RESIDUE THEORY: SUMS INVOLVING BINOMIAL COEFFICIENTS AND ROOTS OF TRANSCENDENTAL EQUATIONS

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE
(APPLIED MATHEMATICS)
FACULTY OF GRADUATE STUDIES
MAHIDOL UNIVERSITY
2010

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Thesis
entitled

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was submitted to the Faculty of Graduate Studies, Mahidol University
for the degree of Master of Science (Applied Mathematics)
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ACKNOWLEDGEMENTS

I would firstly like to express my sincere gratitude and appreciation to my advisor, Dr. Aram Tangboonduangjit for his guidance, constant encouragement and kindness. His advice is essential to complete this thesis successfully.

I would like to thank my co-advisor, Associate Professor Dr. Pachara Chaisuriya, for her kindness and valuable comments as well as useful instruction with warm kindness.

I would like to thank all of my teachers who taught me at every level of study. I also thank all office workers in the Department of Mathematics, Mahidol University for providing me with necessary facilities. I would also like to thank all of my friends for their friendship support, and encouragement.

Finally, I would like to express gratitude to my family for their love, sincerity, encouragement, and generous support throughout my life.

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RESIDUE THEORY: SUMS INVOLVING BINOMIAL COEFFICIENTS AND ROOTS OF TRANSCENDENTAL EQUATIONS.

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ABSTRACT

In this work, we studied a novel application of residue theory in the subject of meromorphic functions to find the exact values of infinite series involving real roots of some transcendental equations. Moreover, we showed how to use residue theory to prove some classic identities about the finite sum of binomial coefficients.

KEY WORDS : RESIDUE THEORY/TRANSCENDENTAL EQUATION/ BINOMIAL COEFFICIENTS/

22 pages.

ทฤษฎีส่วนตกค้าง : อนุกรมที่เกี่ยวข้องกับสัมประสิทธิ์ทวินามและรากของสมการอดิสัย
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บทคัดย่อ

ในงานวิจัยนี้เราได้ศึกษาเกี่ยวกับการประยุกต์ในอีกแนวทางหนึ่งของทฤษฎีส่วนตกค้างในส่วนของฟังก์ชัน meromorphic เพื่อหาค่าที่แท้จริงของอนุกรมอนันต์ที่เกี่ยวข้องกับรากที่เป็นจำนวนจริงของบางสมการอดิสัย นอกจากนี้เรายังแสดงให้เห็นถึงการใช้ทฤษฎีส่วนตกค้างเพื่อพิสูจน์ถึงเอกลักษณ์ดั้งเดิมที่เกี่ยวข้องกับอนุกรมจำกัดของสัมประสิทธิ์ทวินามโดยวิเคราะห์ในมุมมองของการใช้ทฤษฎีส่วนตกค้างสำหรับหาค่าของอนุกรม

22 หน้า

CONTENTS

	Page
ACKNOWLEDGEMENTS	**iii**
ABSTRACT (ENGLISH)	**iv**
ABSTRACT (THAI)	**v**
LIST OF FIGURES	**vii**
CHAPTER I INTRODUCTION	**1**
1.1 Objectives	1
1.2 Organization of the study	1
CHAPTER II LITERATURE REVIEW	**3**
CHAPTER III THEORETICAL BACKGROUND	**4**
3.1 Residue Theory	4
CHAPTER IV SUMMATION INVOLVING TRANSCENDENTAL EQUATIONS	**8**
CHAPTER V SUMMATION INVOLVING BINOMIAL COEFFICIENTS	**14**
CHAPTER VI CONCLUSION AND DISCUSSION	**18**
REFERENCES	**21**
BIOGRAPHY	**22**

LIST OF FIGURES

Figure **Page**

4.1 The square path C_k 9

6.1 Positive roots of $\cot x = x$ 20

CHAPTER I

INTRODUCTION

The usual application of residue theory is to evaluate the definite integrals of various types. It provides a straightforward, yet efficient, method to compute these integrals. Another application of the residue theory is to find the exact values of certain convergent infinite series. In particular we shall present an application of the residue theory in the aspect of finding the exact values of infinite series involving real roots of some transcendental equations and of finite series involving the binomial coefficients. We also present some numerical methods that can be used to confirm the legitimacy of our results.

1.1 Objectives

The main objectives of this study are

1. To show an alternative application of the residue theory.
2. To find the exact values of the sums involving real roots of some transcendental equations.
3. To use the residue theory to evaluate the finite sums involving binomial coefficients.
4. To compare the exact results with approximated ones using numerical methods.

1.2 Organization of the study

We will organize this study into six chapters. An introduction of the study is contained in Chapter 1. In the next chapter, we give the literature reviews. In Chapter 3, we describe the theoretical background of this study. We separate two different types of sums in Chapter 4 and 5: one involving the real roots of transcendental equations and the other involving binomial coefficients. We conclude the study and discuss some open problems in the last chapter.

CHAPTER II

LITERATURE REVIEW

In 1971, Ricardo [1] proposed the following theorem

Theorem 2.1 *If $|z| < \frac{(k-1)^{k-1}}{k^k}$ for some positive integer $k > 1$, then*

$$\sum_{n=0}^{\infty} \binom{kn}{n} z^n = \frac{1+W}{1-(k-1)W},$$

where W is the unique root of the equation $w - z(1+w)^k = 0$ inside the circle $|w| = \frac{1}{k-1}$.

Ricardo proved this theorem using the residue theory. The method of this proof inspires us to explore further the applications of the residue theory. In 1982, Bak and Newman [2] introduced the application of the contour integral method. They showed how to use the residue theorem to find infinite sums of some rational expressions and to estimate the sums involving binomial coefficients. In 1997, Antimirov, Kolyshkin and Vaillancourt [3] considered infinite series of the following forms

$$S_1 = \sum_{k=-\infty}^{\infty} f(k), \qquad S_2 = \sum_{k=-\infty}^{\infty} (-1)^k f(k),$$

$$S_3 = \sum_{k=-\infty}^{\infty} (-1)^k f(k) e^{\imath ak}, \qquad S_4 = \sum_{k=-\infty}^{\infty} f(k) e^{\imath ak},$$

$$S_5 = \sum_{k=1}^{\infty} f(k), \qquad S_6 = \sum_{k=1}^{\infty} (-1)^k f(k),$$

where $f(z) = P_n(z)/Q_m(z)$ with $P_n(z)$ and $Q_m(z)$ being polynomials of degrees n and m, respectively with $m \geq n+2$. Again, they used the residue theory to obtain the exact values of these sums.

CHAPTER III

THEORETICAL BACKGROUND

In this chapter, we give some basic knowledge and theoretical background that will be used throughout the thesis. We review some definitions and theorems that relate to infinite sums and contour integrals.

3.1 Residue Theory

Definition 3.1 *[4] A point z_0 is called a singular point of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. A singular point z_0 is said to be isolated if, in addition, there is a deleted neighborhood $B'(z_0, \varepsilon)$ of z_0 throughout which f is analytic.*

When z_0 is an isolated singularity of a function f, there is a positive number R such that f is analytic at each point z in the deleted neighborhood $B'(z_0, R)$. Consequently, in that neighborhood $f(z)$ is represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m},$$

which converges uniformly on every compact subset of $B'(z_0, R)$. The coefficients a_n, b_m have certain integral representations. In particular,

$$b_m = \frac{1}{2\pi i} \int\limits_C f(z)(z - z_0)^{m-1} dz,$$

where C is any positively oriented simple closed contour around z_0 and lying in the punctured disk $B'(z_0, R)$. When $m = 1$, this expression for b_m can be written

$$b_1 = \frac{1}{2\pi i} \int\limits_C f(z) dz.$$

Definition 3.2 *Let $f(z)$ have a non-removable isolated singularity at the point z_0. Then $f(z)$ has the Laurent series representation for all z in some punctured* *disk $B'(z_0, R)$. given by $f(z) = \sum\limits_{k=-\infty}^{\infty} a_n(z-z_0)^n$. The coefficient a_{-1} of $\frac{1}{z-z_0}$ is called the residue of $f(z)$ at z_0.*

Theorem 3.3 *[3] Suppose that*

(i) C is a simple closed contour, described in the counterclockwise direction;

(ii) C_k $(k = 1, 2, \ldots, n)$ are simple closed contour interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of all points inside C and exterior to each C_k, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

Theorem 3.4 *(Residue theorem)[4] Let C be a positively oriented simple closed contour. If a function f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \ldots, n)$ inside C, then*

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{3.1}$$

Proof. Let the point z_k, $k = 1, 2, \ldots, n$ be centers of positively oriented circles C_k which are interior to C and are so small that no two of them have points in common. The circles C_k, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain. Hence by Theorem 3.3 (deformation of contours,) we have

$$\int_C f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

This reduces to equation (3.1) because

$$\oint_C f(z)dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad (k = 1, 2, \ldots, n),$$

and the proof is complete. ■

Remark. The calculus of residue is the way to find the residue of the function $f(z)$ at the pole $z = a$ of order k, which in general can be found by the following formula:

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-a)^k f(z) \right]. \tag{3.2}$$

In this study, $P_n(z)$ and $Q_m(z)$ are always polynomials of degree n and m, respectively, with $m > n$.

As usual, the binomial coefficients are defined by

$$\binom{n}{m} = \begin{cases} \frac{n!}{(n-m)!m!}, & \text{if } n \geq m \ ; \\ 0, & \text{if } n < m, \end{cases}$$

where n and m are nonnegative integers.

As a corollary to Theorem 3.4, we have a relationship between contour integral and binomial coefficients as follows.

Corollary 3.5 *[2] Given positive integers n and k with $k \leq n$, then*

$$\binom{n}{k} = \frac{1}{2\pi i} \int_C \frac{(1+z)^n}{z^{k+1}} dz, \tag{3.3}$$

where C is a unit circle centered at the origin.

Definition 3.6 *[3] A system of closed paths C_n $(n = 1, 2, 3, ...)$ is called regular if the following three conditions are satisfied :*

(a) The path C_1 contains the point $z = 0$ and each path C_n lies inside the region bounded by the path C_{n+1}.

(b) The distance, d_n, from C_n to the origin increases without bound as n increases.

(c) The quotient of the length, l_n, of C_n to the distance d_n remains bounded; i.e., there exists a constant $A > 0$ such that

$$\frac{l_n}{d_n} \leq A \qquad \textit{for all} \quad n \in \mathbb{N}.$$

Theorem 3.7 *[3] Let $F(z)$ be an entire function such that the poles, γ_k, of $\frac{F'(z)}{F(z)}$ tend to infinity as $k \to \infty$. Also let C_k be a regular system of paths. If*

$$\lim_{k\to\infty} \oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz = 0, \tag{3.4}$$

then

$$\sum_k \operatorname*{Res}_{z=\gamma_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) = -\sum_k \operatorname*{Res}_{z=z_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right), \tag{3.5}$$

where z_k are the zeros of the polynomial $Q_m(z)$ and $z_k \neq \gamma_l$ for all k and l.

Proof. By the residue theorem

$$\oint\limits_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz = 2\pi i \left(\sum_k \operatorname*{Res}_{z=\gamma_k} \left[\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right] + \sum_k \operatorname*{Res}_{z=z_k} \left[\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right] \right), \tag{3.6}$$

where γ_k are the pole of $F'(z)/F(z)$ and z_k are the zeros of $Q_m(z)$ inside the path C_k. Consider the limit of (3.6) as $k \to \infty$ and using (3.4), we obtain (3.5). ■

CHAPTER IV

SUMMATION INVOLVING TRANSCENDENTAL EQUATIONS

In this chapter, we study the exact values of infinite series involving the real roots of some transcendental equations. We consider how to use the residue theorem to find the summation. We exploit the method much further for finding the sum and proving some interesting formulas. In what follows we let $f(z) = \frac{P_n(z)}{Q_m(z)}$ be a rational function with degree $P_n(z) = n$ and degree $Q_m(z) = m$ and $m \geq n + 2$.

Theorem 4.1 *Let $f(z) = \frac{P_n(z)}{Q_m(z)}$ with $m \geq n+2$. If γ_k are the roots of equation* $\cot z = -Cz$, *where C is a constant with $C \geq -1$, then*

$$\sum_{k=-\infty}^{\infty} f(\gamma_k) = -\sum_k \operatorname*{Res}_{z=z_k} \left[\frac{P_n(z)}{Q_m(z)} \left(\frac{-\sin z + Cz\cos z + C\sin z}{\cos z + Cz\sin z} \right) \right], \tag{4.1}$$

where z_k are the zeros of the polynomial $Q_m(z)$.

Proof. Let

$$F(z) = \cos z + Cz\sin z.$$

Then

$$F'(z) = -\sin z + Cz\cos z + C\sin z.$$

Therefore,

$$\frac{F'(z)}{F(z)} = \frac{-\sin z + Cz\cos z + C\sin z}{\cos z + Cz\sin z}.$$

Next we consider C_k to be the square with vertices A_k, B_k, D_k, E_k at the points $\left(\pm\frac{(2k+1)\pi}{2}, \pm\frac{(2k+1)\pi}{2}\right)$, for each $k = 1, 2, \ldots$ then C_k is the regular system of closed paths. By the residue theorem, we obtain

$$\oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz = 2\pi i \left[\sum_k \operatorname*{Res}_{z=z_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) + \sum_k \operatorname*{Res}_{z=\gamma_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) \right],$$

where z_k are the poles of $\frac{P_n(z)}{Q_m(z)}$ and γ_k are the poles of $\frac{F'(z)}{F(z)}$.
Next, we want to show that the integral on the left-hand side tends to zero as $k \to \infty$.

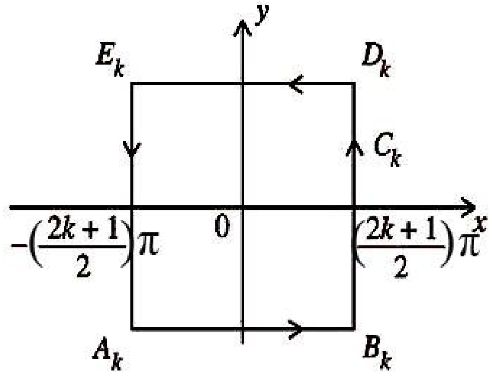

Figure 4.1: The square path C_k

We have

$$\begin{aligned}
\left|\oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz\right| &= \left|\oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{-\sin z + Cz\cos z + C\sin z}{\cos z + Cz\sin z} dz\right| \\
&\leq \oint_{C_k} \left|\frac{P_n(z)}{Q_m(z)}\right| \frac{|-\sin z + C(z\cos z + \sin z)|}{|\cos z + Cz\sin z|} |dz| \\
&\leq \oint_{C_k} \left|\frac{P_n(z)}{Q_m(z)}\right| \frac{|-1 + C(z\cot z + 1)|}{|\cot z + Cz|} |dz| \\
&\leq \oint_{C_k} \left|\frac{P_n(z)}{Q_m(z)}\right| \frac{|C-1| + |C|\,|z\cot z|}{|Cz| - |\cot z|} |dz| \\
&\leq \oint_{C_k} \left|\frac{P_n(z)}{Q_m(z)}\right| \frac{\frac{|C-1|}{|z|} + |C|\,|\cot z|}{|C| - \frac{|\cot z|}{|z|}} |dz|\,.
\end{aligned}$$

Since

$$|\cot z|^2 = \left|\frac{\cos z}{\sin z}\right|^2 = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y},$$

where $z = x + iy$ and

$$\frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y} \leq \frac{1 + \sinh^2 y}{\sinh^2 y} = \frac{\cosh^2 y}{\sinh^2 y} = \coth^2 y,$$

we have $|\cot z| \leq \coth \pi/2 = 1.090331411...$ for all z lying on the horizontal sides of C_k.

On the other hand, since

$$\frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y} = \frac{\sinh^2 y}{1 + \sinh^2 y} \leq 1,$$

we have $|\cot z| \leq 1$ for all z lying on the vertical sides of C_k.

This shows that $\cot z$ is bounded in the square paths C_k and therefore there exists $M_1 > 0$ such that

$$\frac{\frac{|C-1|}{|z|} + |C|\,|\cot z|}{|C| - \frac{|\cot z|}{|z|}} < M_1.$$

Thus,

$$\begin{aligned} \left| \oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz \right| &\leq M_1 \oint_{C_k} \left| \frac{P_n(z)}{Q_m(z)} \right| |dz| \\ &\leq M_1 (8k+4)\pi \max_{z \in C_k} \left| \frac{P_n(z)}{Q_m(z)} \right|. \end{aligned}$$

Since $\left| \frac{P_n(z)}{Q_m(z)} \right| \leq \frac{A}{|z|^2}$ for some $A > 0$, it follows that

$$\lim_{z \to \infty} \frac{P_n(z)}{Q_m(z)} = 0.$$

Therefore, there exists $M_2 > 0$ such that

$$\left| \oint_{C_k} \frac{P_n(z)}{Q_m(z)} \left(\frac{-\sin z + Cz\cos z + C\sin z}{\cos z + Cz\sin z} \right) dz \right| \leq M_2 \cdot \max_{z \in C_k} \left| \frac{P_n(z)}{Q_m(z)} \right| \to 0 \text{ as } k \to \infty.$$

Therefore

$$\sum_k \operatorname*{Res}_{z=\gamma_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) = -\sum_k \operatorname*{Res}_{z=z_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right).$$

Hence, we obtain the formula

$$\sum_{k=-\infty}^{\infty} \frac{P_n(\gamma_k)}{Q_m(\gamma_k)} = -\sum_k \operatorname*{Res}_{z=z_k} \left[\frac{P_n(z)}{Q_m(z)} \left(\frac{-\sin z + Cz\cos z + C\sin z}{\cos z + Cz\sin z} \right) \right].$$

■

Theorem 4.2 *Let* $f(z) = \frac{P_n(z)}{Q_m(z)}$ *with* $m \geq n+2$. *If* γ_k *are the roots of equation* $\csc z = -Cz$, *where* C *is a constant with* $C \geq -1$ *,then*

$$\sum_{k=-\infty}^{\infty} f(\gamma_k) = -\sum_k \operatorname*{Res}_{z=z_k} \left[\frac{P_n(z)}{Q_m(z)} \left(\frac{Cz\cos z + C\sin z}{1 + Cz\sin z} \right) \right], \tag{4.2}$$

where z_k are the zeros of the polynomial $Q_m(z)$.

Proof. Let

$$F(z) = 1 + Cz\sin z.$$

Then

$$F'(z) = Cz\cos z + C\sin z.$$

Therefore,

$$\frac{F'(z)}{F(z)} = \frac{Cz\cos z + C\sin z}{1 + Cz\sin z}.$$

Similarly, we let C_k be the square with vertices A_k, B_k, D_k, E_k at the points $\left(\pm\frac{(2k+1)\pi}{2}, \pm\frac{(2k+1)\pi}{2}\right)$, then C_k is the regular system of closed paths.

We use the residue theorem to obtain that

$$\oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz = 2\pi i \left[\sum_k \operatorname*{Res}_{z=z_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)}\right) + \sum_k \operatorname*{Res}_{z=\gamma_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)}\right)\right],$$

where z_k are the poles of $\frac{P_n(z)}{Q_m(z)}$ and γ_k are the poles of $\frac{F'(z)}{F(z)}$.

Next, we want to show that the integral on the left-hand side tends to zero as $k \to \infty$.

We have

$$\begin{aligned}
\left|\oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz\right| &= \left|\oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{Cz\cos z + C\sin z}{1 + Cz\sin z} dz\right| \\
&\le \oint_{C_k} \left|\frac{P_n(z)}{Q_m(z)}\right| \frac{|Cz\cot z + C)|}{|\csc z + Cz|} |dz| \\
&\le \oint_{C_k} \left|\frac{P_n(z)}{Q_m(z)}\right| \frac{|Cz\cot z| + |C|}{|Cz| - |\csc z|} |dz| \\
&\le \oint_{C_k} \left|\frac{P_n(z)}{Q_m(z)}\right| \frac{|C\cot z| + \left|\frac{C}{z}\right|}{|C| - \frac{|\csc z|}{|z|}} |dz|.
\end{aligned}$$

From the proof of Theorem 4.1 we can see that $\cot z$ is bounded and therefore we can easily show that $\csc z$ is also bounded. Since

$$|\csc z|^2 = \frac{1}{|\sin z|^2} = \frac{1}{\sin^2 x + \sinh^2 y} \le \frac{1}{\sinh^2 y} < \frac{1}{y^2} \le \frac{4}{\pi^2} < 1,$$

We have

$$\left| \oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{Cz\cos z + C\sin z}{1 + Cz\sin z} dz \right| \to 0 \text{ as } k \to \infty.$$

Therefore,

$$\sum_k \operatorname*{Res}_{z=\gamma_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) = -\sum_k \operatorname*{Res}_{z=z_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right).$$

And hence,

$$\sum_{k=-\infty}^{\infty} \frac{P_n(\gamma_k)}{Q_m(\gamma_k)} = -\sum_k \operatorname*{Res}_{z=z_k} \left[\frac{P_n(z)}{Q_m(z)} \left(\frac{Cz\cos z + C\sin z}{1 + Cz\sin z} \right) \right].$$

■

Theorem 4.3 *Let $f(z) = \frac{P_n(z)}{Q_m(z)}$ with $m \geq n+2$. If γ_k are the roots of equation $\sec z = -Cz$, where C is a constant with $C \geq -1$,then*

$$\sum_{k=-\infty}^{\infty} f(\gamma_k) = -\sum_k \operatorname*{Res}_{z=z_k} \left[\frac{P_n(z)}{Q_m(z)} \left(\frac{-Cz\sin z + C\cos z}{1 + Cz\cos z} \right) \right], \tag{4.3}$$

where z_k are the zeros of the polynomial $Q_m(z)$.

Proof. Let

$$F(z) = 1 + Cz\cos z.$$

Then

$$F'(z) = -Cz\sin z + C\cos z.$$

Therefore,

$$\frac{F'(z)}{F(z)} = \frac{-Cz\sin z + C\cos z}{1 + Cz\cos z}.$$

Similarly, we let C_k be the square with vertices A_k, B_k, D_k, E_k at the points $\left(\pm\frac{(2k+1)\pi}{2}, \pm\frac{(2k+1)\pi}{2}\right)$, then C_k is the regular system of closed paths.
We use the residue theorem to obtain that

$$\oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz = 2\pi i \left[\sum_k \operatorname*{Res}_{z=z_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) + \sum_k \operatorname*{Res}_{z=\gamma_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) \right],$$

where z_k are the poles of $\frac{P_n(z)}{Q_m(z)}$ and γ_k are the poles of $\frac{F'(z)}{F(z)}$.
Next, we want to show that the integral on the left-hand side tends to zero as

$k \to \infty$.

We have

$$\begin{aligned}
\left| \oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} dz \right| &= \left| \oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{-Cz \sin z + C \cos z}{1 + Cz \cos z} dz \right| \\
&\leq \oint_{C_k} \left| \frac{P_n(z)}{Q_m(z)} \right| \frac{|-Cz \tan z + C)|}{|\sec z + Cz|} |dz| \\
&\leq \oint_{C_k} \left| \frac{P_n(z)}{Q_m(z)} \right| \frac{|Cz \tan z| + |C|}{|Cz| - |\sec z|} |dz| \\
&\leq \oint_{C_k} \left| \frac{P_n(z)}{Q_m(z)} \right| \frac{|C \tan z| + \left|\frac{C}{z}\right|}{|C| - \frac{|\sec z|}{|z|}} |dz| .
\end{aligned}$$

Similarly, it can be shown that $\tan z$ and $\sec z$ are bounded in the square paths C_k and therefore there exists $M_1 > 0$ such that

$$\frac{|C \tan z| + \left|\frac{C}{z}\right|}{|C| - \frac{|\sec z|}{|z|}} < M_1.$$

Thus

$$\left| \oint_{C_k} \frac{P_n(z)}{Q_m(z)} \frac{-Cz \sin z + C \cos z}{1 + Cz \cos z} dz \right| \to 0 \text{ as } k \to \infty.$$

Therefore,

$$\sum_k \operatorname*{Res}_{z=\gamma_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right) = -\sum_k \operatorname*{Res}_{z=z_k} \left(\frac{P_n(z)}{Q_m(z)} \frac{F'(z)}{F(z)} \right).$$

And hence,

$$\sum_{k=-\infty}^{\infty} \frac{P_n(\gamma_k)}{Q_m(\gamma_k)} = -\sum_k \operatorname*{Res}_{z=z_k} \left[\frac{P_n(z)}{Q_m(z)} \left(\frac{-Cz \sin z + C \cos z}{1 + Cz \cos z} \right) \right].$$

$\blacksquare$

CHAPTER V

SUMMATION INVOLVING BINOMIAL COEFFICIENTS

In this chapter we discuss the identities about the sum of binomial coefficients. We start with the connection between the binomial coefficients and contour integral.Then we show how to evaluate the sum by using the residue theory.

Proposition 5.1 *If n and m are positive integers, then the identity*

$$\sum_{k=m}^{n} \binom{n}{k}\binom{n-m}{k-m} = \binom{2n-m}{n-m} \tag{5.1}$$

holds for $m \leq n$.

Proof. We note that

$$\binom{n}{k} \quad \text{is the coefficient of} \quad z^k \quad \text{in} \quad (1+z)^n$$

and

$$\binom{n-m}{k-m} \quad \text{is the coefficient of} \quad \frac{1}{z^{k-m}} \quad \text{in} \quad \left(1+\frac{1}{z}\right)^{n-m}$$

and if we consider the expansion of $(1+z)^n\left(1+\frac{1}{z}\right)^{n-m}$, then we have

$$\sum_{k=m}^{n} \binom{n}{k}\binom{n-m}{k-m} \text{ being the constant term in } (1+z)^n\left(1+\frac{1}{z}\right)^{n-m}.$$

By using the residue theorem and letting C be a unit circle centered at the origin, we obtain

$$\begin{aligned}
\sum_{k=m}^{n} \binom{n}{k}\binom{n-m}{k-m} &= \operatorname*{Res}_{z=0}\left[\frac{(1+z)^n\left(1+\frac{1}{z}\right)^{n-m}}{z}\right] \\
&= \frac{1}{2\pi i}\int_C (1+z)^n\left(1+\frac{1}{z}\right)^{n-m}\frac{dz}{z} \\
&= \frac{1}{2\pi i}\int_C \frac{(1+z)^{2n-m}}{z^{n-m+1}}dz.
\end{aligned}$$

By Corollary 3.5, we have

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^{2n-m}}{z^{n-m+1}}dz = \binom{2n-m}{n-m}.$$

Hence, the result follows. ■

Proposition 5.2 *If n and m are positive integers, then the identity*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{mn}{k} = \binom{(m+1)n}{mn} \tag{5.2}$$

holds.

Proof. Similarly, we note that

$$\binom{n}{k} \quad \text{is the coefficient of} \quad z^k \quad \text{in} \quad (1+z)^n$$

and

$$\binom{mn}{k} \quad \text{is the coefficient of} \quad \frac{1}{z^k} \quad \text{in} \quad \left(1+\frac{1}{z}\right)^{mn}$$

and if we consider the expansion of $(1+z)^n\left(1+\frac{1}{z}\right)^{mn}$, then we have

$$\sum_{k=0}^{n}\binom{n}{k}\binom{mn}{k} \text{ being the constant term in } (1+z)^n\left(1+\frac{1}{z}\right)^{mn}.$$

By the same argument as before, we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\binom{mn}{k} &= \frac{1}{2\pi i}\int_C (1+z)^n\left(1+\frac{1}{z}\right)^{mn}\frac{dz}{z} \\ &= \frac{1}{2\pi i}\int_C \frac{(1+z)^{mn+n}}{z^{mn+1}}dz.\end{aligned}$$

By Corollary 3.5, we have

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^{mn+n}}{z^{mn+1}}dz = \binom{mn+n}{mn}.$$

Hence, we get the result. ■

Proposition 5.3 *If n is a positive integer and $|z|<1$, then*

$$\sum_{k=0}^{\infty}\binom{n+k}{k}z^k = \frac{1}{(1-z)^{n+1}}. \tag{5.3}$$

Proof. From Corollary 3.5, we have

$$\binom{n+k}{k} = \frac{1}{2\pi i}\int\limits_C \frac{(1+w)^{n+k}}{w^{k+1}}dw.$$

So

$$\binom{n+k}{k} z^k = \left[\frac{1}{2\pi i}\int\limits_C \frac{(1+w)^{n+k}}{w^{k+1}}dw\right] z^k$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{n+k}{k} z^k &= \frac{1}{2\pi i}\int\limits_C \sum_{k=0}^{\infty}\frac{(1+w)^{n+k}z^k}{w^{k+1}}dw \\ &= \frac{1}{2\pi i}\int\limits_C (1+w)^n \sum_{k=0}^{\infty}\left[\frac{(1+w)z}{w}\right]^k dw.\end{aligned}$$

We sum the geometric series and use the residue theorem to obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{n+k}{k} z^k &= \frac{1}{2\pi i}\int\limits_C \frac{\frac{(1+w)^n}{1-z}}{w-\frac{z}{1-z}}dw \\ &= \operatorname*{Res}_{w=\frac{z}{1-z}}\left[\frac{(1+w)^n}{1-z}\right] \\ &= \frac{1}{(1-z)^{n+1}}.\end{aligned}$$

■

Proposition 5.4 *If n is a positive integer, then*

$$\sum\nolimits_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{\frac{n}{2}}\binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even.}\end{cases} \tag{5.4}$$

Proof. We note that

$$\binom{n}{k} \quad \text{is the coefficient of} \quad z^k \quad \text{in the expansion of} \quad (1+z)^n$$

and

$$(-1)^k\binom{n}{k} \quad \text{is the coefficient of} \quad \frac{1}{z^k} \quad \text{in the expansion of} \quad \left(1-\frac{1}{z}\right)^n.$$

If we consider the expansion of $(1+z)^n\left(1-\frac{1}{z}\right)^n$, then we obtain that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 \text{ is the constant term in } (1+z)^n\left(1-\frac{1}{z}\right)^n.$$

By using the residue theorem with C being a unit circle centered at the origin, we have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 &= \frac{1}{2\pi i}\int\limits_C (1+z)^n\left(1-\frac{1}{z}\right)^n\frac{dz}{z}\\
&= \frac{1}{2\pi i}\int\limits_C \frac{(1+z)^n(z-1)^n}{z^{n+1}}dz\\
&= \frac{1}{2\pi i}\int\limits_C \frac{(z^2-1)^n}{z^{n+1}}dz\\
&= \operatorname*{Res}_{z=0}\left[\frac{(z^2-1)^n}{z^{n+1}}\right]\\
&= \begin{cases} 0, & \text{if } n \text{ is odd;}\\ (-1)^{\frac{n}{2}}\binom{n}{\frac{n}{2}}, & \text{if n is even.}\end{cases}
\end{aligned}$$

Hence, we obtain the result. ■

CHAPTER VI

CONCLUSION AND DISCUSSION

The purpose of this chapter is to accumulate all the main results that we have proved in Chapter 4 and Chapter 5. In Chapter 4, we proved the theorem for finding the sums involving the real roots of some transcendental equations by using the residue theory. We obtain the following results.

Theorem 6.1 *Let* $f(z) = \frac{P_n(z)}{Q_m(z)}$ *with* $m \geq n+2$. *If* γ_k *are the real roots of equation* $\cot z = -Cz$, *where* C *is a constant with* $C \geq -1$, *then*

$$\sum_{k=-\infty}^{\infty} f(\gamma_k) = -\sum_k \operatorname*{Res}_{z=z_k}\left[\frac{P_n(z)}{Q_m(z)}\left(\frac{-\sin z + Cz\cos z + C\sin z}{\cos z + Cz\sin z}\right)\right],$$

where z_k *are the zeros of the polynomial* $Q_m(z)$.

Theorem 6.2 *Let* $f(z) = \frac{P_n(z)}{Q_m(z)}$ *with* $m \geq n+2$. *If* γ_k *are the real roots of equation* $\csc z = -Cz$, *where* C *is a constant with* $C \geq -1$ *,then*

$$\sum_{k=-\infty}^{\infty} f(\gamma_k) = -\sum_k \operatorname*{Res}_{z=z_k}\left[\frac{P_n(z)}{Q_m(z)}\left(\frac{Cz\cos z + C\sin z}{1 + Cz\sin z}\right)\right],$$

where z_k *are the zeros of the polynomial* $Q_m(z)$.

Theorem 6.3 *Let* $f(z) = \frac{P_n(z)}{Q_m(z)}$ *with* $m \geq n+2$. *If* γ_k *are the real roots of equation* $\sec z = -Cz$, *where* C *is a constant with* $C \geq -1$ *,then*

$$\sum_{k=-\infty}^{\infty} f(\gamma_k) = -\sum_k \operatorname*{Res}_{z=z_k}\left[\frac{P_n(z)}{Q_m(z)}\left(\frac{-Cz\sin z + C\cos z}{1 + Cz\cos z}\right)\right],$$

where z_k *are the zeros of the polynomial* $Q_m(z)$.

In Chapter 5, we considered the sums of binomial coefficients and proved the identities of these sums by using the residue theory. We obtain the results:

Proposition 6.4 *If n and m are positive integers, then the identity*

$$\sum_{k=m}^{n}\binom{n}{k}\binom{n-m}{k-m}=\binom{2n-m}{n-m}$$

holds for $m \leq n$.

Proposition 6.5 *If n and m are positive integers, then the identity*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{mn}{k}=\binom{(m+1)n}{mn}$$

holds.

Proposition 6.6 *If n is a positive integer and $|z|<1$, then*

$$\sum_{k=0}^{\infty}\binom{n+k}{k}z^k=\frac{1}{(1-z)^{n+1}}.$$

Proposition 6.7 *If n is a positive integer, then*

$$\textstyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=\begin{cases}0, & \text{if } n \text{ is odd;}\\ (-1)^{\frac{n}{2}}\binom{n}{\frac{n}{2}}, & \text{if } n \text{ is even.}\end{cases}.$$

Some examples now follow, using above theorems.

Example Find $\sum_{k=-\infty}^{\infty}\frac{1}{{\gamma_k}^2+1}$ where γ_k are all real roots of $\cot x = x$.

<u>Solution</u> From equation (4.1) in Theorem 4.1 with $C=-1$, we obtain

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{1}{{\gamma_k}^2+1} &= -\sum_k \operatorname*{Res}_{z=z_k}\left[\frac{1}{z^2+1}\left(\frac{-\sin z-(z\cos z+\sin z)}{\cos z - z\sin z}\right)\right]\\ &= -\left[\left(\operatorname*{Res}_{z=i}+\operatorname*{Res}_{z=-i}\right)\left(\frac{-2\sin z - z\cos z}{(z^2+1)(\cos z - z\sin z)}\right)\right].\end{aligned}$$

By computing the residue on the right-hand side, we obtain

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{1}{{\gamma_k}^2+1} &= -\left(\frac{-2\sin i - i\cos i}{(2i)(\cos i - i\sin i)}+\frac{2\sin i + i\cos i}{(-2i)(\cos i - i\sin i)}\right)\\ &= \frac{2\sin i + i\cos i}{i\cos i + \sin i}=\frac{2\sinh 1+\cosh 1}{\sinh 1+\cosh 1}\approx 1.432332358.\end{aligned}$$

Remark. Summation of the series will be confirmed by using the graph plotting method.(see [3]) We plot the graph $y=\cot x$ and $y=x$ in the same axes and see that γ_k, which are the intersection points of two curves are the real roots of

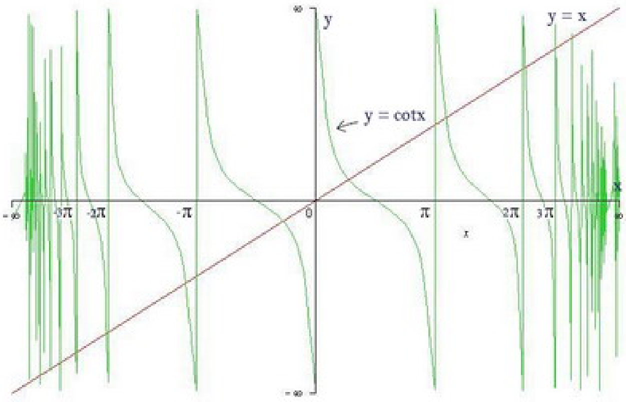

Figure 6.1: Positive roots of $\cot x = x$

$\cot x = x$.(Figure 2.) It follows from the graph that $\lim\limits_{k\to\infty}(\gamma_{k+1} - \gamma_k) = \pi$ and $\gamma_k \approx (k-1)\pi, k \geq 6$. Then

$$\sum_{k=1}^{\infty}\frac{1}{{\gamma_k}^2+1} = \frac{1}{{\gamma_1}^2+1}+\frac{1}{{\gamma_2}^2+1}+\frac{1}{{\gamma_3}^2+1}+\frac{1}{{\gamma_4}^2+1}+\frac{1}{{\gamma_5}^2+1}+\sum_{k=6}^{\infty}\frac{1}{{\gamma_k}^2+1},$$

where

$$\gamma_1 \approx 0.8603, \quad \gamma_2 \approx 3.4256, \quad \gamma_3 \approx 6.4373, \quad \gamma_4 \approx 9.5293, \quad \gamma_5 \approx 12.6453.$$

Hence, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{{\gamma_k}^2+1} \approx 1.432516821$, which corresponds approximately to the exact computation obtained above.

Moreover, it will be interesting if this concept can be extended to other transcendental equations as well.

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