CHAPTER III

FORMULATION OF THE BOUNDARY ELEMENT METHOD

As described in chapter I, the boundary element method has many advantages. One of those is that the method generates smaller size of equation system matrix which is easier to be computed in the computing step. In order to give the readers more understanding of this advantage of the method, this chapter we explain the way to construct the equation matrix system. In this, we begin with explanation of the relation between the domain of the problem at hand and the formulas expressed in chapter II.

Let us consider the $j-\text{th}$ element of the discretized boundary having end point j and $j+1$, and length l_j as in Figure 3.1

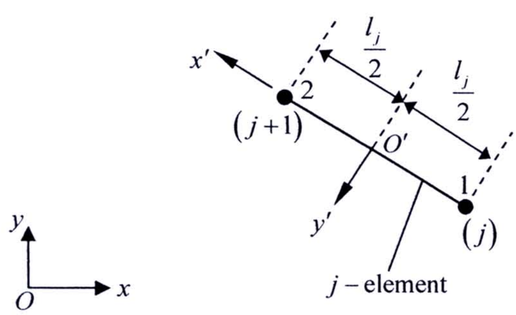

Figure 3.1 Global (Oxy) and local $(O'x'y')$ systems of axes for the $j-\text{th}$ element

In the local system of axes $O'x'y'$ its geometry is described by

$$x' = x', \quad \left(-\frac{l}{2} \le x' \le \frac{l}{2}\right) \tag{3.1}$$

$$y' = 0$$

whereas in the global system of axes Oxy by the following equation;

$$x = \frac{x_{j+1} + x_j}{2} + \left(\frac{x_{j+1} - x_j}{l_j}\right)x' \tag{3.2}$$

$$y = \frac{y_{j+1} + y_j}{2} + \left(\frac{y_{j+1} - y_j}{l_j} \right) x' \tag{3.3}$$

The positive direction of x' on the element is from point j towards point $j+1$, sine the nodes and extreme points of the discretized boundary are numbered in the counter clockwise. Expressions that map the global coordinates onto the integration interval $[-1,1]$ are obtained by introducing to Equations (3.2) and (3.3) the geometric relation

$$\xi = \frac{x'}{l_j/2} \tag{3.4}$$

where

$$l_j = \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2} \tag{3.5}$$

and then, the resulting coordinate transformation becomes

$$x = \frac{x_{j+1} + x_j}{2} + \frac{x_{j+1} - x_j}{2} \xi \tag{3.6}$$

$$y = \frac{y_{j+1} + y_j}{2} + \frac{y_{j+1} - y_j}{2} \xi \tag{3.7}$$

where $-1 \le \xi \le 1$.

Equation (3.6) and (3.7) may be rewritten as

$$x(\xi) = \frac{1}{2}(1-\xi) x_j + \frac{1}{2}(1+\xi) x_{j+1} \tag{3.8}$$

$$y(\xi) = \frac{1}{2}(1-\xi) y_j + \frac{1}{2}(1+\xi) y_{j+1} \tag{3.9}$$

The variation of the boundary quantity u (or $q = \partial u/\partial n$) is linear on the element. Hence, its distribution in the local system of the axes is given by the expression;

$$u = \frac{u^{j+1} + u^j}{2} + \left(\frac{u^{j+1} - u^j}{l_j} \right) x' \tag{3.10}$$

or

$$u = \frac{u^{j+1} + u^j}{2} + \left(\frac{u^{j+1} - u^j}{2} \right) \xi \tag{3.11}$$

or

$$u(\xi) = \frac{1}{2}(1-\xi) u^j + \frac{1}{2}(1+\xi) u^{j+1} \tag{3.12}$$

and

$$q(\xi) = \frac{1}{2}(1-\xi) q^j + \frac{1}{2}(1+\xi) q^{j+1} \tag{3.13}$$

Using local numbering for the nodes where j and $j+1$ are renamed to 1 and 2, respectively. Equations(3.8), (3.9), (3.12) and (3.13) may all be expressed through the following general equation.

$$f(\xi)=\phi_1(\xi)f_1+\phi_2(\xi)f_2 \tag{3.14}$$

where f_1 and f_2 are the values of the function $f(x)$ at the nodes 1 and 2, and $f(\xi)$ represents any of the functions $x(\xi)$, $y(\xi)$, $u(\xi)$ or $q(\xi)$. The functions $\phi_1(\xi)$ and $\phi_2(\xi)$ are given as

$$\phi_1=\frac{1}{2}(1-\xi) \tag{3.15}$$

$$\phi_2=\frac{1}{2}(1+\xi) \tag{3.16}$$

and they express the influence of the nodal values f_j and f_{j+1} on $f(\xi)$ for the linear element. They are the functions of linear interpolation and they are referred to as *linear shape functions*.

After discretizing the boundary into N elements. The boundary integral equation (2.47) becomes

$$C_i u_i=\sum_{j=1}^{N}\int_{\Gamma_j} u^* q\,ds-\sum_{j=1}^{N}\int_{\Gamma_j} u q^* ds \tag{3.17}$$

where $C_i=\dfrac{\alpha_i}{2\pi}$, $u^*=-\dfrac{1}{2\pi}\ln r$ and $q^*=\dfrac{\partial u^*}{\partial n}$.

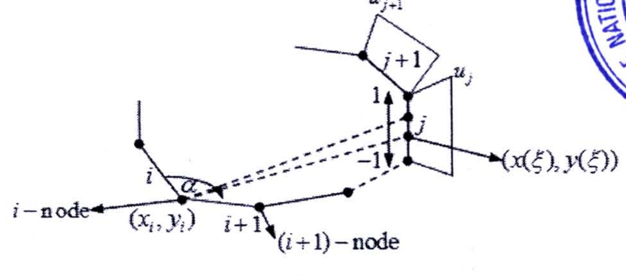

Figure 3.2 Nodal point location and relative distance for element discretization

We examine now the integrals over the $j-\text{th}$ element. Using the linear approximation of (3.14) for values u and q, the line integral appearing in the first summation of (3.17) may be written as

$$\int_{\Gamma_j} u^* q\,ds = \int_{-1}^{1} u^*\left[\phi_1(\xi)q_1 + \phi_2(\xi)q_2\right]\frac{l_j}{2}d\xi$$

$$= g_1^{ij}q_1 + g_2^{ij}q_2 \tag{3.18}$$

where

$$g_1^{(ij)} = \frac{l_j}{2}\int_{-1}^{1} u^*\phi_1(\xi)\,d\xi \tag{3.19}$$

$$g_2^{ij} = \frac{l_j}{2}\int_{-1}^{1} u^*\phi_2(\xi)\,d\xi \tag{3.20}$$

and

$$u^* = -\frac{1}{2\pi}\ln r \tag{3.21}$$

$$r = \sqrt{\left[x(\xi)-x_i\right]^2 + \left[y(\xi)-y_i\right]^2} \tag{3.22}$$

In a similar fashion, the integral appearing the second sum of (3.17) may be written as

$$\int_{\Gamma_j} u q^*\,ds = \int_{-1}^{1} q^*\left[\phi_1(\xi)u_1 + \phi_2(\xi)u_2\right]\frac{l_j}{2}d\xi$$

$$= h_1^{(ij)}u_1 + h_2^{(ij)}u_2 \tag{3.23}$$

where

$$h_1^{(ij)} = \frac{l_j}{2}\int_{-1}^{1} q^*\phi_1(\xi)\,d\xi \tag{3.24}$$

$$h_2^{ij} = \frac{l_j}{2}\int_{-1}^{1} q^*\phi_2(\xi)\,d\xi \tag{3.25}$$

and

$$q^* = \frac{\partial u^*}{\partial n} = -\frac{1}{2\pi}\frac{\cos\theta}{r}.$$

Substituting (3.18) and (3.23) back into (3.17), we obtain

$$C_i u_i + \sum_{j=1}^{N}\widehat{H}_{ij}u_j = \sum_{j=1}^{N} G_{ij}q_j \tag{3.26}$$

where $$\widehat{H}_{ij} = \begin{cases} h_1^{(ij)} + h_2^{(iN)} & \text{for } j = 1 \\ h_1^{(ij)} + h_2^{i(j-1)} & \text{for } j = 2,3,4,\ldots,N \end{cases}$$

$$G_{ij} = \begin{cases} g_1^{(ij)} + g_2^{(iN)} & \text{for } j = 1 \\ g_1^{(ij)} + g_2^{i(j-1)} & \text{for } j = 2,3,4,\ldots,N. \end{cases}$$

Let $\widehat{H} = \left[\widehat{H}_{ij}\right]_{N\times N}$, $G = \left[G_{ij}\right]_{N\times N}$, u and q be vectors of dimension N. Equation (3.26) may be written in a matrix form as

$$Hu = Gq \tag{3.27}$$

where $$H_{ij} = \begin{cases} \widehat{H}_{ij}, & i \neq j \\ \widehat{H}_{ij} + c_i, & i = j \end{cases}$$

We assume a mixed boundary condition. In this case the part Γ_1 of the boundary on which u is described and the part Γ_2 on which q is described, are discretized into N_1 and N_2 linear elements, respectively $(\Gamma_1 \cup \Gamma_2 = \Gamma, \Gamma_1 \cap \Gamma_2 \neq \phi, \text{ and } N_1 + N_2 = N)$. Before solving the system of equations, it is important to separate the unknown from known quantities of equation (3.27), which may be written after partitioning of the matrices in the following way

$$\begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3.28}$$

where u_1 and q_2 denote the prescribed quantities on Γ_1 and Γ_2, respectively, q_1 and u_2 denote the corresponding unknown ones. Carrying out the multiplications and moving all unknowns to the left hand side of the equation, we obtain the system as

$$Ax = B \tag{3.29}$$

where

$$A = \begin{bmatrix} H_2 & -G_1 \end{bmatrix}, \quad x = \begin{bmatrix} u_2 \\ q_1 \end{bmatrix}, \quad B = -H_1 u_1 + G_2 q_2$$

A is a square matrix with dimensions $N \times N$. B and x are vectors with dimension N. Therefore, knowing all the boundary quantities on Γ, the solution u can be computed at any point $P(x,y)$ in the domain Ω by equation (3.17) for $c_i = 1$ as the following

$$u_p = \sum_{j=1}^{N} \int_{\Gamma_j} u^* q\, ds - \sum_{j=1}^{N} \int_{\Gamma_j} u q^*\, ds \tag{3.30}$$

where $u^* = -\dfrac{1}{2\pi} \ln r$ and $q^* = \dfrac{\partial u^*}{\partial n}$.