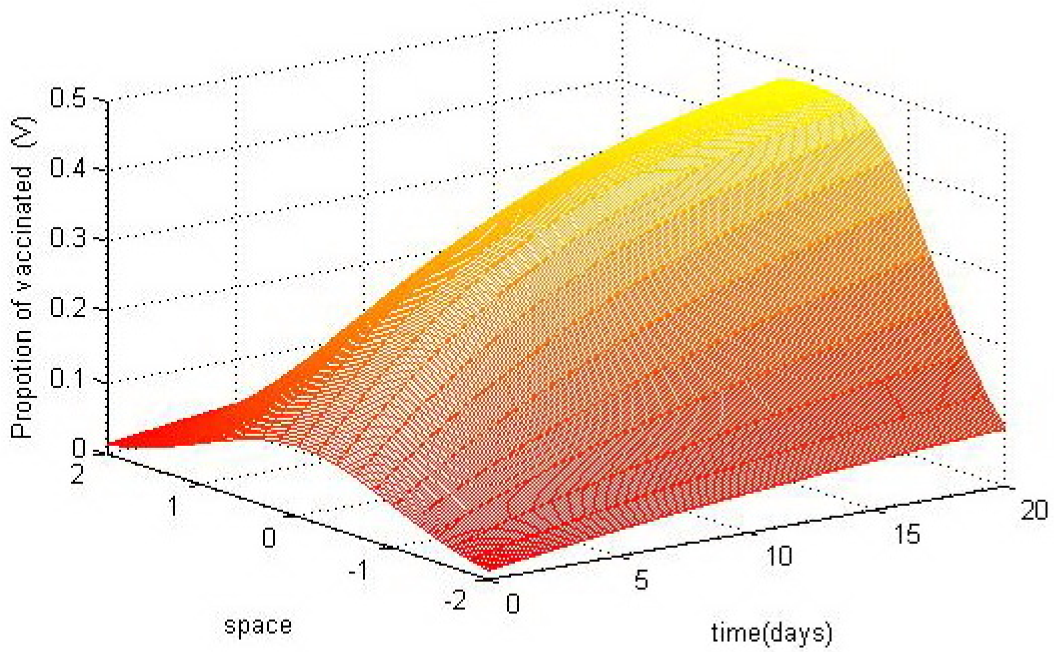

Figure 4.6 Three-dimensional distribution with initial condition (i) of proportion vaccinated; $\beta_v = 0.1,\ h = 0.2,\ \ell$=0.1, ϕ=0.05.

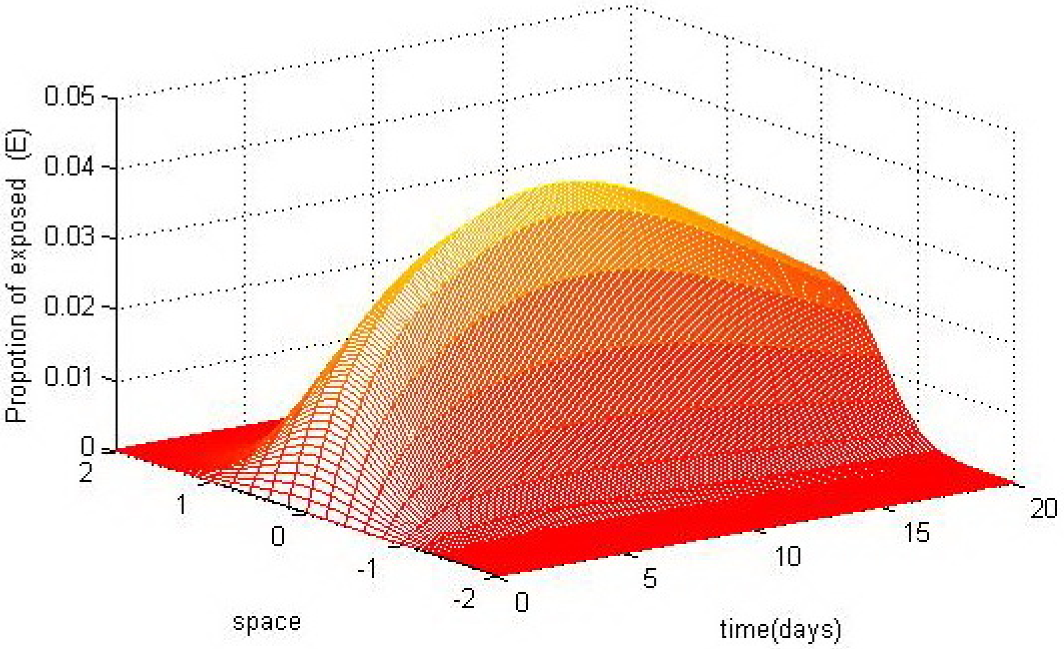

Figure 4.7 Three-dimensional distribution with initial condition (i) of proportion exposed; $\beta_v = 0.1$, $h = 0.2$, ℓ=0.1, ϕ=0.05.

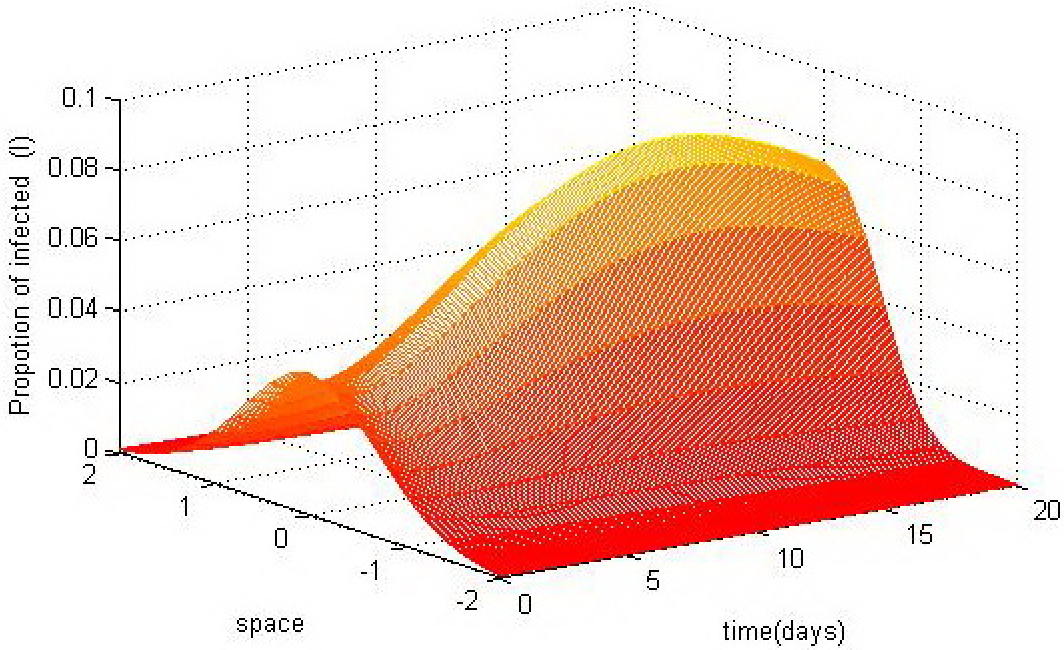

Figure 4.8 Three-dimensional distribution with initial condition (i) of proportion infected; $\beta_v = 0.1$, $h = 0.2$, ℓ=0.1, ϕ=0.05.

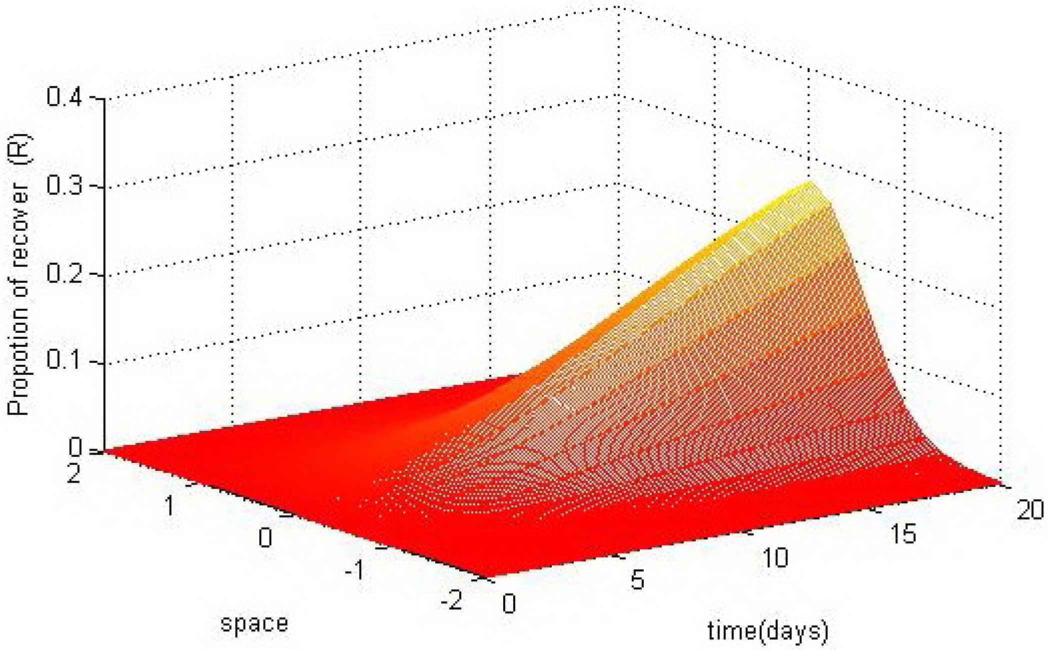

Figure 4.9 Three-dimensional distribution with initial condition (i) of proportion recover; $\beta_v = 0.1,\ h = 0.2,\ \ell=0.1,\ \phi=0.05.$

In figure 4.5-4.9, give three dimensional plots of proportion susceptible, vaccinated, exposed, infected recover population for t= 0 day until 20 days. As time increases, they found that the proportion of susceptible population decrease whereas the proportion of vaccinated, exposed, infected, recover population increase until time t = 15 day after which the proportion of susceptible become less than the proportion of vaccinated population. For the proportion of exposed population decrease after time t = 10 day. For the proportion of infected population decrease after time t = 15 day.

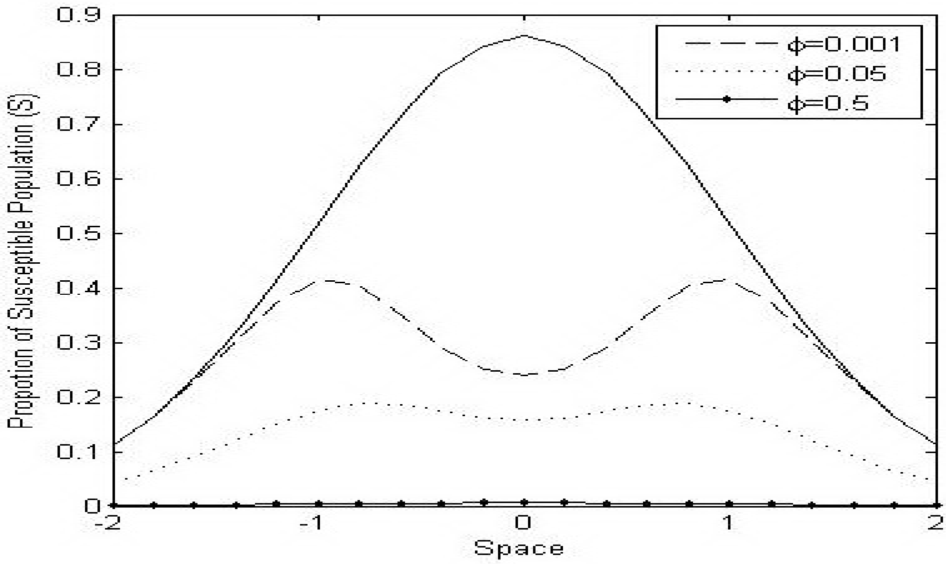

Figure 4.10 Solution with initial condition (i) and with diffusion of $\beta_v = 0.1$, $\ell = 0.1$ for cases $\phi = 0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

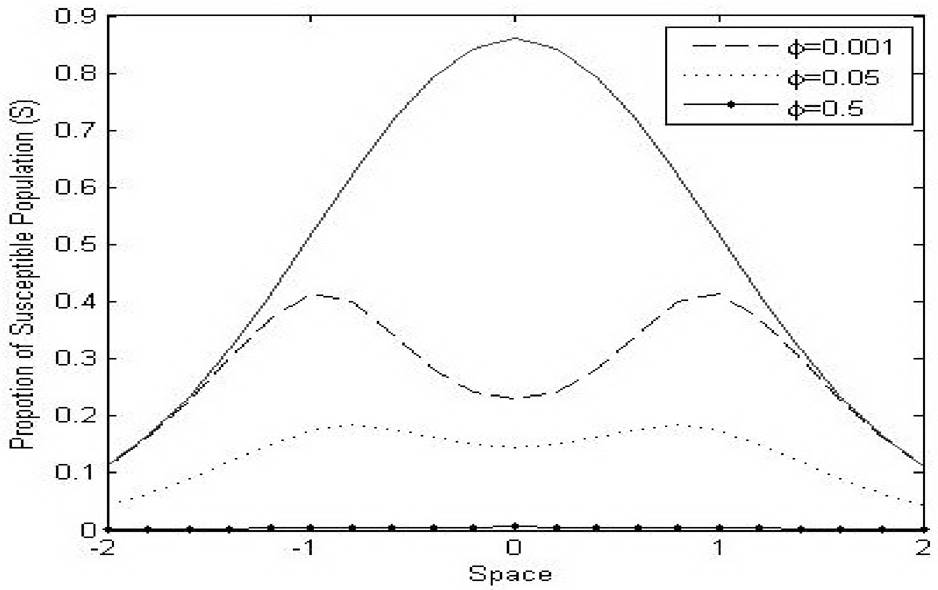

Figure 4.11 Solution with initial condition (i) and with diffusion of $\beta_v = 0.2$, $\ell = 0.1$ for cases $\phi = 0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

Figure 4.10-4.11, show the output with initial condition (i) on $t = 0$ day they found that the proportion of susceptible population is spread in all over domain [-2,2] with peak value as 0.1 at x=0. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001,\ 0.05$ and 0.5 the proportion of susceptible population is spread all over in domain [-2,2] with peak value as 0.415 at x= -1,1 ,as 0.188 at x= -0.8,0.8 as 0.006 at x= -1,1 respectively. On $t = 20$ day for $\beta_v = 0.2$ case $\phi = 0.001,\ 0.05$ and 0.5 the proportion of susceptible population is spread all over in domain [-2,2] with peak value as 0.081 at x= -1,1 as 0.413 at x= -0.8,0.8 as 0.184 at x = -1,1 respectively. The decrement of susceptible population enlarges the number rate of vaccinated. For $\beta_v = 0.1,\ 0.2$ at same ϕ they found that the decrement of vaccinated population enlarges the number ability to cause infection by vaccination individuals (β_v).

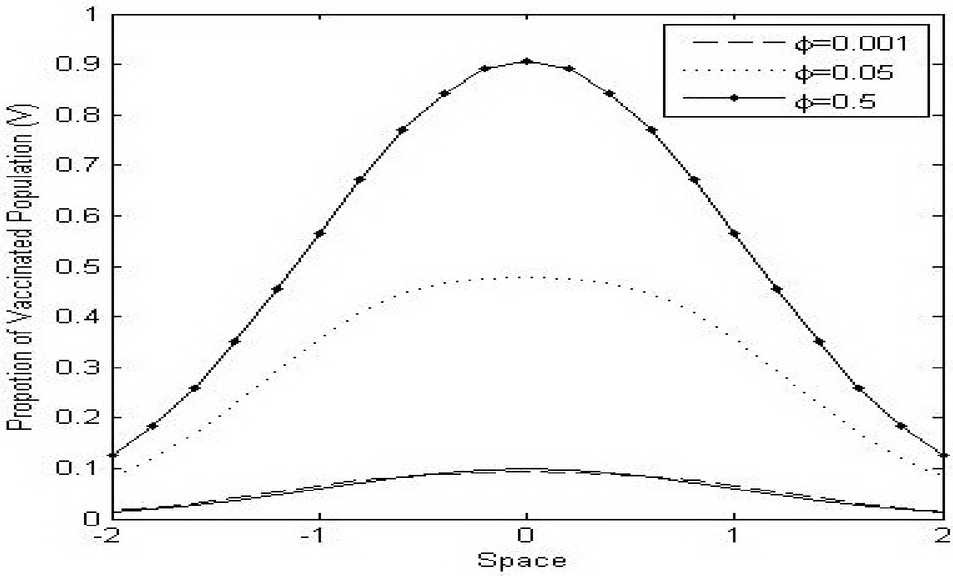

Figure 4.12 Solution with initial condition (i) and with diffusion of $\beta_v = 0.1$, ℓ=0.1 for cases $\phi = 0.001,\ 0.05, 0.5$ at t = 20 days and t = 0 day (—).

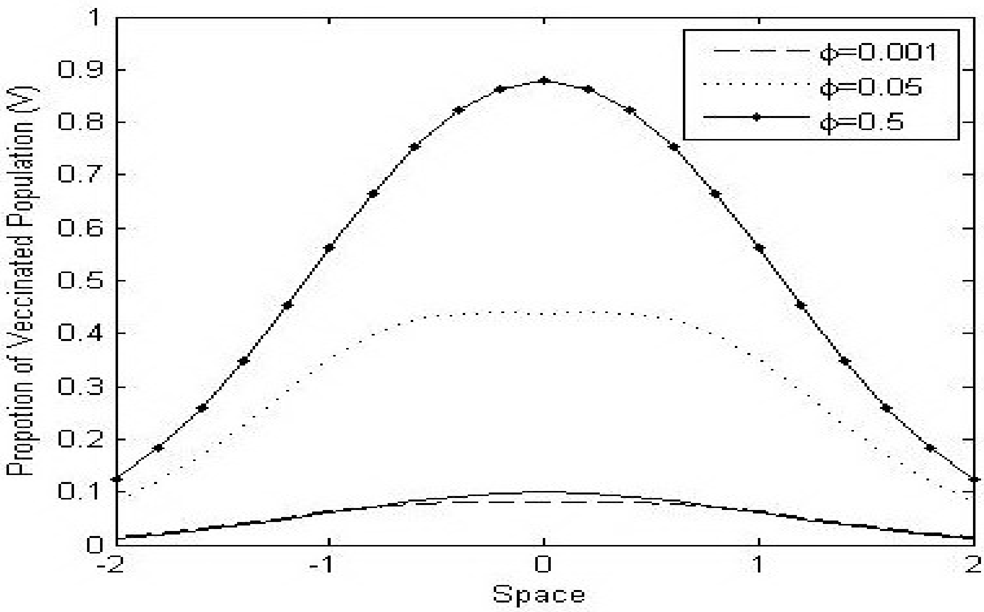

Figure 4.13 Solution with initial condition (i) and with diffusion of $\beta_v = 0.2$, ℓ=0.1 for cases $\phi = 0.001,\ 0.05, 0.5$ at t = 20 days and t = 0 day (—).

Figure 4.12-4.13**,** show the output with initial condition (i) on $t = 0$ day they found that the proportion of vaccinated population is spread in all over domain [-2, 2] with peak value as 0.1 at $x = 0$. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001, 0.05, 0.5$ the proportion of vaccinated population is spread all over in domain [-2, 2] with peak value as 0.093 at $x=0$, as 0.478 at $x=0$, as 0.91 at $x=0$ respectively. On $t=20$ day for $\beta_v=0.2$ case $\phi=0.001,\ 0.05,\ 0.5$ the proportion of vaccinated population is spread all over in domain [-2, 2] with peak value as 0.081 at $x=0$, as 0.439 at $x=\ -0.2, 0.2$, as 0.878 at $x=0$, respectively. The increment of vaccinated population enlarges the number rate of vaccinated. For $\beta_v=0.1,\ 0.2$ at same ϕ they found that the decrement of vaccinated population enlarges the number ability to cause infection by vaccination individuals (β_v).

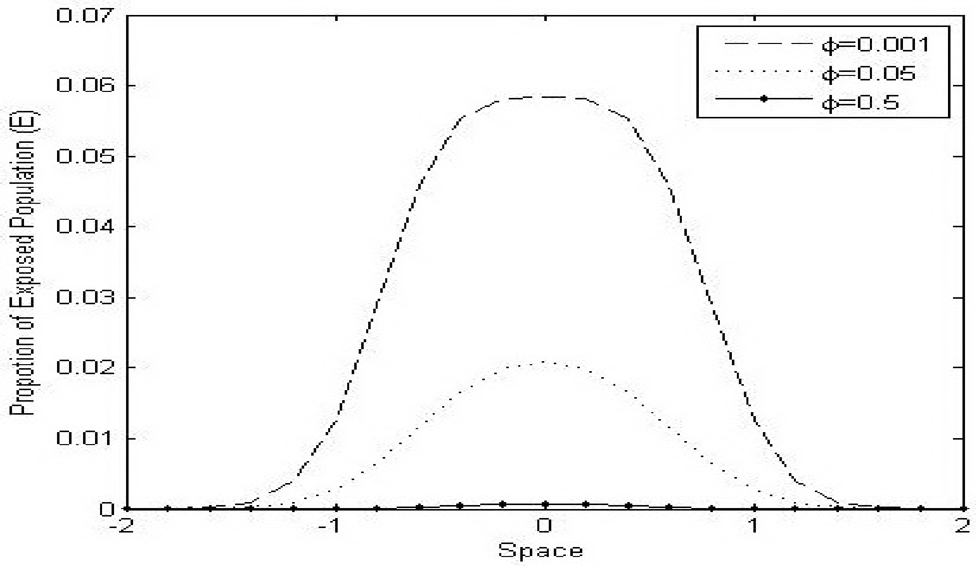

Figure 4.14 Solution with initial condition (i) and with diffusion of $\beta_v=0.1$, ℓ=0.1 for cases $\phi=0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

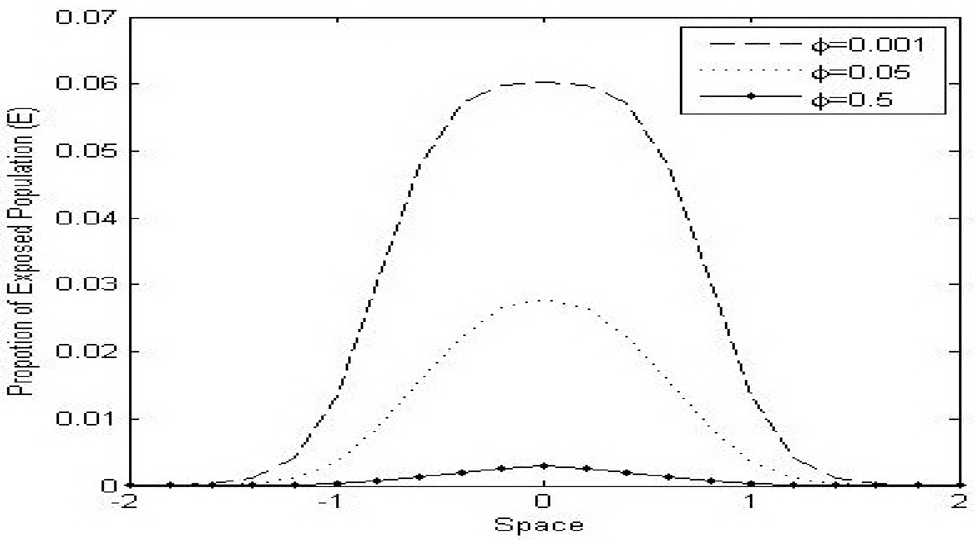

Figure 4.15 Solution with initial condition (i) and with diffusion of $\beta_v=0.2$, ℓ=0.1 for cases $\phi=0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

Figure 4.14-4.15**,** show the output with initial condition (i) on $t = 0$ day they found that the proportion of exposed population there are not spreading in domain. On $t = 20$ day for $\beta_v = 0.1$ case the proportion of exposed population is spreads in domain [-1.6, 1.6] with peak value as 0.058 at $x = 0$. Case $\phi = 0.05$ the proportion of exposed population is spreads in domain [-1.6, 1.6] with peak value as 0.021 at $x = 0$. Case $\phi = 0.05$ the proportion of proportion of exposed population is spreads in domain [-1, 1] with peak value as 0.001 at $x = 0$. On $t = 20$ day for $\beta_v = 0.2$ case $\phi = 0.001$ the proportion of exposed population is spreads in domain [-1.6, 1.6] with peak value as 0.06 at $x = 0$. Case $\phi = 0.05$ the proportion of exposed population is spreads in domain [-1.6, 1.6] with peak value as 0.028 at $x = 0$. Case $\phi = 0.5$ the exposed population is spreads in domain [-1.2, 1.2] with peak value as 0.003 at $x = 0$. The increment of exposed population enlarges the number rate of ability to cause infection by vaccination individuals (β_v). And the space for case $\phi = 0.5$ wide enlarges the number rate of ability to cause infection by vaccination individuals (β_v).

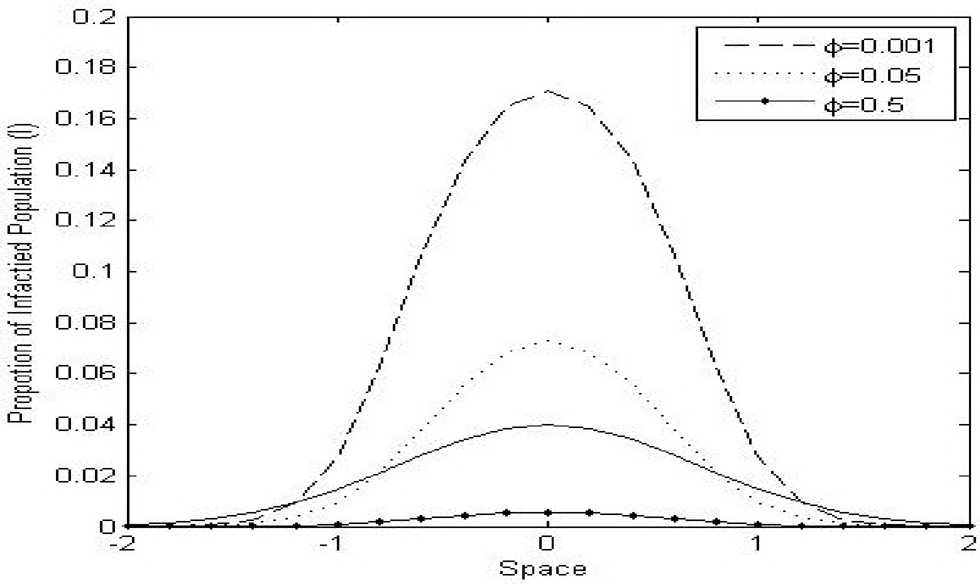

Figure 4.16 Solution with initial condition (i) and with diffusion of $\beta_v = 0.1$, ℓ=0.1for cases $\phi = 0.001, 0.05, 0.5$ at t = 20 days and t = 0 day (—).

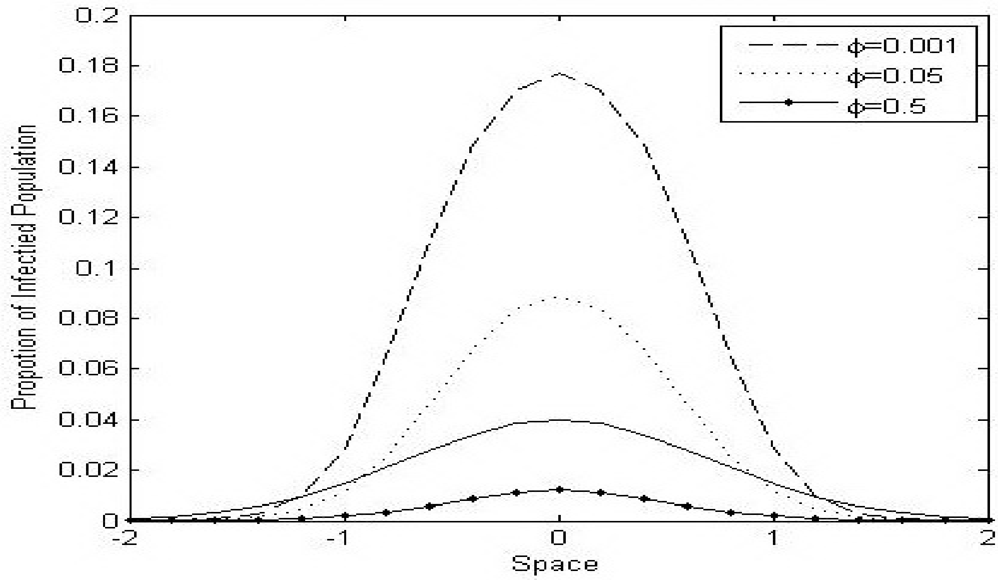

Figure 4.17 solution with initial condition (i) and with diffusion of $\beta_v = 0.2$, ℓ=0.1 for cases $\phi = 0.001$, 0.05, 0.5 at t = 20 days and t = 0 day (—).

Figure 4.16-4.17, Show the output with initial condition (i) on $t = 0$ day they found that the infected proportion of population is spread in all over domain [-2, 2] with peak value as 0.04 at $x = 0$. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001$ the proportion of infected population is spreads all over domain [-2, 2] with peak value as 0.171 at $x = 0$. Case $\phi = 0.05$ the proportion of infected population is spreads in domain [-1.8, 1.8] with peak value as 0.073 at $x = 0$. Case $\phi = 0.5$ the proportion of infected population is spreads in domain [-1.6, 1.6] with peak value as 0.001 at $x = 0$. On $t = 20$ day for $\beta_v = 0.2$ case $\phi = 0.001$ the proportion of infected population is spreads in domain [-2, 2] with peak value as 0.177 at $x = 0$. Case $\phi = 0.05$ the proportion of infected population is spreads in domain [-1.8, 1.8] with peak value as 0.089 at $x = 0$. Case $\phi = 0.5$ the proportion of infected population is spreads in domain [-1.8, 1.8] with peak value as 0.012 at $x = 0$. The increment of proportion infected population enlarges the number rate of vaccinated. The increment of proportion infected population enlarges the number rate of ability to cause infection by vaccination individuals (β_v). And the space for case $\phi = 0.5$ wide enlarges the number rate of ability to cause infection by vaccination individuals (β_v).

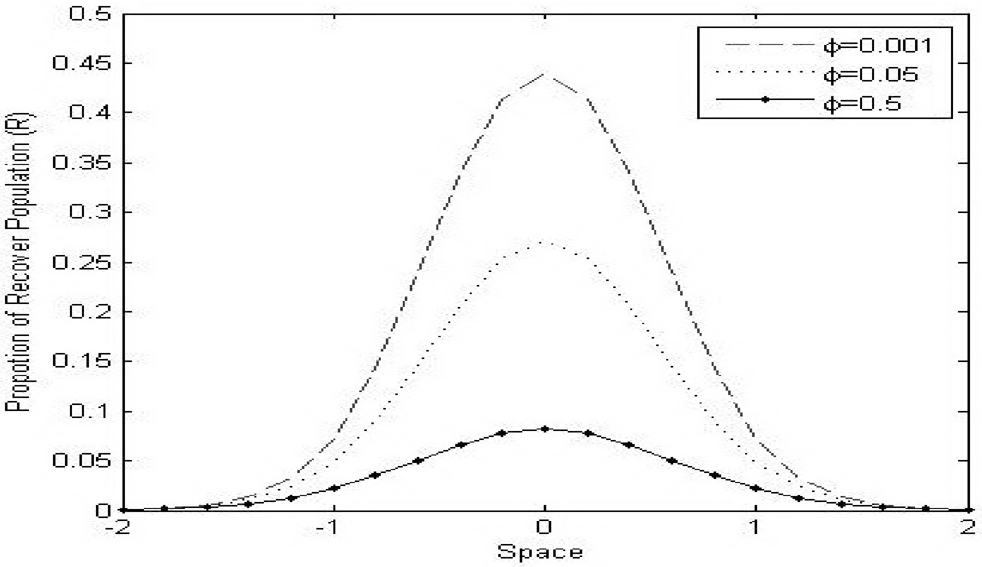

Figure 4.18 Solution with initial condition (i) and with diffusion of $\beta_v = 0.1$, ℓ=0.1 for cases $\phi = 0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

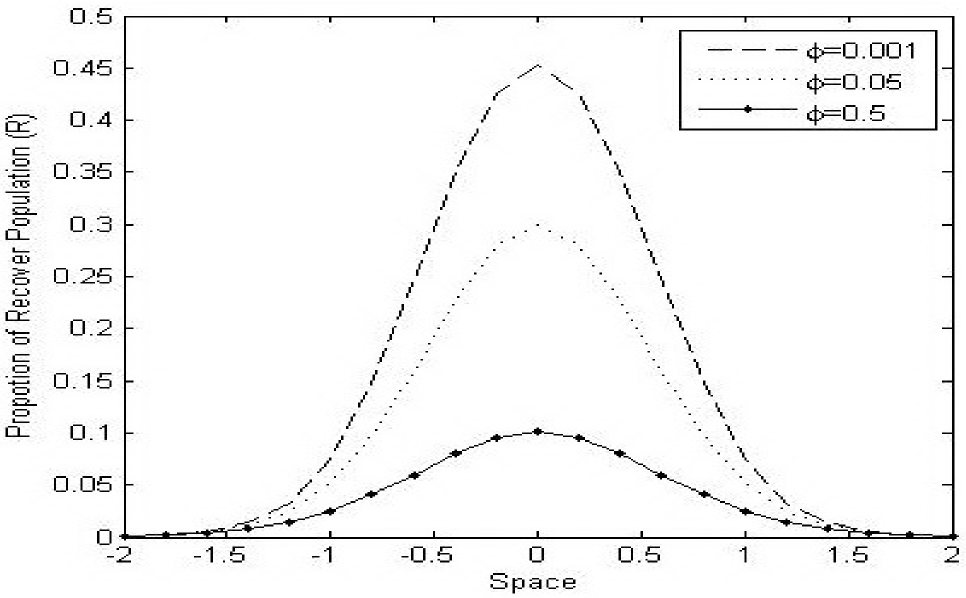

Figure 4.19 Solution with initial condition (i) and with diffusion of $\beta_v = 0.2$, ℓ=0.1 for cases $\phi = 0.001, 0.05, 0.5$ at t = 20 days and t = 0 day (—).

Figure 4.18-4.19, show the output with initial condition (i) on $t = 0$ day they found that the proportion of recover population there are not spreading in domain. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001,\ 0.05,$ and 0.5 the proportion of recover population is spread all over in domain [-2, 2] with peak value as 0.44 at $x = 0$, as 0.271 at $x = 0$, as 0.81 at $x = 0$ respectively. For $\beta_v = 0.2$ case $\phi = 0.001,\ 0.05,$ and 0.5 the proportion of recover population is spread all over in domain [-2, 2] with peak value as 0.452 at $x = 0$, as 0.299 at $x = 0$, as 0.101 at $x = 0$ respectively. The increment of proportion recover population enlarges the number rate of vaccinated. For $\beta_v = 0.1,\ 0.2$ at same ϕ they found that the increment of recover population enlarges the number ability to cause infection by vaccination individuals (β_v).

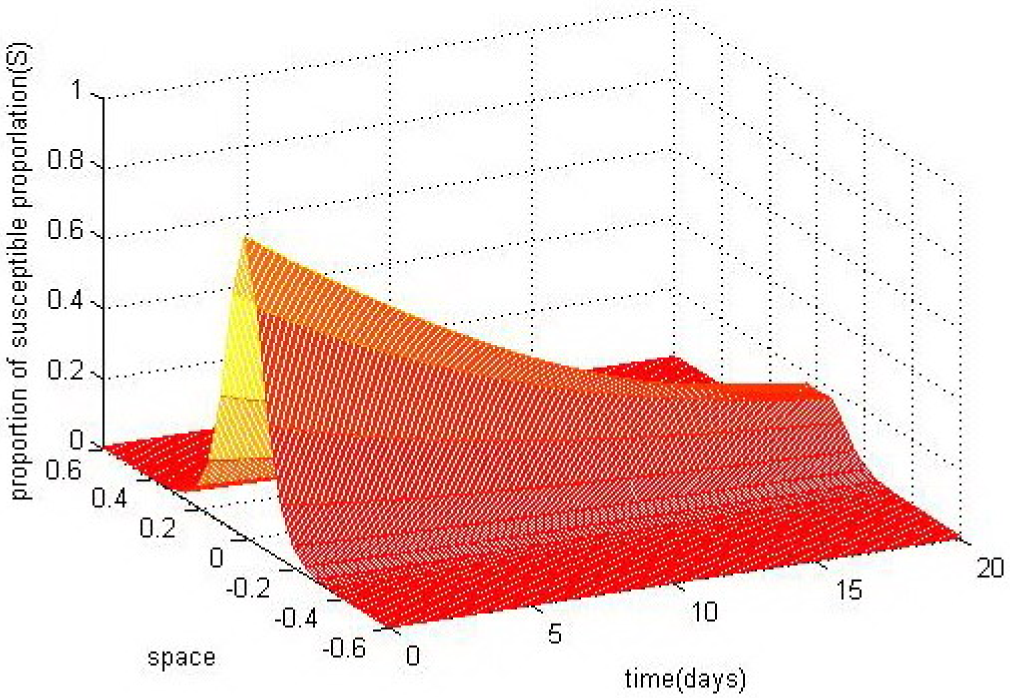

Figure 4.20 Three-dimensional distribution with initial condition (ii) of proportion susceptible; $\beta_v = 0.1,\ h = 0.05,\ \ell$=0.1, ϕ=0.05.

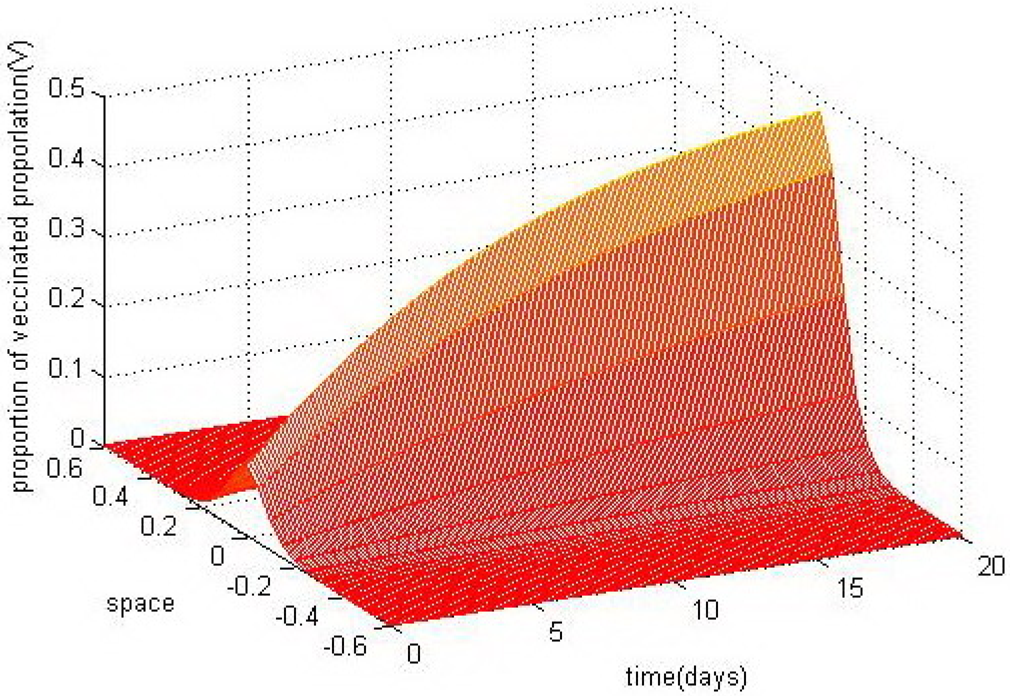

Figure 4.21 Three-dimensional distribution with initial condition (ii) of proportion vaccinated; $\beta_v = 0.1,\ h = 0.05,\ \ell$=0.1, ϕ=0.05.

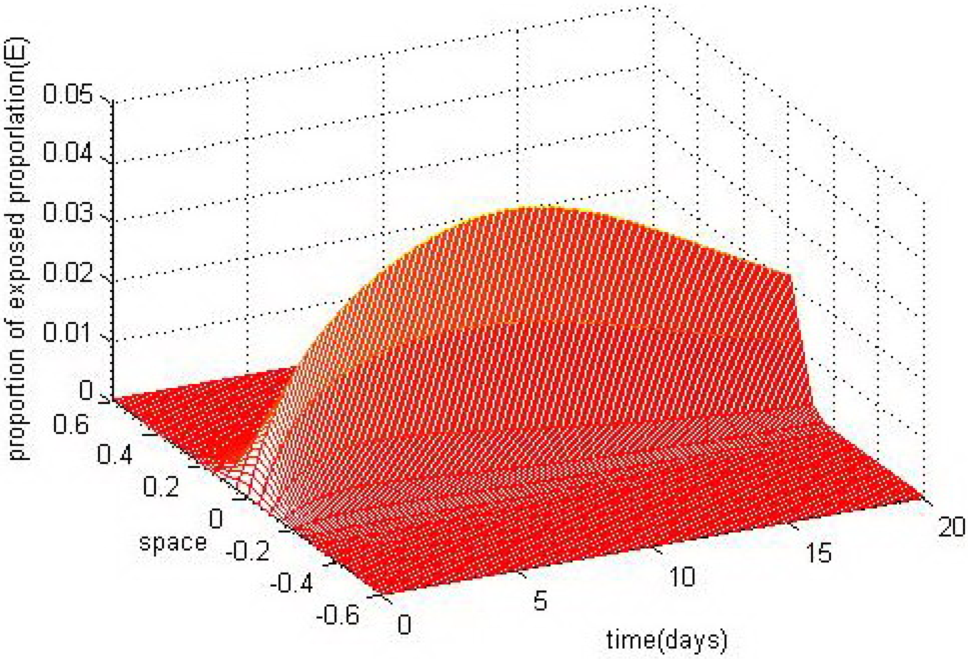

Figure 4.22 Three-dimensional distribution with initial condition (ii) of proportion exposed; $\beta_v = 0.1,\ h = 0.05,\ \ell$=0.1, ϕ=0.05.

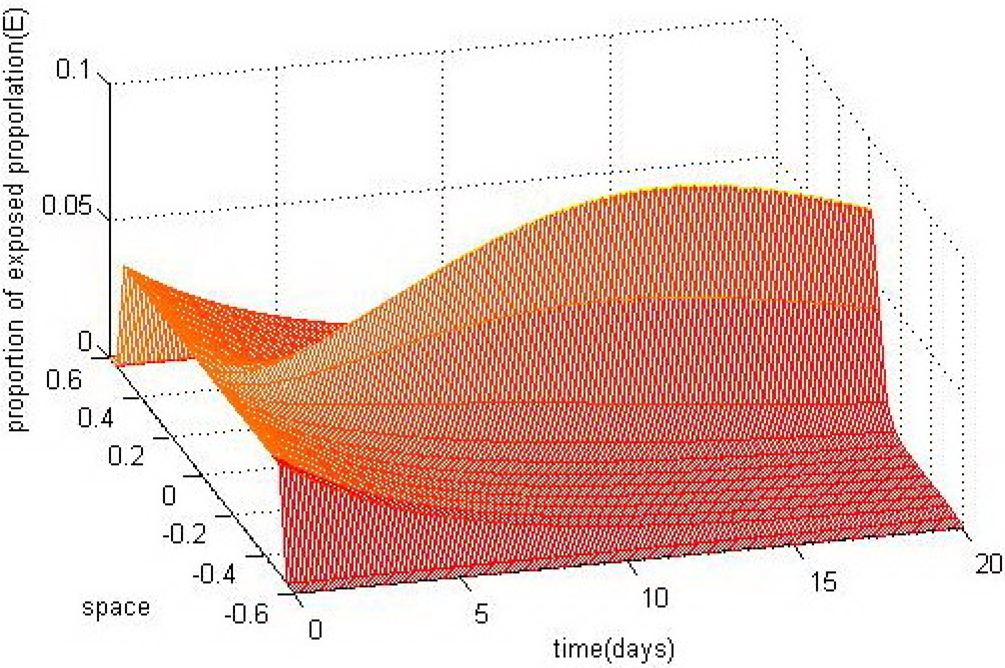

Figure 4.23 Three-dimensional distribution with initial condition (ii) of proportion infected; $\beta_v = 0.1,\ h = 0.05,\ \ell$=0.1, ϕ=0.05.

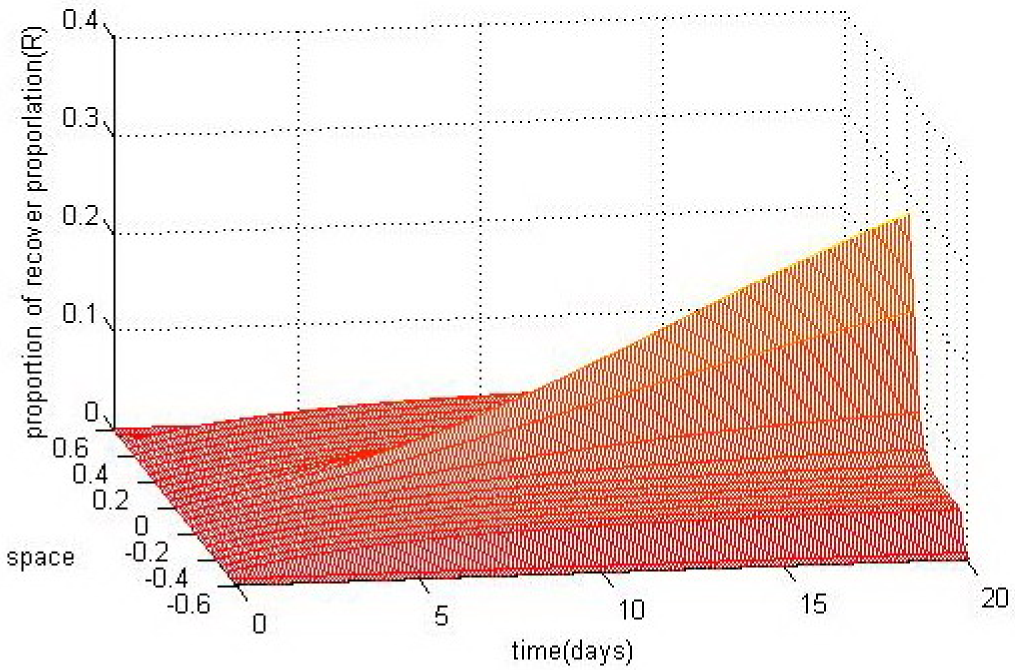

Figure 4.24 Three-dimensional distribution with initial condition (ii) of proportion recovers; $\beta_v = 0.1$, $h = 0.05$, ℓ=0.1, ϕ=0.05.

Figure 4.20-4.24, give three dimensional plots of proportion susceptible, vaccinated, exposed, infected recover population for t= 0 day until 20 days. As time increases, they found that the proportion of susceptible population decrease whereas the proportion of vaccinated, exposed, infected, recover population increase until time t = 15 day after which the proportion of susceptible become less than the proportion of vaccinated population. For the proportion of exposed population decrease after time t = 10 day. For the proportion of infected population decrease after time t = 15 day.

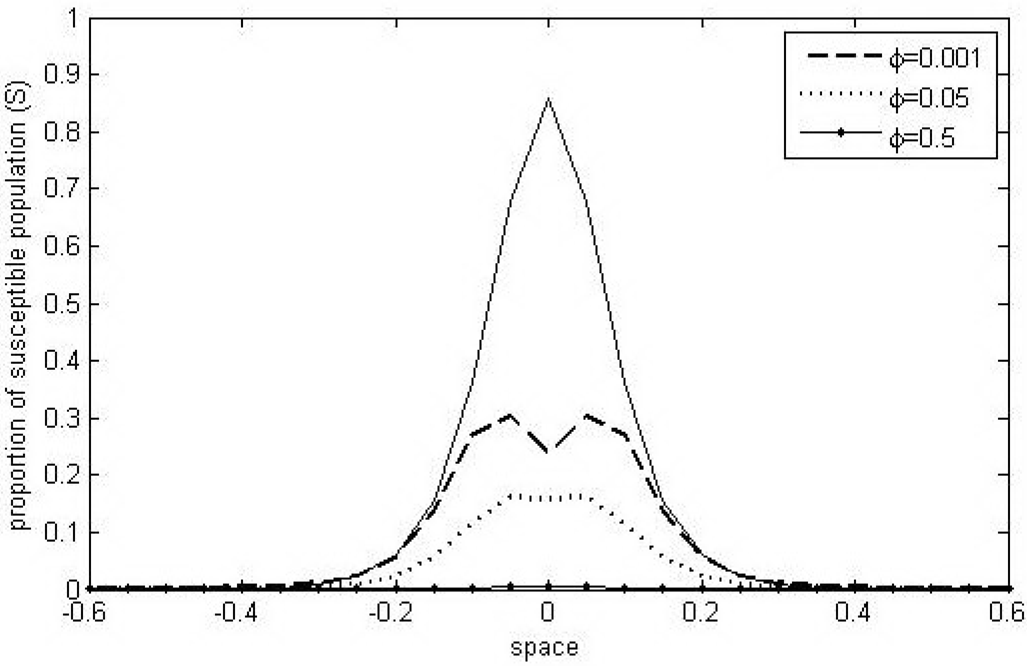

Figure 4.25 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.1$, ℓ=0.1for cases $\phi = 0.001, 0.05, 0.5$ at t = 20 days and t = 0 day (——).

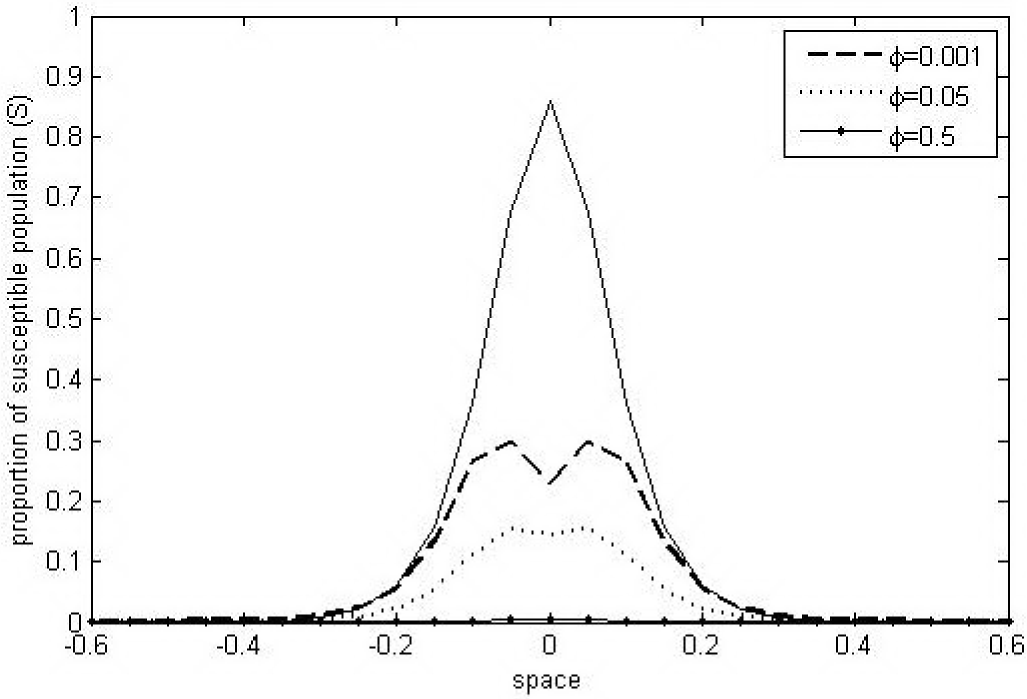

Figure 4.26 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.2$, ℓ=0.1 for cases $\phi = 0.001, 0.05, 0.5$ at t = 20 days and t = 0 day (——).

Figure 4.25-4.26, show the output with initial condition (ii) on $t = 0$ day they found that the proportion of susceptible population is spread in domain [-0.4, 0.4] with peak value as 0.86 at $x = 0$. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001,\ 0.05, \text{and } 0.5$ the proportion of susceptible is spread all over in domain [-0.6, 0.6] with peak value as 0.304 at $x = -0.05,\ 0.05$, as 0.161 at $x = -0.05,\ 0.05$ as 0.005 at x= 0, respectively. On $t = 20$ day for $\beta_v = 0.2$ case $\phi = 0.001,\ 0.05, \text{and } 0.5$ the proportion of susceptible is spread all over in domain [-0.6, 0.6] with peak value as 0.229 at $x = -0.05,\ 0.05$ as 0.154 at $x = -0.05,\ 0.05$ as 0.005 at $x = 0$ respectively. The decrement of proportion of susceptible population enlarges the number rate of vaccinated. For $\beta_v = 0.1,\ 0.2$ at same ϕ they found that the increment the proportion of susceptible population enlarges the number ability to cause infection by vaccination individuals (β_v).

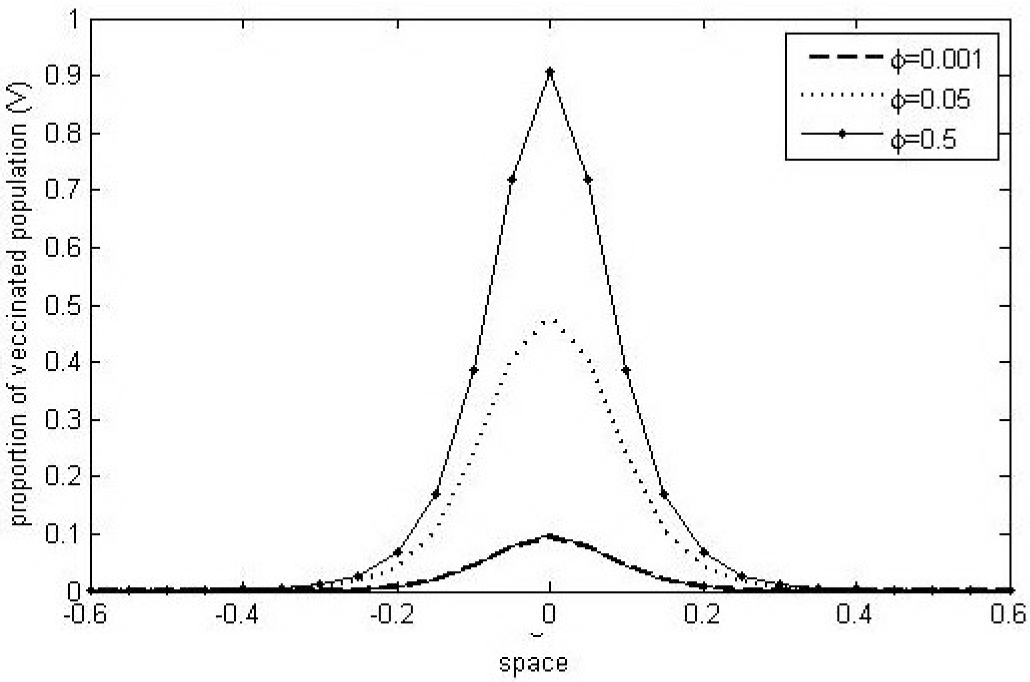

Figure 4.27 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.1$, ℓ=0.1 for cases $\phi = 0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

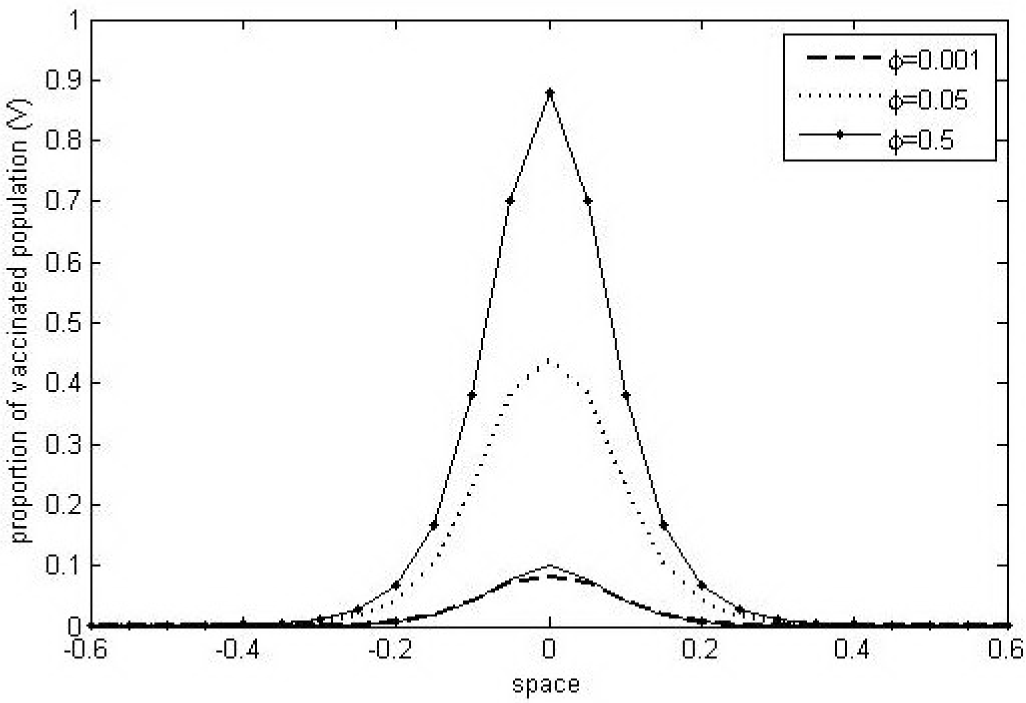

Figure 4.28 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.2$, ℓ=0.1for cases $\phi = 0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

Figure 4.27-4.28, show the output with initial condition (ii) on $t = 0$ day they found that the proportion of vaccinated population is spread in all over domain [-0.4, 0.4] with peak value as 0.1 at $x = 0$. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001$ the proportion of vaccinated population is spread all over in domain [-0.3, 0.3] with peak value as 0.093 at $x = 0$, case $\phi = 0.05,\ 0.5$ the proportion of vaccinated population is spread all over in domain [-0.05, 0.05] and [-0.6, 0.6] with peak value as 0.478 at $x = 0$, as 0.91 at $x = 0$ respectively. On $t = 20$ day for $\beta_v = 0.2$ case $\phi = 0.001$ the proportion of vaccinated population is spread all over in domain [-0.3, 0.3] with peak value as 0.081 at $x = 0$, case $\phi = 0.05,\ 0.5$ the proportion of vaccinated population is spread all over in domain [-0.5, 0.5] with peak value as 0.439 at $x = 0$, as 0.878 at $x = 0$ respectively. The increment of vaccinated population enlarges the number rate of vaccinated. The decrement of vaccinated population enlarges the number ability to cause infection by vaccination individuals (β_v). And they found that the domain wide enlarges the number rate of vaccinated.

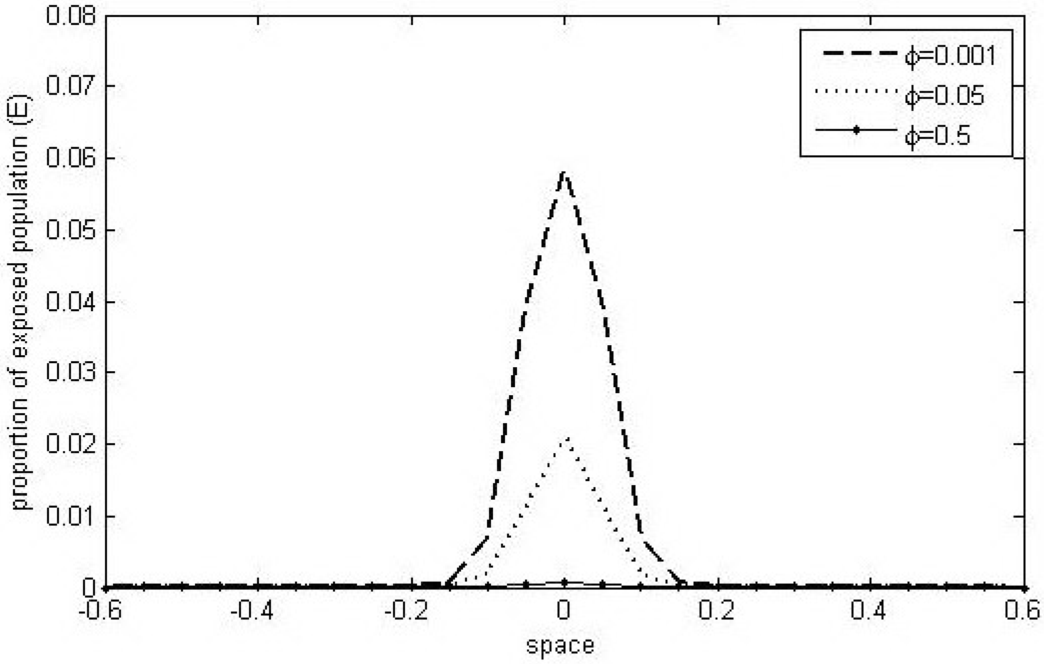

Figure 4.29 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.1$, ℓ=0.1for cases $\phi = 0.001, \ 0.05, 0.5$ at t = 20 days and t = 0 day (—).

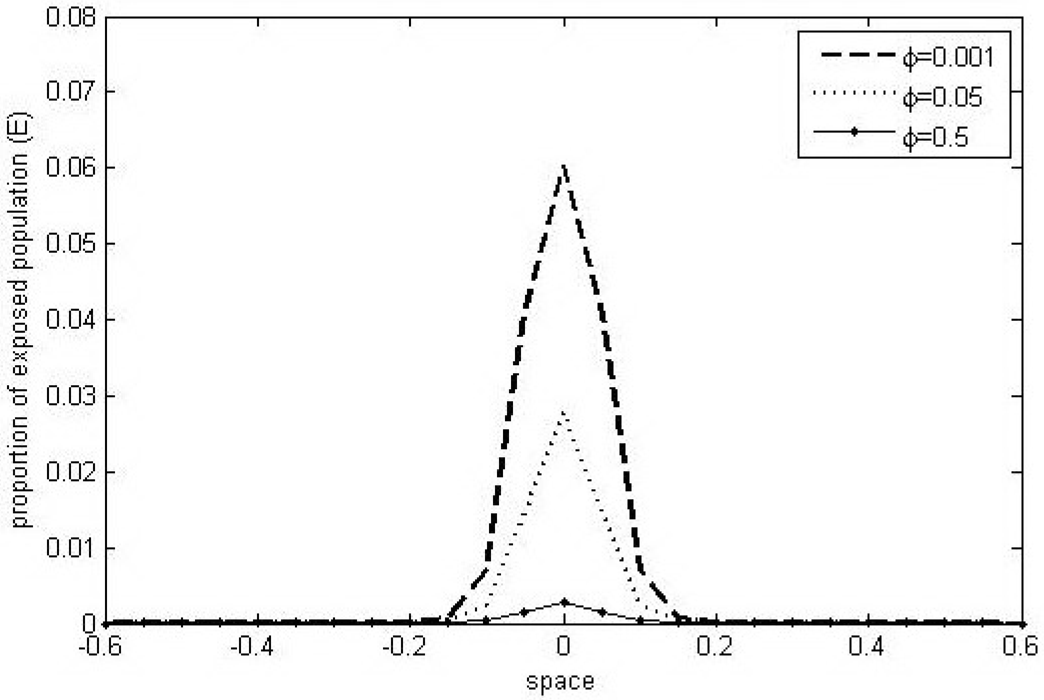

Figure 4.30 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.2$, ℓ=0.1for cases $\phi = 0.001, \ 0.05, 0.5$ at t = 20 days and t = 0 day (—).

Figure 4.29-4.30, show the output with initial condition (ii) on $t = 0$ day they found that the proportion of exposed population there are not spreading in domain. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001$ the proportion of exposed population is spread all over in domain [-0.2, 0.2] with peak value as 0.058 at $x=0$, case $\phi = 0.05,\ 0.5$ the proportion of exposed population is spread all over in domain [-0.15, 0.15] and [-0.075, 0.075] with peak value as 0.021 at $x=0$, as 0.001 at $x=0$ respectively. On $t=20$ day for $\beta_v = 0.2$ case $\phi = 0.001$ the proportion of exposed population is spread all over in domain [-0.2, 0.2] with peak value as 0.06 at $x=0$, case $\phi = 0.05,\ 0.5$ the proportion of exposed population is spread all over in domain [-0.2, 0.2] and [-0.15, 0.15] with peak value as 0.028 at $x=0$, as 0.003 at $x=0$ respectively. The increment of exposed population enlarges the number rate of ability to cause infection by vaccination individuals (β_v). And the space for case $\phi = 0.5$ wide enlarges the number rate of ability to cause infection by vaccination individuals (β_v).

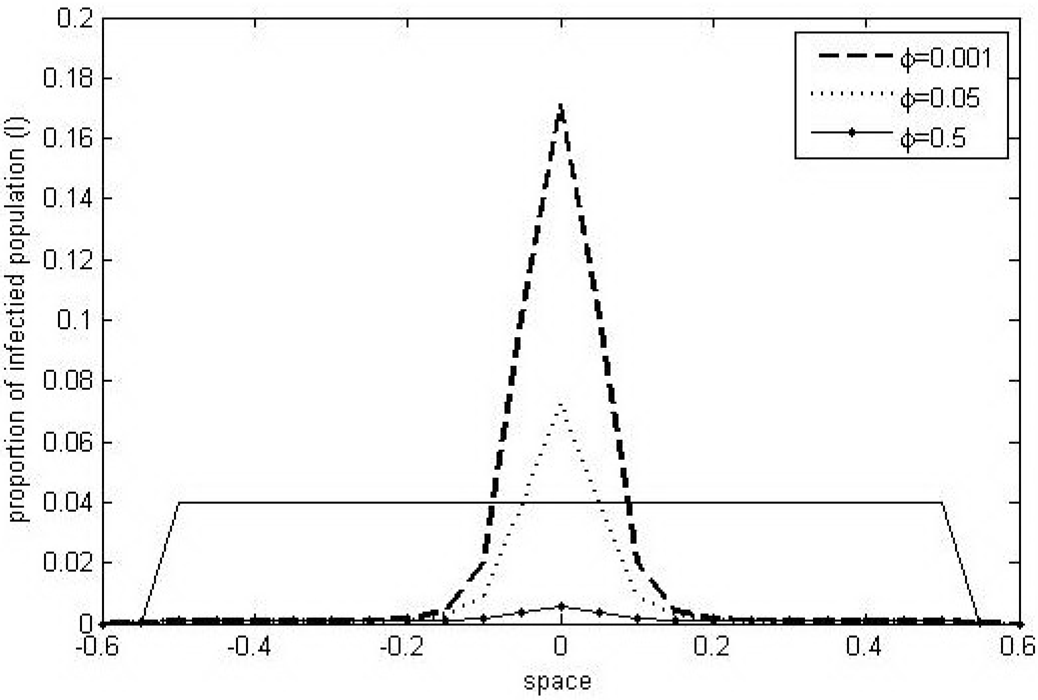

Figure 4.31 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.1,\ \ell=0.1$ for cases $\phi = 0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

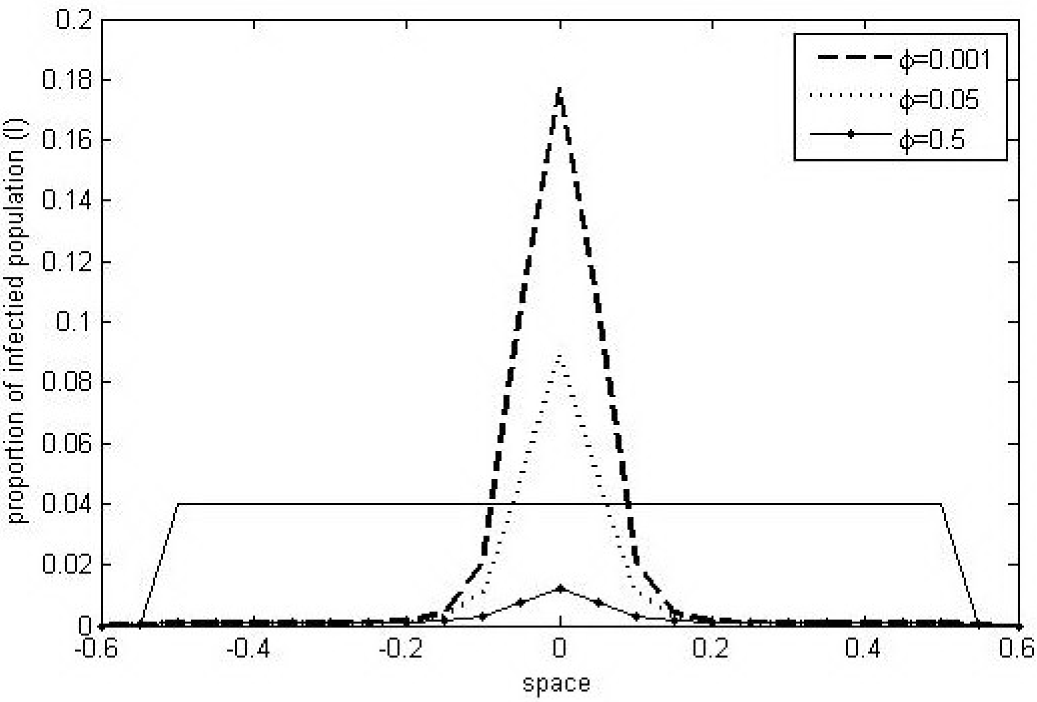

Figure 4.32 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.2$, ℓ=0.1for cases $\phi = 0.001,\ 0.05,\ 0.5$ at t = 20 days and t = 0 day (—).

Figure 4.31-4.32, show the output with initial condition (ii) on $t = 0$ day they found that the proportion of infected population is spread in domain [-0.55, 0.55] with peak value as 0.04 in domain [-0.5, 0.5]. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001,\ 0.05$ and 0.5 the proportion of infected population is spread all over in domain [-0.25, 0.25], [-0.2, 0.2] and [-0.15, 0.15] with peak value as 0.171 at $x = 0$, as 0.073 at $x = 0$, and as 0.006 at $x = 0$ respectively. On $t = 20$ day for $\beta_v = 0.2$ case $\phi = 0.001,\ 0.05$ and 0.5 the proportion of infected population is spread all over in domain [-0.25, 0.25], [-0.2, 0.2] and [-0.2,0.2] with peak value as 0.177 at $x = 0$, as 0.089 at $x = 0$,and as 0.012 at $x = 0$ respectively. The increment of proportion infected population enlarges the number rate of vaccinated. The increment of proportion infected population enlarges the number rate of ability to cause infection by vaccination individuals (β_v).And the space for case $\phi = 0.5$ wide enlarges the number rate of ability to cause infection by vaccination individuals (β_v).

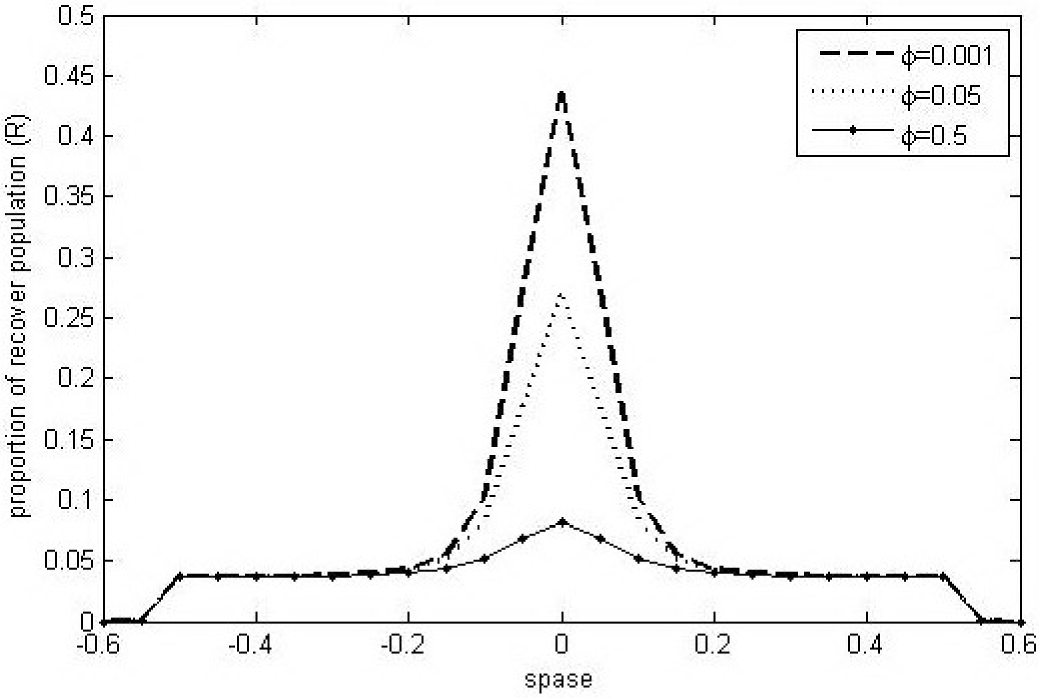

Figure 4.33 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.1$, ℓ=0.1 for cases $\phi = 0.001$, 0.05, 0.5 at t = 20 days and t = 0 day (—).

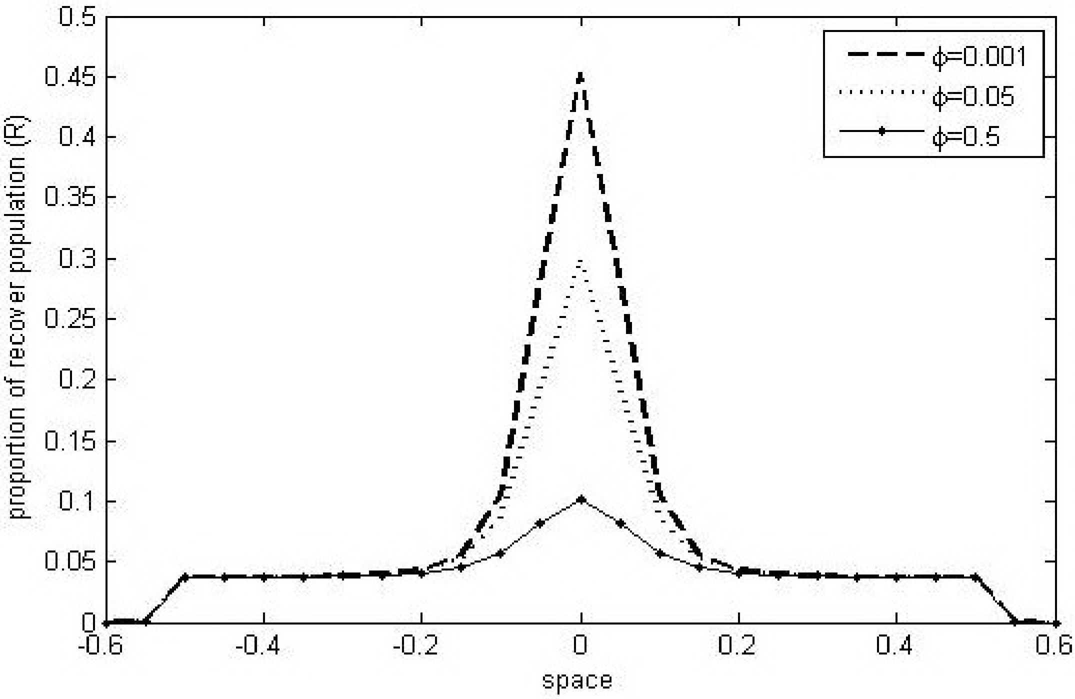

Figure 4.34 Solution with initial condition (ii) and with diffusion of $\beta_v = 0.2$, ℓ=0.1 for cases $\phi = 0.001$, 0.05, 0.5 at t = 20 days and t = 0 day (—).

Figure 4.33-4.34, show the output with initial condition (i) on $t = 0$ day they found that the proportion of recover population there are not spreading in domain. On $t = 20$ day for $\beta_v = 0.1$ case $\phi = 0.001$, 0.05 and 0.5 the proportion of recover population is spread all over in domain [-0.5, 0.5] with peak value as 0.44 at $x=0$, as 0.271 at $x=0$, as 0.081 at x= 0 respectively. for $\beta_v = 0.1$ case $\phi = 0.001,\ 0.05$ and 0.5 the proportion of recover population is spread all over in domain [-0.5,0.5] with peak value as 0.452 at $x=0$, as 0.299 at $x=0$, as 0.101 at $x=0$ respectively. The increment of proportion recover population enlarges the number rate of vaccinated. For $\beta_v = 0.1,\ 0.2$ at same ϕ they found that the increment of recover population enlarges the number ability to cause infection by vaccination individuals (β_v).

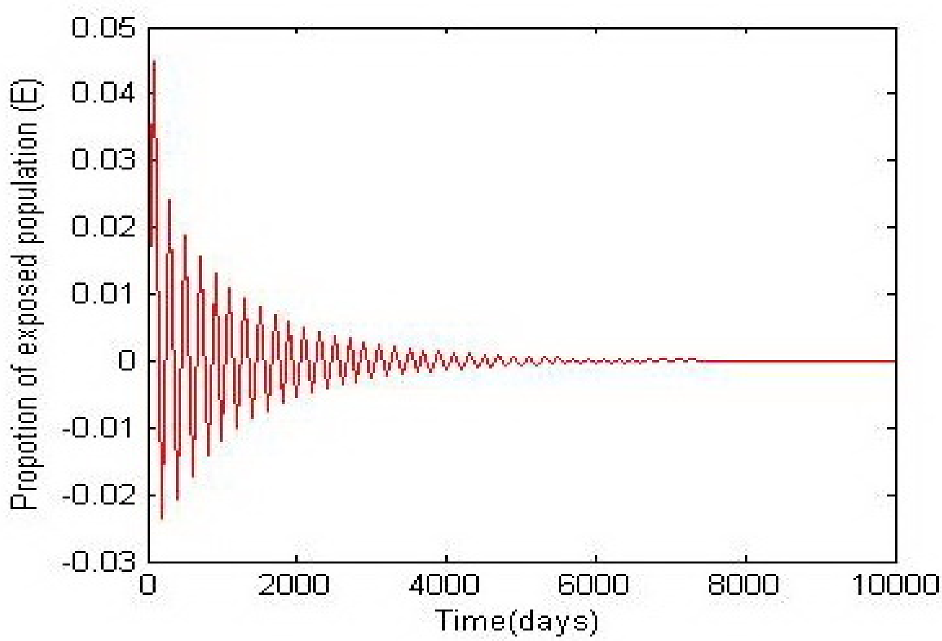

Figure 4.35 Profile of the non-standard finite-difference method (3.122) at $x=1$ for $\beta_v = 0.1,\quad \ell=100$ and cases $\phi = 0.001$

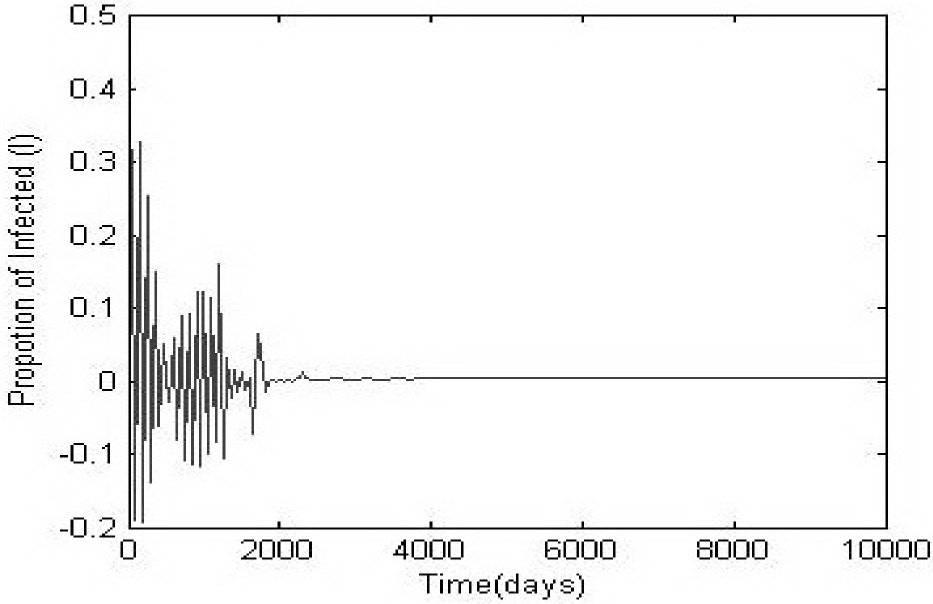

Figure 4.36 Profile of the non-standard finite-difference method (3.122) at $x = 1$ for $\beta_v = 0.1$, $\ell = 50$, and cases $\phi = 0.001$,

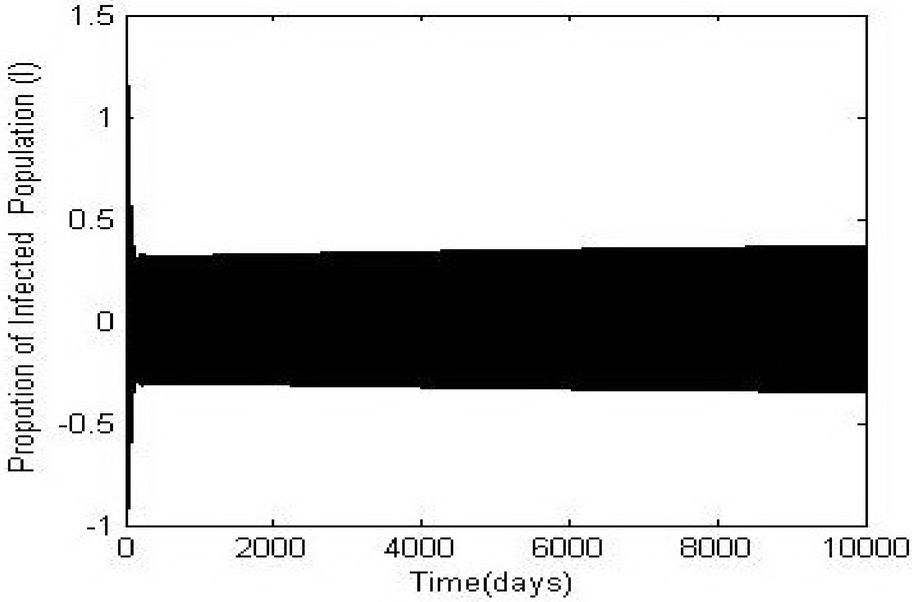

Figure 4.37 Profile of the standard finite-difference method (3.61)-(3.65) at $x = 1$ for $\beta_v = 0.1$, $\ell = 3.8$, and cases $\phi = 0.001$,

Table 4.1 Stability intervals of the standard finite-difference methods and non-standard finite-difference methods for $\beta_v = 0.1$ and $h = 0.2$ for initial condition (i)

Rate of vaccination (ϕ)	Interval of stability of standard finite-difference	Interval of stability of non-standard finite-difference
0.001	(0,2.872]	(0,4.048]
0.05	(0,2.825]	(0,3.054]
0.5	(0,2.065]	(0,2.999]

The table shows the interval of stability for the standard finite-difference methods and non-standard finite-difference methods when $d_1 = 0.05,\ d_2 = 0.05,\ d_3 = 0.025,\ d_4 = 0.001,$ $d_5 = 0.0,$ and $-2 \le x \le 2$. In the standard finite-difference method case ϕ=0.001, is chosen in the interval $\ell \in (0, 2.872]$ for space $-2 \le x \le 2$. The negative values of exposed (E) and infected (I) individuals and the oscillations began for $\ell > 2.872$ (converged) at $x = 0$. Case ϕ=0.05 the method is chosen in the interval $\ell \in (0, 2.825]$ for space $-2 \le x \le 2$. The negative values of exposed (E) individuals and the oscillations began for $\ell > 2.825$ (converged) at $x =$ -2,2. Case ϕ=0.5 the method is chosen in the interval $\ell \in (0, 2.065]$ for space $-2 \le x \le 2$. The negative values of exposed (E) individuals and the oscillations began for $\ell > 2.065$ (converged) at $x = 0.2,$ -0.2. The values in the stability interval with overflow occurring was increased further.

In the stability interval of nonstandard finite-difference method case ϕ=0.001, is chosen in the interval $\ell \in (0, 4.048]$ for space $-2 \le x \le 2$. The negative values of exposed (E) and infected (I) individuals and the oscillations began for $\ell > 4.048$ (converged) at $x = -2, 2$. Case ϕ=0.05 the method is chosen in the interval $\ell \in (0, 3.054]$ for space $-2 \le x \le 2$. The negative values of exposed (E) individuals and the oscillations began for $\ell > 3.054$ (converged) at $x = 2,$ -2. Case ϕ=0.5 the method is chosen in the interval $\ell \in (0, 2.999]$ for space $-2 \le x \le 2$. The negative values of exposed (E) individuals and the oscillations began for $\ell > 2.999$ (converged) at $x = -2,\ 2$.

Fig 4.35-4.36 illustrate the convergence stability profile of E and I in case $\phi = 0.001,\ \beta_v = 0.1,\ \ell = 50$ and $\ell = 100$ respectively. The standard finite-difference method (3.61)-(3.65) for $\beta_v = 0.1$, , $\ell = 3.8,$ and cases $\phi = 0.001$ diverge (Fig 4.37). The evidence shows that the nonstandard finite - difference method. As above described and Table 4.1, it is verified that nonstandard finite - difference method has a much better stability property than standard finite-difference method.

CHAPTER 5 CONCLUSION

5.1 Conclusion

The standard finite-difference and non- standard finite-difference scheme have been developed and implemented in this paper for computing the solutions of the *SVEIR* model with vaccination in one dimension (3.3). The numerical experiments showed that the numerical solution obtained from the constructed method is coincided with the solution in [9]. The local truncation errors of standard finite-difference method (3.13) - (3.17) are illustrated. The orders of this method are $o(h^2+\ell)$ as $h,\ell \to 0.$ Furthermore, the von Neumann is used to investigate the stability of the method (3.13)-(3.17). This yields the condition of stability (3.38) - (3.42). The local truncation error of non-standard finite-difference method (3.86), (3.89), (3.92), (3.95) and (3.98) are elucidated. The local truncation error in this case are second-order in time and space $o(h^2+\ell^2)$ as $h,\ell \to 0.$

The numerical results of the model are examined by the standard and non-standard finite-difference method with initial condition (i) and (ii) as shown in section 4.2 and 4.3. The non- standard finite-difference has interval of stability longer than the standard finite-difference method. The concrete evidence is manifested in Table 4.1. In case of initial condition (i) and (ii), for all case considered the decrement of proportion of susceptible population enlarges the number rate of vaccinated. For $\beta_v = 0.1,\ 0.2$ at same ϕ they found that the increment the proportion of susceptible population enlarges the number ability to cause infection by vaccination individuals (β_v).The decrement of vaccinated population enlarges the number ability to cause infection by vaccination individuals (β_v). And they found that the domain wide enlarges the number rate of vaccinated. The increment of exposed population enlarges the number rate of ability to cause infection by vaccination individuals (β_v). The space for case $\phi = 0.5$ wide enlarges the number rate of ability to cause infection by vaccination individuals (β_v).The increment of proportion infected population enlarges the number rate of vaccinated. The increment of proportion infected population enlarges the number rate of ability to cause infection by vaccination individuals (β_v). And the space for case $\phi = 0.5$ wide enlarges the number rate of ability to cause infection by vaccination individuals (β_v).The increment of proportion recover population enlarges the number rate of vaccinated. For $\beta_v = 0.1,\ 0.2$ at same ϕ they found that the increment of recover population enlarges the number ability to cause infection by vaccination individuals (β_v). Thus case $\beta_v = 0.1$ has a much better efficiency property than $\beta_v = 0.2$.

We can summarized the work as the following

- The increasing of vaccination rate (ϕ) can be reduced the proportion of infected and exposed population.
- The inflated of vaccination efficiency (β_v) is essential decrease proportion of exposed and infected population.
- Diffusion in the model can help to stabilize the system.
- The initial condition of population definitely plays an importance role in the spread of disease.
- The dynamic depends on the diffusion rate.

The further work, we will develop the model into space and time and apply to physical situation.

REFERENCES

1. Disease Prevention Control, **Influenza Virus** [Online], Available : http:// www. ddc.moph.go.th/km/showimgkm.php?id=32 [2012, May 29].

2. Keitel, W.A., Cate, T.R. and Couch, R.B., 1988, "Efficacy of Sequential Annual Vaccination with Inactivated Influenza Virus Vaccine", **American Journal of Epidemiology**, Vol. 127, pp. 53-64.

3. Moghadas, S.M. and Gumel, A.B., 2003, "A Mathematical Study of A Model for Childhood Diseases with Non-Permanent Immunity", **Journal of Computational and Applied Mathematics**, Vol. 157, pp. 347-363.

4. Jodar, L., Villanueva, R., Arenas, A. and Gonzalez, G., 2008, "Nonstandard Numerical Methods for a Mathematical Model for Influenza Disease", **Mathematics and Computers in Simulation**, Vol. 79, pp. 622-633.

5. Arenas, A., Morano, J. and Cortes, J., 2008, "Non-Standard Numerical Method for a Mathematical Model of RSV Epidemiological Transmission", **Computers and Mathematics with Applications**, Vol. 56, pp. 670-678.

6. Samsuzzoha, M., Singh, M. and Lucy, D., 2011,"Numerical Study of a Diffusive Epidemic Model of Influenza with Variable Transmission Coefficient", **Applied Mathematics Modelling**, Vol. 35, pp. 5507-5523.

7. Samsuzzoha, M., Singh, M. and Lucy, D., 2012, "A Numerical Study on an Influenza Epidemic Model with Vaccination and Diffusion", **Applied Mathematic and Computation**, Vol. 219, pp. 122-141.

8. Samsuzzoha, M., Singh, M. and Lucy, D., 2013, "Uncertainty and Sensitivity Analysis of the Basic Reproduction Number of a Vaccinated Epidemic Model of Influenza", **Applied Mathematical Modelling**, Vol. 37, pp. 903-915.

9. Gerald, C.F. and Wheatley, P.O., **Finite Difference** [Online], Availabe : http:// staff.unila.ac.id/tiryono/files/2011/09/BedaHingga-FDM.pdf [2012, May 29].

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