CHAPTER V
NUMERICAL RESULTS

In this chapter we present some examples of solving the Laplace equation by using BEM and meshless method as detailed in the previous chapters. Several examples are similar to Lucha [18] (and some examples of Toutip [8]). The numerical results obtained from BEM and meshless method are compared and discussed with those from analytical solution in terms of accuracy. The numerical results are shown as the tables and graphs. The absolute errors are used for this investigation. In addition, meshless methods are compared among their methods; Kansa and Hermite, by considering the same shape parameter. Furthermore, both methods are compared to any value of shape parameter.

In meshless methods, the inverse multiquadric radial basis function (IMQ) is used to calculate the solution at each interior point. In BEM, 3-Gauss integration is used for the approximate integration.

Several tested problems with two boundary conditions; Dirichlet boundary condition and mixed boundary conditions are considered.

In the below section, MLM1, MLM2, BEM and Exact represent, respectively Kansa's method, Hermite-based method, boundary element method and exact solution.

5.1 A square domain with Dirichlet boundary condition

Example 5.1.1 A square domain covering $0 \le x \le 1$ and $0 \le y \le 1$ is studied. Dirichlet boundary condition is imposed on the four edges of this domain using the following analytical solution:

$$u(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\left(1-(-1)^n\right)\left(\sin(n\pi x)\right)\sinh(n\pi y)}{n\left(\sinh(n\pi)\right)}$$

Consider the Laplace equation

$$\nabla^2 u = 0,\ 0 \le x \le 1,\ 0 \le y \le 1$$

Dirichlet boundary condition

$u = 0 \quad 0 \le x \le 1, y = 0$, $\quad u = 0 \quad x = 1, 0 \le y \le 1$

$u = 1 \quad 0 \le x \le 1, y = 1$, $\quad u = 0 \quad x = 0, 0 \le y \le 1$

as shown in Figure 5.1.1

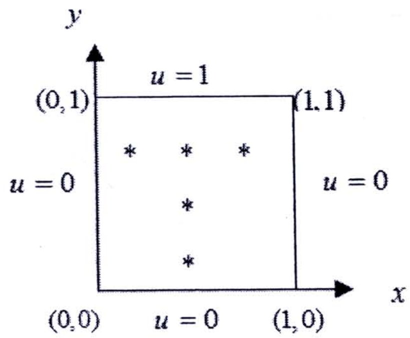


Figure 5.1.1 Square domain with Dirichlet boundary condition for Example 5.1.1

Table 5.1.1 shows the results obtained from BEM and MLM1 at each interior point. The meshless method, shape parameter $\varepsilon = 3$ is used for solving this problem since this value yields the best solutions. Also, the errors of solution from both methods are compared in this table. Results for this example are plotted in Figure 5.1.2

Table 5.1.1 Comparison between the numerical results obtained from two methods, applied to Example 5.1.1

Internal point	**BEM**	**MLM1**	**Exact**	**Error**	
				BEM	**MLM1**
(0.25,0.75)	0.4025	0.4311	0.4320	0.0295	0.0009
(0.50,0.75)	0.5409	0.5406	0.5405	0.0004	0.0001
(0.50,0.50)	0.2382	0.2500	0.2500	0.0118	0.0000
(0.75,0.75)	0.4025	0.4330	0.4320	0.0295	0.0010
(0.50,0.25)	0.0859	0.0954	0.0954	0.0095	0.0000

Each curve in Figure 5.1.2 represents results of two methods and analytical solution; BEM, MLM1 and Exact. The results of the present comparison show that MLM1 is more accurate than BEM solutions.

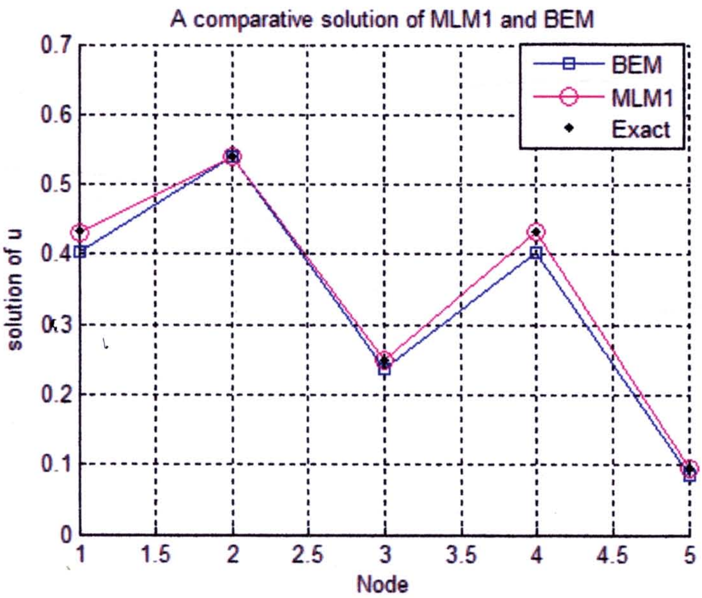


Figure 5.1.2 A comparative solution of Example 5.1.1

Figure 5.1.3 shows the errors of solutions from both methods, it can be seen that those of solution from BEM are more than those of solution from meshless method.

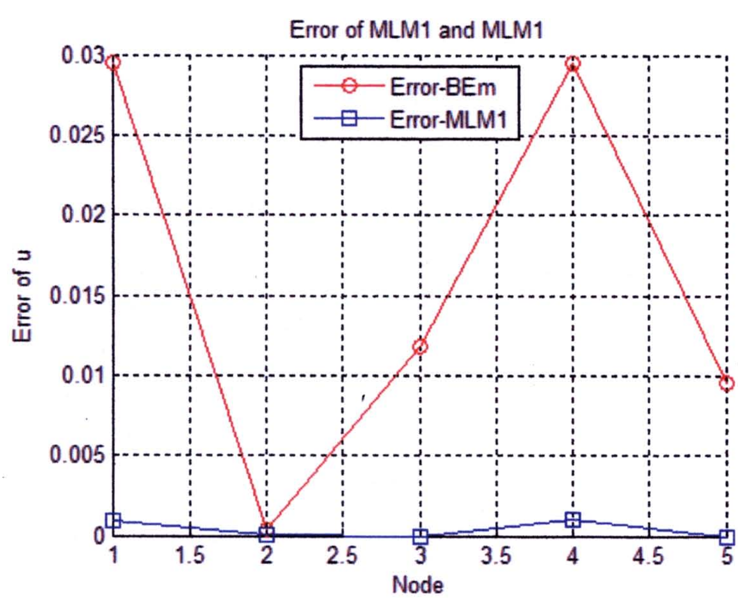


Figure 5.1.3 Error of MLM1 and BEM for Example 5.1.1

Example 5.1.2 A square domain covering $0 \le x \le 1$ and $0 \le y \le 1$ is studied. Dirichlet boundary condition is imposed on the four edges of this domain using the following analytical solution: $u(x,y) = x^2 - y^2$.

Consider the Laplace equation

$$\nabla^2 u = 0,\ 0 \le x \le 1,\ 0 \le y \le 1$$

Dirichlet boundary condition

$$u = -y^2 \quad x = 0, 0 \le y \le 1, \qquad u = 1 - y^2 \quad x = 1, 0 \le y \le 1$$

$$u = x^2 - 1 \quad 0 \le x \le 1, y = 1, \qquad u = x^2 \quad 0 \le x \le 1, y = 0$$

as shown in Figure 5.1.4

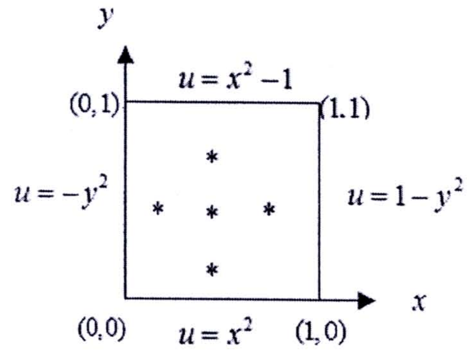


Figure 5.1.4 Square domain with Dirichlet boundary condition for Example 5.1.2

Table 5.1.2 shows results obtained from BEM and MLM1. Shape parameter $\varepsilon = 3$ is used for this example. Besides, errors of solutions from both methods are compared. Results for this example are plotted in Figure 5.1.5. It is similar to Example 5.1.1, MLM1 are more accurate than BEM solutions.

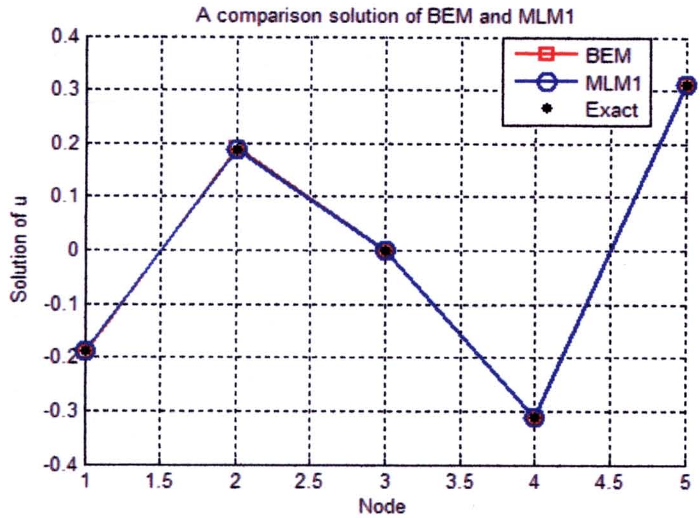


Figure 5.1.5 A comparative solution of Example 5.1.2

Figure 5.1.6 shows the errors of solution obtained from BEM and meshless method. It can be seen that those of solution obtained from BEM are higher those of solution obtained from meshless method.

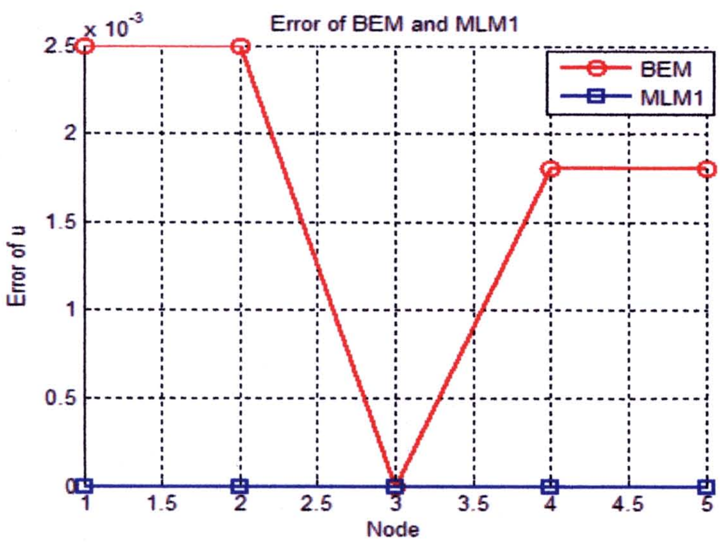


Figure 5.1.6 Error of MLM1 and BEM for Example 5.1.2

Table 5.1.2 Comparison between the numerical results obtained from two methods, applied to Example 5.1.2

Internal point	BEM	MLM1	Exact	Error	
				BEM	MLM1
(0.25,0.50)	-0.1900	-0.1875	-0.1875	0.0025	0.0000
(0.50,0.25)	0.1900	0.1875	0.1875	0.0025	0.0000
(0.50,0.50)	0.0000	0.0000	0.0000	0.0000	0.0000
(0.50,0.75)	-0.3107	-0.1325	-0.1325	0.0018	0.0000
(0.75,0.50)	0.3107	0.3125	0.3125	0.0018	0.0000

According to the table 5.1.2, errors of MLM1 closed to zero when considering four decimals.

Example 5.1.3 A square domain covering $0 \le x \le 2, 0 \le y \le 4$ is studied. Dirichlet boundary condition is imposed on the four edges of this domain using the following analytical solution: $u(x,y) = x^3 - 3xy^2$.

Consider the Laplace equation

$$\nabla^2 u = 0, \quad 0 \le x \le 2, 0 \le y \le 4$$

with the following boundary condition

$$u = 0 \qquad x = 0, 0 \le y \le 4, \quad u = 8 - 6y^2 \quad x = 2, 0 \le y \le 4$$

$$u = x^3 - 48x \quad 0 \le x \le 2, y = 4, \quad u = x^3 \qquad 0 \le x \le 2, y = 0$$

as shown in Figure 5.1.7

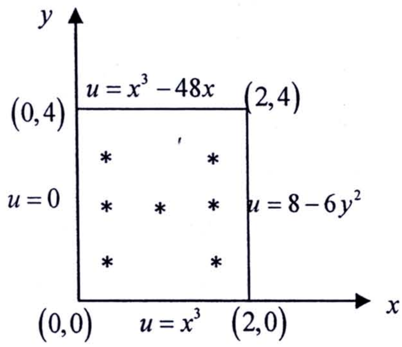


Figure 5.1.7 Square domain with Dirichlet boundary condition for Example 5.1.3

Table 5.1.3 shows results obtained from BEM and MLM1. Shape parameter $\varepsilon = 1$ is used for this example. Besides, errors of solution from both methods are compared in this table. Results for this example are plotted in Figure 5.1.8. MLM1 are more accurate than BEM solutions.

Table 5.1.3 Comparison between the numerical results obtained from two methods, applied to Example 5.1.3

Internal point	BEM	MLM1	Exact	Error	
				BEM	MLM1
(0.5,1.0)	-1.4016	-1.3747	-1.3750	0.0266	0.0003
(0.5,2.0)	-5.9257	-5.8746	-5.8750	0.0507	0.0004
(0.5,3.0)	-13.4689	-13.3734	-13.3750	0.0939	0.0016
(1.0,3.0)	-26.1107	-25.9974	-26.0000	0.1107	0.0026
(1.5,1.0)	-1.4043	-1.1246	-1.1250	0.2793	0.0004
(1.5,2.0)	-14.9475	-14.6246	-14.6250	0.3225	0.0004
(1.5,3.0)	-37.4047	-37.1229	-37.1250	0.2797	0.0021

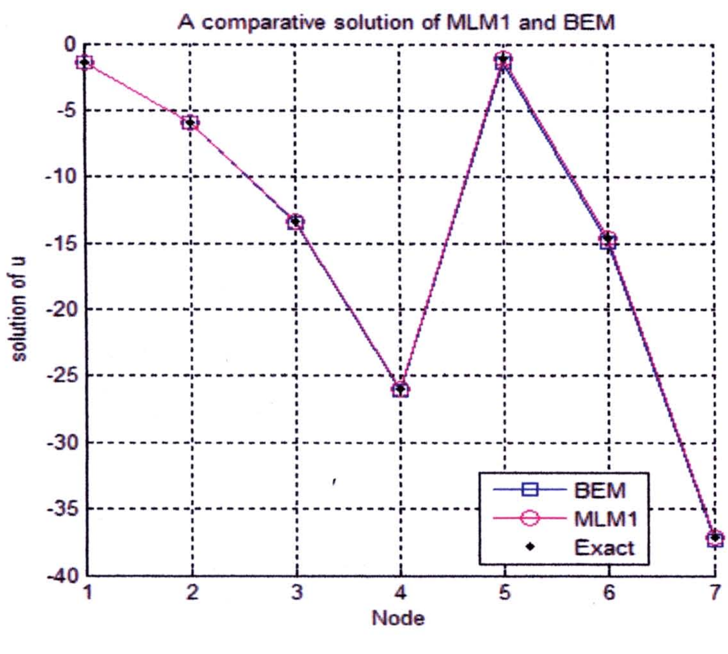


Figure 5.1.8 A comparative solution of Example 5.1.3

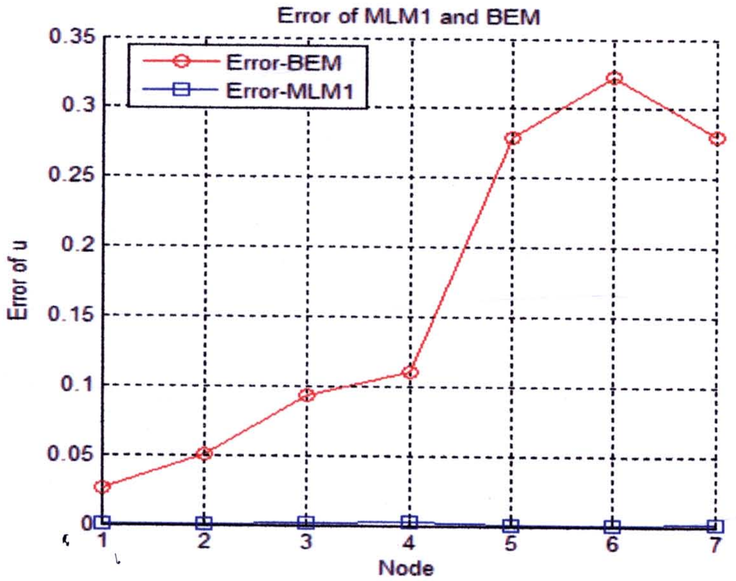


Figure 5.1.9 Error of MLM1 and BEM for Example 5.1.3

Example 5.1.4 A square domain covering $0 \le x \le 1$ and $0 \le y \le 1$ is studied. Dirichlet boundary condition is imposed on the four edges of this domain using the following analytical solution: $u(x,y) = x^2 + y(1-y)$.

Consider the Laplace equation

$$\nabla^2 u = 0, \quad 0 \le x \le 1, 0 \le y \le 1$$

with boundary condition

$$u = y(1-y) \quad x = 0, 0 \le y \le 1, \qquad u = 1 + y(1-y) \quad x = 1, 0 \le y \le 1$$

$$u = x^2 \qquad 0 \le x \le 1, y = 1, \qquad u = x^2 \qquad 0 \le x \le 1, y = 0$$

as shown in Figure 5.1.10

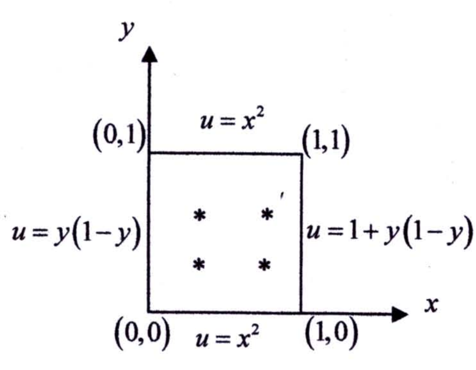


Figure 5.1.10 Square domain with Dirichlet boundary condition for Example 5.1.4

Table 5.1.4 shows results obtained from BEM and MLM1. Shape parameter $\varepsilon = 3$ is used for this example. And errors of solution from both methods are compared in this table. Results for this example are plotted in Figure 5.1.11. It is similar to Example 5.1.3; MLM1 are more accurate than BEM solutions.

Table 5.1.4 Comparison between the numerical results obtained from two methods, applied to Example 5.1.4

Internal point	BEM	MLM1	Exact	Error	
				BEM	MLM1
(0.4,0.6)	0.400003	0.400000	0.400000	0.000003	0.000000
(0.4,0.4)	0.400003	0.400000	0.400000	0.000003	0.000000
(0.6,0.4)	0.599999	0.600000	0.600000	0.000001	0.000000
(0.6,0.6)	0.599999	0.600000	0.600000	0.000001	0.000000

From table 5.1.4, results of both methods are compared with analytical solution; those of both methods are a little difference. In terms of accuracy, solution obtained from MLM1 is better than those obtained from BEM. The errors of MLM1 converge to zero for considering six decimals.

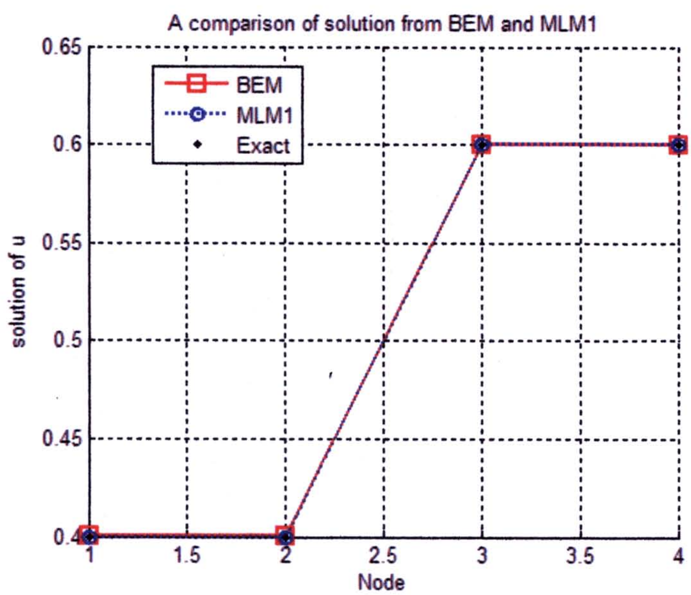


Figure 5.1.11 A comparative solution of Example 5.1.4

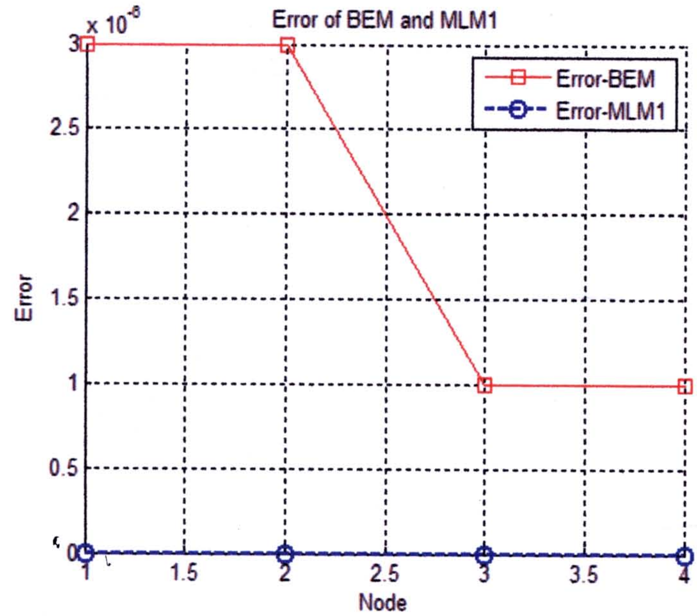


Figure 5.1.12 Error of MLM1 and BEM for Example 5.1.4

Example 5.1.5 A square domain covering $0 \le x \le 6$ and $0 \le y \le 6$ is studied. Dirichlet boundary condition is imposed on the four edges of this domain using the following analytical solution: $u(x,y) = 300 - 50x$.

Consider the Laplace equation

$$\nabla^2 u = 0,\ 0 \le x \le 6,\ 0 \le y \le 6$$

with the following boundary condition

$$u = 300 \quad x = 0, 0 \le y \le 6, \qquad u = 0 \quad x = 6, 0 \le y \le 6$$

$$u = 300 - 50x \quad 0 \le x \le 6, y = 6 \quad , u = 300 - 50x \quad 0 \le x \le 6, y = 0$$

as shown in Figure 5.1.13

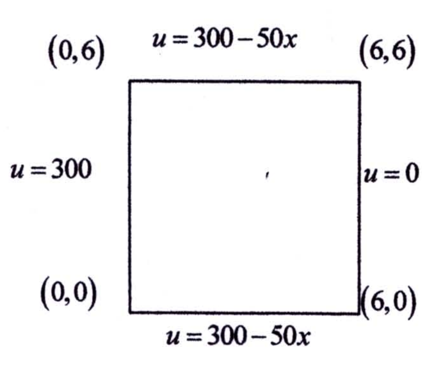


Figure 5.1.13 Square domain with Dirichlet boundary condition for Example 5.1.5

Table 5.1.5 shows results obtained from BEM and MLM1. Shape parameter $\varepsilon = 3$ is used for this example. Also, errors of solution from both methods are compared in this table. Results for this example are plotted in Figure 5.1.14. It is similar to Example 5.1.4, MLM1 are more accurate than BEM solutions.

Table 5.1.5 Comparison between the numerical results obtained from two methods, applied to Example 5.1.5

Internal point	BEM	MLM1	Exact	Error	
				BEM	MLM1
(2,2)	199.9958	199.9985	200.0000	0.0042	0.0015
(2,4)	199.9958	199.9985	200.0000	0.0042	0.0015
(4,2)	100.0056	100.0001	100.0000	0.0056	0.0001
(4,4)	100.0056	100.0001	100.0000	0.0056	0.0001

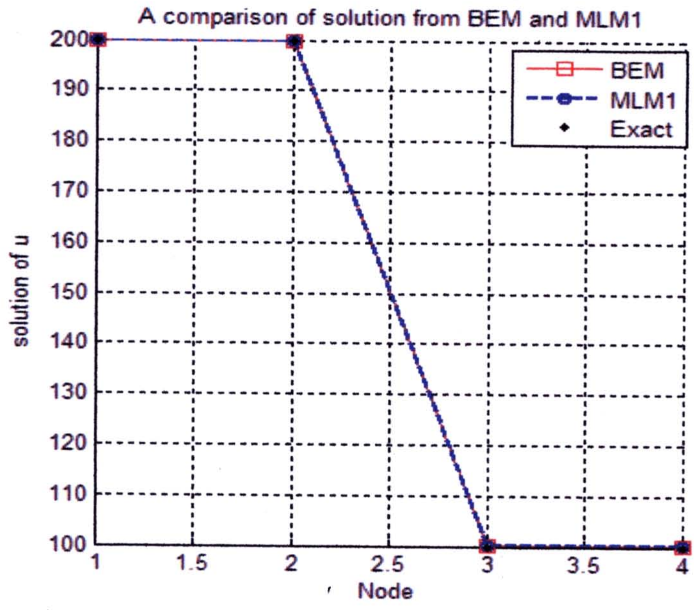


Figure 5.1.14 A comparative solution of Example 5.1.5

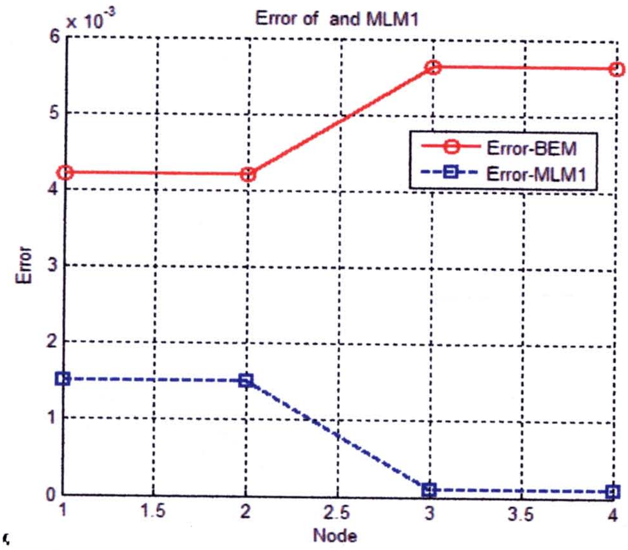


Figure 5.1.15 Error of MLM1 and BEM for Example 5.1.5

5.2 A square domain with mixed boundary conditions

Example 5.2.1 A square domain covering $0 \le x \le 1$ and $0 \le y \le 1$ is studied. A mixed boundary conditions are imposed on the four edges of this domain using the following analytical solution: $u(x,y) = 1 + x + 2y$.

Consider the Laplace equation

$$\nabla^2 u = 0, \quad 0 \le x \le 1 \ , \quad 0 \le y \le 1$$

with boundary condition

$$u = 1 + 2y \quad x = 0, 0 \le y \le 1, \qquad u = 2 + 2y \quad x = 1, 0 \le y \le 1$$

$$q = 2 \qquad 0 \le x \le 1, y = 1, \qquad q = -2 \qquad 0 \le x \le 1, y = 0$$

as shown in Figure 5.2.1

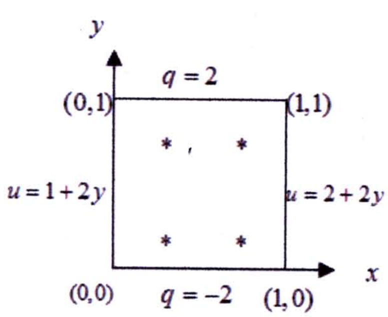


Figure 5.2.1 Square domain with mixed boundary conditions for Example 5.2.1

Table 5.2.1 shows results obtained from BEM and MLM1. Shape parameter $\varepsilon = 3$ is used for this example. Besides, errors of solution from both methods are compared. Results for this example are plotted in Figure 5.2.2. It can be seen that MLM1 are more accurate than BEM solutions.

Table 5.2.1 Comparison between the numerical results obtained from two methods, applied to Example 5.2.1

Internal point	**BEM**	**MLM1**	**Exact**	**Error**	
				BEM	**MLM1**
(0.25,0.25)	1.7552	1.7501	1.7500	0.0052	0.0001
(0.75,0.25)	2.2589	2.2500	2.2500	0.0089	0.0000
(0.75,0.75)	3.2444	3.2489	3.2489	0.0056	0.0011
(0.25,0.75)	2.7407	2.7490	2.7490	0.0093	0.0010

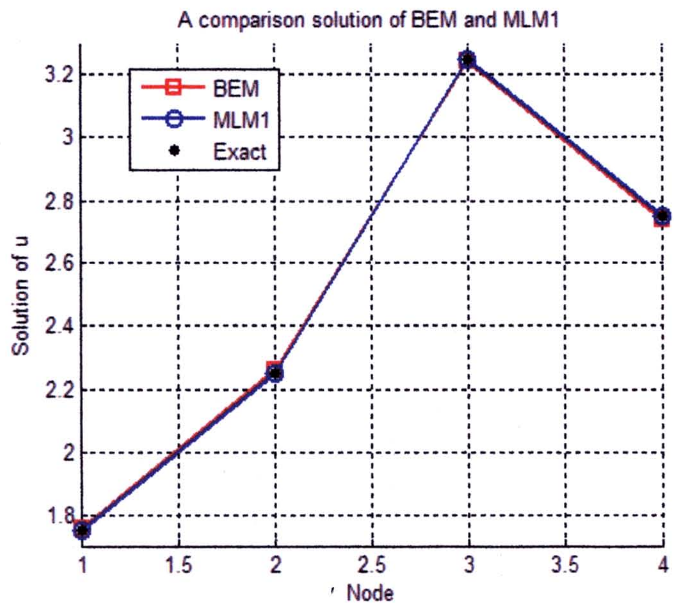


Figure 5.2.2 A comparative solution of Example 5.2.1

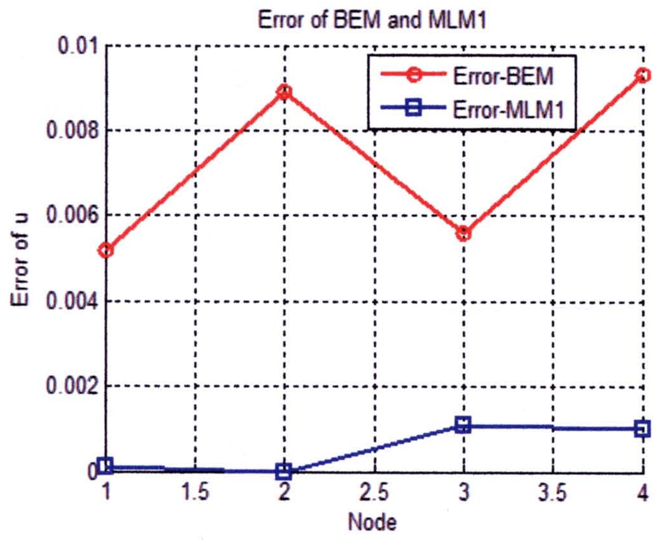


Figure 5.2.3 Error of MLM1 and BEM for Example 5.2.1

Example 5.2.2 A square domain covering $0 \le x \le 6$ and $0 \le y \le 6$ is studied. Dirichlet boundary condition is imposed on the four edges of this domain using the following analytical solution: $u(x,y) = 300 - 50x$.

Consider the Laplace equation

$$\nabla^2 u = 0,\ 0 \le x \le 6, 0 \le y \le 6$$

with boundary condition

$$u = 300 \quad x = 0, 0 \le y \le 6, \qquad u = 0 \quad x = 6, 0 \le y \le 6$$

$$q = 0 \quad 0 \le x \le 6, y = 6, \qquad q = 0 \quad 0 \le x \le 6, y = 0$$

as shown in Figure 5.2.4

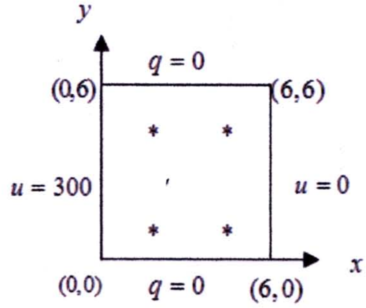


Figure 5.2.4 Square domain with mixed boundary conditions for Example 5.2.2

Table 5.2.2 shows results obtained from BEM and MLM1. Shape parameter $\varepsilon = 1.5$ is used for this example. Besides, errors of solution from both methods are compared. Results for this example are plotted in Figure 5.2.5. MLM1 are more accurate than BEM solutions.

Table 5.2.2 Comparison between the numerical results obtained from two methods, applied to Example 5.2.2

Internal point	BEM	MLM1	Exact	Error	
				BEM	MLM1
(0.20,0.20)	200.3143	200.0629	200.0000	0.3143	0.0629
(0.20,0.40)	200.3143	200.0626	200.0000	0.3143	0.0.626
(0.40,0.20)	99.6871	100.0576	100.0000	0.3129	0.0576
(0.40,0.40)	99.6871	100.0573	100.0000	0.3129	0.0573

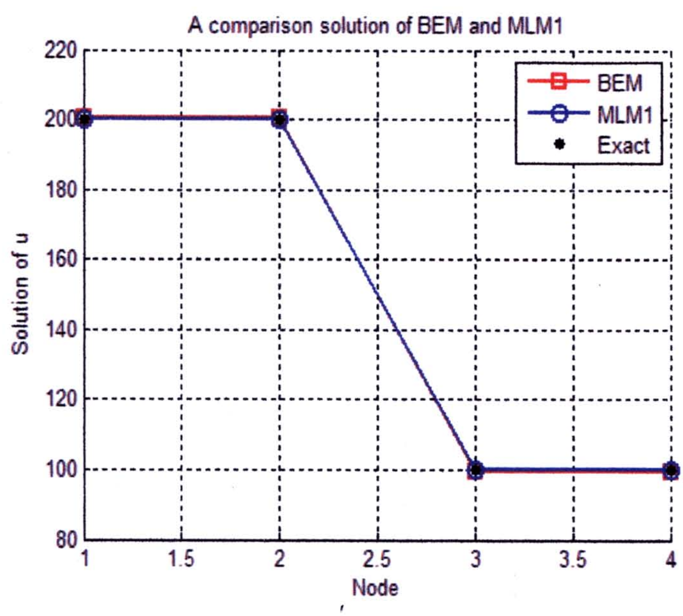


Figure 5.2.5 A comparative solution of Example 5.2.2

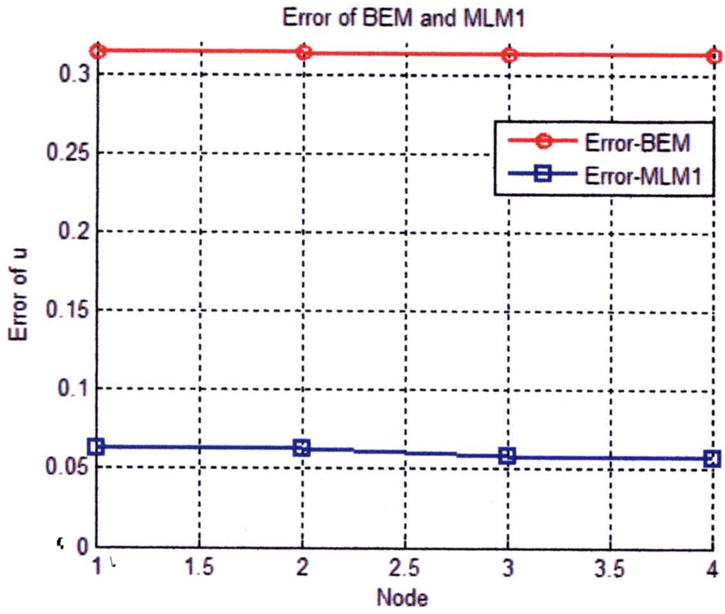


Figure 5.2.6 Error of MLM1 and BEM for Example 5.2.2

Example 5.2.3 Consider the region as shown in Figure 5.2.6 with Laplace equation and boundary conditions

$$u = 0 \text{ on } y = 0$$

$$u = 0 \text{ on } (x-4)^2 + y^2 = 1, y \geq 0$$

$$q = 0 \text{ on } x = 4, 1 < y < 2$$

$$u = 2 \text{ on } y = 2$$

$$u = y \text{ on } x = 0$$

This problem without the exact solution. However, a widely accepted benchmark solution is available for this problem.

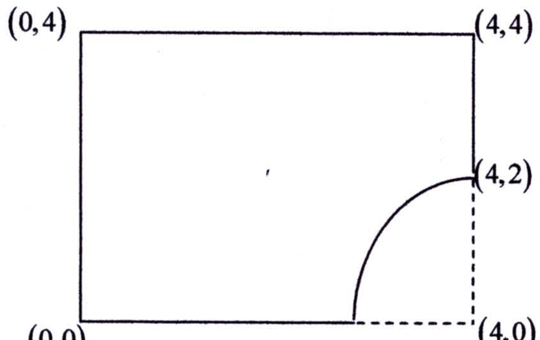


Figure 5.2.7 Domain with mixed boundary conditions for Example 5.2.3

Table 5.2.3 shows results obtained from both methods. Shape parameter $\varepsilon = 2.6$ is used for this example. There are 400 points uniformly distributed in the domain. Results for this example are plotted in Figure 5.2.8, and these results are compared with SDPR. It can be seen that MLM1 solutions are more accurate than BEM solutions. SDPR denotes the benchmark solution in this example.

Table 5.2.3 Comparison between the numerical results obtained from two methods, applied to Example 5.2.3

Internal point	SDPR	MLM1	BEM
(0.75,0.25)	0.2450	0.2461	0.2450
(0.75,1.75)	1.7450	1.7449	1.7399
(2.25,0.25)	0.1960	0.1958	0.1979
(2.25,1.00)	0.8750	0.8751	0.8786

Although we do not have the exact solution, from table 5.2.3, MLM1 and standard method are quite similar. Therefore MLM1 can be used for solving problems.

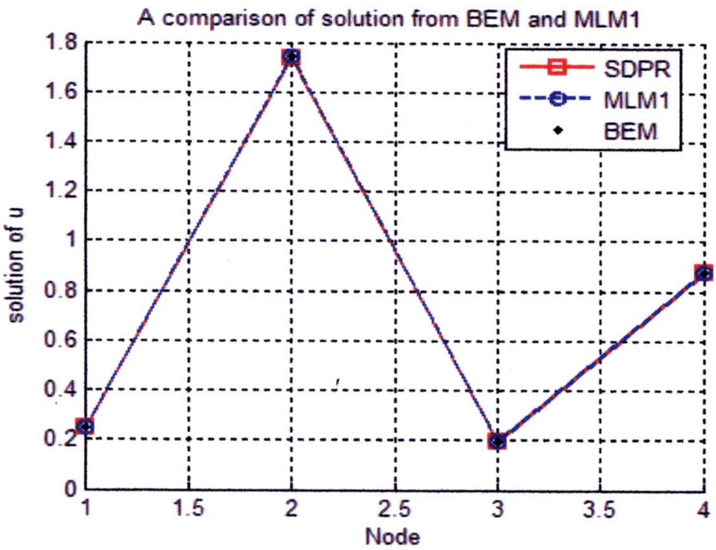


Figure 5.2.8 A comparative solution of Example 5.2.3

Example 5.2.4 Solve the Laplace equation $\nabla^2 u = 0$ for elliptic region R, shown in Figure 5.2.9, with the semi-axes 2 and 1 respectively, and the boundary condition $u = \dfrac{x^2 + y^2}{2}$ on curve. Because of the symmetry about both x and y axes, we shall consider the quarter region.

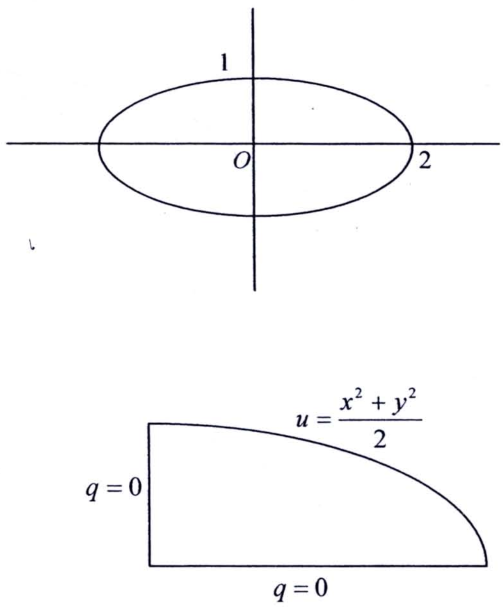


Figure 5.2.9 The quarter region for Example 5.2.4

Table 5.2.4 shows results obtained from both methods. Shape parameter $\varepsilon = 1.5$ is used for this example. These results are compared with SDPR. It can be seen that MLM1 solutions are similar to SDPR solutions. SDPR denotes benchmark solution in this example.

Table 5.2.4 Comparison between the numerical results obtained from MLM1 and SDPR, applied to Example 5.2.4

Internal point	**MLM1**	**SDPR**
(0,0)	0.9775	0.9506
(1,0.5)	1.0984	1.0755

Although we do not have the exact solution, from table 5.2.4, MLM1 and standard method are quite similar. Therefore MLM1 can be used for solving problems.

As it can be seen from these examples, the numerical results obtained from MLM1are better than those obtained from the BEM. Therefore, we can get more accurate results by using MLM1.

5.3 A comparison of results obtained from MLM1 and MLM2

In Example 5.1.1, the results of MLM1 and MLM2 are compared for several shape parameters ε and errors of these methods are shown as table 5.3.1, 5.3.2, respectively. Two cases are considered in Table 5.3.1. The first case compares results of both methods for the same shape parameter, it can be seen that results of MLM1 and MLM2 are satisfactory. In terms of accuracy, MLM1 results are more accurate than MLM2. The second case compares results of MLM1 and MLM2 for several shape parameters, it can be seen that the best results of MLM1 are obtained to shape parameter $\varepsilon = 2$ whereas MLM2 obtains best results to shape parameter $\varepsilon = 3$. However, results are slightly different for several shape parameters of each method.

Table 5.3.1 A comparative numerical results of MLM1 and MLM2 in Example 5.1.1, using several shape parameters

Internal point	MLM1				MLM2				Exact
	$\varepsilon = 1.5$	$\varepsilon = 2$	$\varepsilon = 3$	$\varepsilon = 4$	$\varepsilon = 1.5$	$\varepsilon = 2$	$\varepsilon = 3$	$\varepsilon = 4$	
(0.25,0.75)	0.4352	0.4315	0.4304	0.4301	0.4250	0.4297	0.4298	0.4297	0.4320
(0.50,0.75)	0.5407	0.5406	0.5406	0.5405	0.5407	0.5406	0.5406	0.5405	0.5405
(0.75,0.75)	0.4311	0.4326	0.4337	0.4339	0.4394	0.4345	0.4342	0.4343	0.4320
(0.50,0.50)	0.2529	0.2500	0.2500	0.2500	0.2501	0.2500	0.2500	0.2500	0.2489
(0.50,0.25)	0.0981	0.0954	0.0954	0.0954	0.0956	0.0954	0.0954	0.0954	0.0949

Table 5.3.2 A comparative errors of MLM1 and MLM2 in Example 5.1.1, using several shape parameters

Internal point	Error-MLM1				Error-MLM2			
	$\varepsilon=1.5$	$\varepsilon=2$	$\varepsilon=3$	$\varepsilon=4$	$\varepsilon=1.5$	$\varepsilon=2$	$\varepsilon=3$	$\varepsilon=4$
(0.25,0.75)	0.0032	0.0005	0.0016	0.0019	0.0070	0.0023	0.0022	0.0023
(0.50,0.75)	0.0002	0.0001	0.0001	0.0000	0.0002	0.0001	0.0001	0.0000
(0.75,0.75)	0.0009	0.0006	0.0017	0.0019	0.0074	0.0025	0.0022	0.0023
(0.50,0.50)	0.0029	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000
(0.50,0.25)	0.0027	0.0000	0.0000	0.0000	0.0002	0.0000	0.0000	0.0000

In addition, results of two methods are satisfactory if shape parameter ε is chosen in interval $[1.5,4]$.

In a following table, results of MLM2 are shown in Table 5.3.3 to several shape parameters and errors of this method are shown in Table 5.3.4. And Figure 5.3.1 shows a comparison of results of each shape parameter with the analytical solution.

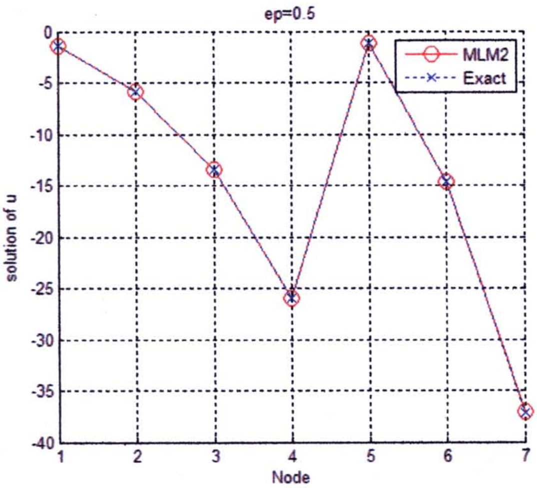
Table 5.3.3 Numerical results of MLM2 in Example 5.1.3, using several shape parameters

Internal point	$\varepsilon=0.5$	$\varepsilon=1$	$\varepsilon=2$	$\varepsilon=3$	$\varepsilon=4$	Exact
(0.5,1.0)	-1.3790	-1.3911	-1.5457	-2.1986	-2.6825	-1.3750
(0.5,2.0)	-5.8740	-5.9064	-6.2708	-6.5466	-6.8922	-5.8750
(0.5,3.0)	-13.3772	-13.4325	-13.2497	-12.448	-12.3638	-13.3750
(1.0,3.0)	-25.9880	-25.9053	-25.6859	-24.2497	-23.4325	-26.0000
(1.5,1.0)	-1.1797	-1.3565	-1.9488	-2.8420	-3.7780	-1.1250
(1.5,2.0)	-14.6046	-14.6531	-14.7294	-14.5592	-14.6610	-14.6250
(1.5,3.0)	-37.0985	-36.8239	-36.4511	-35.2946	-34.2252	-37.1250

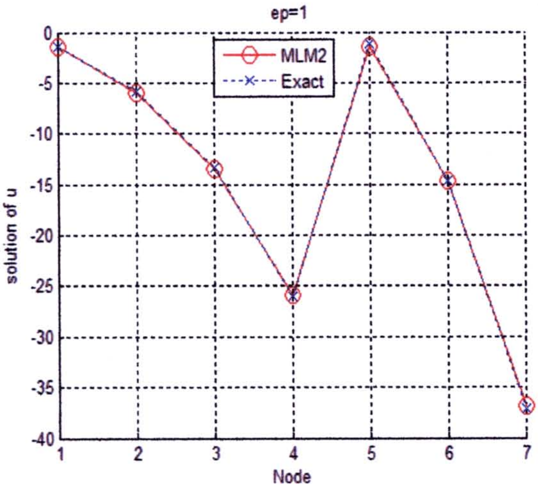
It can be seen from table 5.3.3 that large values of shape parameter ε will reduce accuracy of solution. Table 5.3.4, errors of solution obtained from MLM2 increase if shape parameters are large.

Table 5.3.4 Errors of MLM2 in Example 5.1.3, using several shape parameters

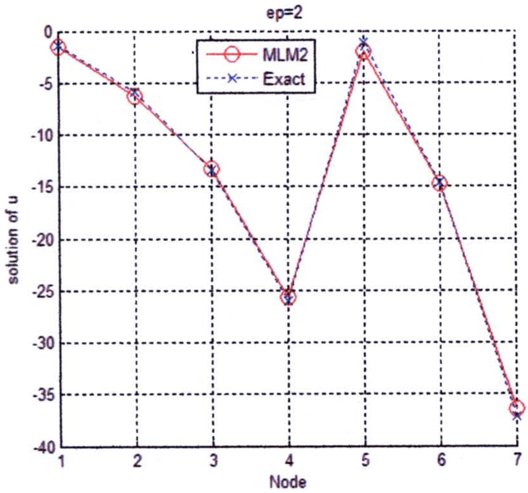
Internal point	$\varepsilon=0.5$	$\varepsilon=1$	$\varepsilon=2$	$\varepsilon=3$	$\varepsilon=4$
(0.5,1.0)	0.0040	0.0161	0.1707	0.8236	1.3075
(0.5,2.0)	0.0010	0.0314	0.3958	0.6716	1.0172
(0.5,3.0)	0.0022	0.0575	0.1253	0.9270	1.0112
(1.0,3.0)	0.0120	0.0947	0.3141	1.7503	2.5675
(1.5,1.0)	0.0547	0.2315	0.8238	1.7170	2.6530
(1.5,2.0)	0.0204	0.0281	0.1044	0.0658	0.0360
(1.5,3.0)	0.0265	0.3011	0.6739	1.8304	2.8998



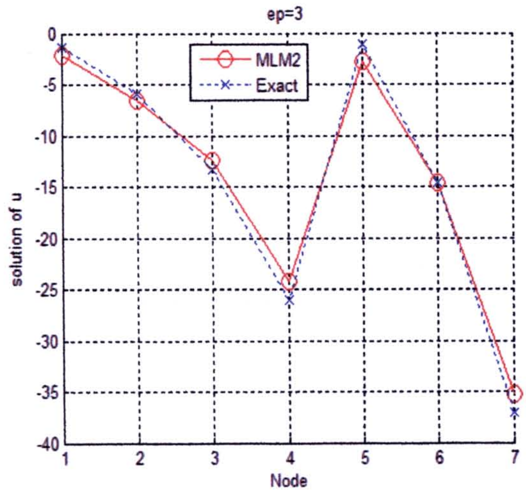
(a)



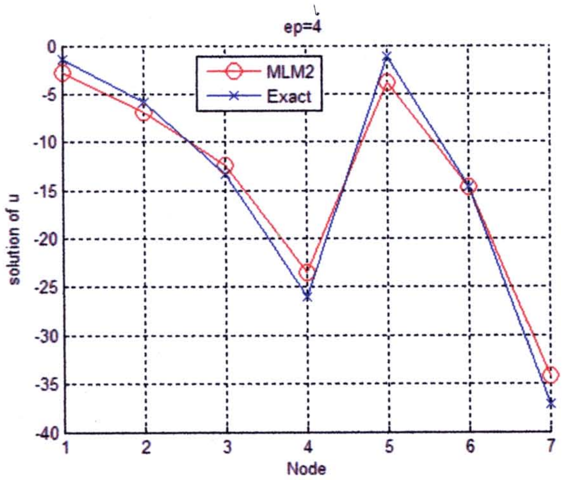
(b)



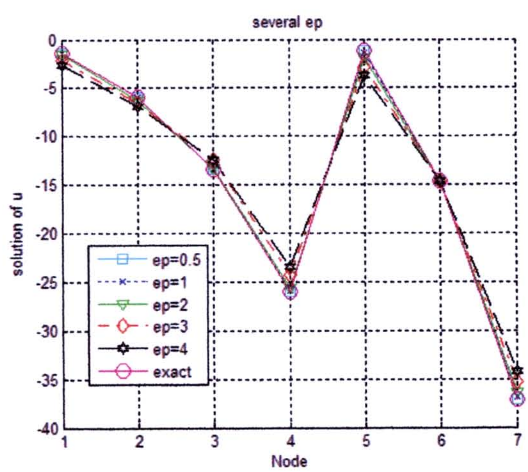
(c)



(d)



(e)



(f)

Figure 5.3.1 A variance of accuracy to any shape parameter for MLM2, ep denotes the shape parameters

According to Figure 5.3.1, it is interesting to note that results of MLM2 are accurate to small shape parameter. Therefore, shape parameters affect the accuracy of the final solutions.