CHAPTER IV
MESHLESS METHOD

Meshless methods are new numerical methods used to calculate in the engineering and science. Element creations are not required for these methods. Before finding the solution of problems, the numbers of nodes in the domain and on the boundary domain are defined.

In this chapter, we introduce two meshless methods; Kansa's method and Hermite–based method. The techniques can be applied to numerical solutions of Laplace equation. Two numerical methods will be a collocation approach based on radial basis functions.

4.1 Kansa's method

We consider a linear elliptic partial differential equation with boundary conditions, where $g(x)$ and $f(x)$ are known. We seek $u(x)$ from

$$Lu(x) = f(x), \qquad x \text{ in } \Omega \tag{4.1}$$

$$Mu(x) = g(x), \qquad x \text{ on } \partial\Omega \tag{4.2}$$

where $\Omega \in \mathbb{R}^s$, $\partial\Omega$ denotes the boundary of domain Ω, L and M are the linear elliptic partial differential operators and operating on the domain Ω and boundary domain $\partial\Omega$, respectively. For Kansa's method, it represents the approximate solution $\tilde{u}(x)$ by the interpolation (2.48), using an RBF expansion as the following

$$\tilde{u}(x) = \sum_{j=1}^{N} c_j \varphi\left(\left\| x - \xi_j \right\|\right) \tag{4.3}$$

We can see that N linear dependent equations are required for solving N unknowns of c_j. Substituting $\tilde{u}(x)$ into (4.1) and (4.2), we obtain the system of equations as

$$L\left(\sum_{j=1}^{N_I} c_j \varphi \left\| x - \xi_j \right\|\right) = \sum_{j=1}^{N_I} c_j L\varphi\left(\left\| x - \xi_j \right\|\right) = f(x_i), i = 1, \ldots, N_I \tag{4.4}$$

$$M\left(\sum_{j=N_I+1}^{N} c_j \varphi \left\| x - \xi_j \right\|\right) = \sum_{j=N_I+1}^{N} c_j M\varphi\left(\left\| x - \xi_j \right\|\right) = g(x_i), i = N_I + 1, \ldots, N \tag{4.5}$$

Above equations, we choose N collocation points on both domain Ω and boundary domain $\partial\Omega$, and divide it into N_I interior points and N_B boundary points $(N = N_I + N_B)$. Let $X = \{x_1, x_2, \ldots, x_N\}$ denotes a set of collocation points, $I = \{I_1, \ldots, I_{N_I}\}$ denotes a set of interior points and $B = \{B_{N_I+1}, \ldots, B_N\}$ denotes a set of boundary points.

The centers ξ_j used in (4.3) are chosen as collocation points. The previous substituting yields a system of linear algebraic equations which can be solved for seeking coefficient c by rewriting (4.4) in matrix form as

$$Ac = F \tag{4.6}$$

where
$$A = \begin{bmatrix} A_L \\ A_M \end{bmatrix} \tag{4.7}$$

$$(A_L)_{ij} = L\varphi(\|x - \xi_j\|)|_{x=x_i}, \quad x_i \in I, \xi_j \in X, \quad i = 1,2,\ldots,N_I, \quad j = N_I + 1,\ldots,N \tag{4.8}$$

$$(A_M)_{ij} = M\varphi(\|x - \xi_j\|)|_{x=x_i}, \quad x_i \in B, \xi_j \in X, \quad j = 1,2,\ldots,N, \quad i = N_I + 1,\ldots,N \tag{4.9}$$

$$F = \begin{bmatrix} f(x_i) \\ g(x_i) \end{bmatrix}, \quad f(x_i); x_i \in I, \quad i = 1,2,\ldots,N_I, \quad g(x_i); x_i \in B, \quad i = N_I + 1,\ldots,N. \tag{4.10}$$

Equation (4.6), the matrix A is non-singular. Therefore, the coefficient c are computed as

$$c = A^{-1}F \tag{4.11}$$

The matrix c is substituted into(4.3) and the approximate solution of $\tilde{u}(x)$ can be determined by

$$\tilde{u}(x) = \sum_{j=1}^{N} A^{-1}F(x_i)\varphi(\|x - \xi_j\|) \tag{4.12}$$

where
$$F = \begin{bmatrix} f(x_i) \\ g(x_i) \end{bmatrix} \begin{matrix} , \quad x_i \in I, \quad i = 1,2,\ldots,N_I \\ , \quad x_i \in B, \quad i = N_I + 1,\ldots,N \end{matrix} \tag{4.13}$$

and $\xi_j \in X, \quad j = 1,2,\ldots,N_I, N_I + 1,\ldots,N$.

4.2 Hermite-based method

This approach is based on the generalized Hemite interpolation method. Considering the PDEs with Dirichlet boundary condition can be expressed in the following form:

$$Lu(x) = f(x) \qquad x \in \Omega \tag{4.14}$$

Dirichlet boundary condition

$$u(x) = g(x) \qquad x \in \partial\Omega . \tag{4.15}$$

where function $f(x)$ and $g(x)$ are known, L denotes linear operator, Ω and $\partial\Omega$ denote domain and boundary domain, respectively.

The solution of (4.14) having Dirichlet boundary condition (4.15) can be approximated with a function $\tilde{u}(x)$ in the following form

$$\tilde{u}(x) = \sum_{j=1}^{N_I} c_j L^{\xi} \varphi(\|x-\xi\|)|_{\xi=\xi_j} + \sum_{j=N_I+1}^{N} c_j \varphi(\|x-\xi_j\|) \tag{4.16}$$

where N denotes the number of collocation points as mentioned in equation (4.4) and (4.5), N_I is the number of interior points of domain Ω, c_j are unknown coefficients and L^{ξ} denotes linear operator which acts on function as function of ξ. Substituting $\tilde{u}(x)$ into (4.14) yields the following system of equations

$$L\left(\sum_{j=1}^{N_I} c_j L^{\xi} \varphi(\|x-\xi\|)|_{\xi=\xi_j} + \sum_{j=N_I+1}^{N} c_j \varphi(\|x-\xi_j\|) \right) = f(x_i), \quad x_i \in I \,, \; i = 1,2,\dots,N_I$$

$$\sum_{j=1}^{N_I} c_j L L^{\xi} \varphi(\|x-\xi\|)|_{\xi=\xi_j} + \sum_{j=N_I+1}^{N} c_j L\varphi(\|x-\xi_j\|) = f(x_i), \quad x_i \in I \,, \; i = 1,2,\dots,N_I \tag{4.17}$$

$$\sum_{j=1}^{N_I} c_j L^{\xi} (\|x-\xi\|)|_{\xi=\xi_j} + \sum_{j=N_I+1}^{N} c_j \varphi(\|x-\xi_j\|) = g(x_i), \quad x_i \in B \,, \; i = N_I+1,\dots,N \tag{4.18}$$

This (4.17) and (4.18) can be written as

$$Ac = F \tag{4.19}$$

where
$$A = \begin{bmatrix} A_{LL^{\xi}} & A_L \\ A_{L^{\xi}} & A \end{bmatrix} \tag{4.20}$$

$$\left(A_{LL^{\xi}}\right)_{ij} = LL^{\xi} \varphi(\|x-\xi\|)|_{x=x_i,\xi=\xi_j}, \quad x_i, \xi_j \in I \,, \; i,j = 1,2,\dots,N_I \tag{4.21}$$

$$\left(A_L\right)_{ij} = L\varphi(\|x-\xi_j\|)|_{x=x_i}, \; x_i \in I, \xi_j \in B \,, \; i = 1,2,\dots,N_I \,, \; j = N_I+1,\dots,N \tag{4.22}$$

$$\left(A_{L^{\xi}}\right)_{ij} = L^{\xi}\varphi\left(\left\|x_i - \xi\right\|\right)|_{\xi=\xi_j}, \quad x_i \in B, \xi_j \in I \;\; j = 1,2,\ldots,N_I, \; i = N_I + 1,\ldots,N \qquad (4.23)$$

$$A_{ij} = \varphi\left(\left\|x_i - \xi_j\right\|\right), \quad x_i, \xi_j \in B, \; i,j = N_I + 1,\ldots,N \qquad (4.24)$$

The matrix A of (4.19) is a non-singular matrix with dimensions $N \times N$. c and F are vectors with dimension N. Therefore, the unknown coefficients c_j are determined and can be represented by inverting system (4.19)

$$c = A^{-1}F \qquad (4.25)$$

Substituting (4.25) into (4.16) can be expressed as

$$\tilde{u}(x) = \left(A^{-1}F\right)\left(\sum_{j=1}^{N_I} L^{\xi}\varphi\left(\left\|x-\xi\right\|\right)|_{\xi=\xi_j} + \sum_{j=N_I+1}^{N} \varphi\left(\left\|x-\xi_j\right\|\right)\right) \qquad (4.26)$$

where $F = \begin{bmatrix} f(x_i) \\ g(x_i) \end{bmatrix} \begin{matrix} , & x_i \in I & , & i = 1,2,\ldots,N_I \\ , & x_i \in B & , & i = N_I + 1,\ldots,N \end{matrix}$ and $\xi_j \in X$

This equation is used to approximate solution of PDEs for any interior points P of domain Ω.

The next chapter, both meshless method and boundary element method are used for solving the Laplace equation with boundary conditions. Their solutions are compared.