



CHAPTER III

THE WEAK SOLUTION OF THE COMPOUND ULTRA-HYPERBOLIC BESSEL EQUATION

In this chapter, we study the weak solution of the compound ultra-hyperbolic Bessel equation. Moreover, such a solution is unique. This work was appeared in [10]. It was a jointed work between Srisombat and Nonlaopon.

3.1 Properties of Ultra-hyperbolic Bessel Kernel

Lemma 3.1.1 $R_{2k,c}^H(x)$ is a homogeneous distribution of order $(\alpha - n - 2|v|)$. In particular, it is a tempered distribution.

Proof. We need to show that $R_{\alpha,c}^H(x)$ satisfies the Euler equation

$$\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{\alpha,c}^H(x) = (\alpha - n - 2|v|) R_{\alpha,c}^H(x).$$

Now

$$\begin{aligned} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} R_{\alpha,c}^H(x) &= \frac{1}{K_{n,v}(\alpha)} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\frac{\alpha-n-2|v|}{2}} \\ &= \frac{\alpha - n - 2|v|}{K_{n,v}(\alpha)} (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\frac{\alpha-n-2|v|-2}{2}} \\ &\quad \times (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2) \\ &= \frac{1}{K_{n,v}(\alpha)} (\alpha - n - 2|v|) \\ &\quad \times (c^2(x_1^2 + \cdots + x_p^2) - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\frac{\alpha-n-2|v|}{2}} \\ &= \frac{(\alpha - n - 2|v|) V^{\frac{\alpha-n-2|v|}{2}}}{K_{n,v}(\alpha)} \\ &= (\alpha - n - 2|v|) R_{\alpha,c}^H(x). \end{aligned}$$



Hence $R_{\alpha,c}^H(x)$ is a homogeneous distribution of order $(\alpha - n - 2|v|)$. Donoghue [3] prove that every homogeneous distribution is a tempered distribution. So $R_{\alpha,c}^H(x)$ is a tempered distribution. This completes the proof. \square

Lemma 3.1.2 *Given the equations*

$$\square_{B,c}^k u(x) = \delta(x), \quad (3.1.1)$$

where $\square_{B,c}^k$ is defined by (1.1.6) and $x \in \mathbb{R}_n^+$, then we obtain $u(x) = R_{2k,c}^H(x)$ as an unique elementary solution of (3.1.1), where $R_{2k,c}^H(x)$ is defined by (2.6.2).

The proof of this lemma is given in [8].

Lemma 3.1.3 *(The B-convolutions of tempered distributions)*

(a) $(\square_{B,c}^k \delta(x)) * u(x) = \square_{B,c}^k u(x)$, where u is any tempered distribution.

(b) Let $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ be defined by (2.6.2),

then $R_{2k,c}^H(x) * R_{2m,c}^H(x)$ exists and is a tempered distribution.

(c) Let $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ be defined by (2.6.2),

then $R_{2k,c}^H(x) * R_{2m,c}^H(x) = R_{2k+2m,c}^H(x)$, where k and m are nonnegative integer.

(d) Let $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ be defined by (2.6.2). If

$R_{2k,c}^H(x) * R_{2m,c}^H(x) = \delta(x)$, then $R_{2k,c}^H(x)$ is an inverse of $R_{2m,c}^H(x)$ in the

B -convolution algebra, denoted by $R_{2k,c}^H(x) = R_{2m,c}^{H*-1}(x)$, moreover, $R_{2m,c}^{H*-1}(x)$ is unique.

Proof.

(a) First, we consider the case $k = 1$, now

$$\square_{B,c} \delta(x) = \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \delta(x)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \delta(x)}{\partial x_i} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \delta(x)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \delta(x)}{\partial x_j} \right)$$

$p + q = n$ and let $\varphi(x)$ be a testing function in the Schwart space S .

By the definition of B -convolution, we have

$$\begin{aligned} \langle \square_{B,c} \delta(x) * u(x), \varphi(x) \rangle &= \langle u(x), \langle \square_{B,c} \delta(x), \varphi(x+y) \rangle \rangle \\ &= \left\langle u(x), \left\langle \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \delta(y)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \delta(y)}{\partial x_i} \right) \right. \right. \\ &\quad \left. \left. - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \delta(y)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \delta(y)}{\partial x_j} \right), \varphi(x+y) \right\rangle \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle u(x), \left\langle \delta(y), \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \varphi(x+y)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \varphi(x+y)}{\partial x_i} \right) \right. \right. \\
&\quad \left. \left. - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \varphi(x+y)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \varphi(x+y)}{\partial x_j} \right) \right\rangle \right\rangle \\
&= \left\langle u(x), \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 \varphi(x)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial \varphi(x)}{\partial x_i} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 \varphi(x)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial \varphi(x)}{\partial x_j} \right) \right\rangle \\
&= \left\langle \frac{1}{c^2} \left(\sum_{i=1}^p \frac{\partial^2 u(x)}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial u(x)}{\partial x_i} \right) - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2 u(x)}{\partial x_j^2} + \frac{2v_j}{x_j} \frac{\partial u(x)}{\partial x_j} \right), \varphi(x) \right\rangle \\
&= \langle \square_{B,c} u(x), \varphi(x) \rangle. \tag{3.1.2}
\end{aligned}$$

It follows that

$$\square_{B,c} \delta(x) * u(x) = \square_{B,c} u(x).$$

Similarly for any k , we can show that

$$\square_{B,c}^k \delta(x) * u(x) = \square_{B,c}^k u(x).$$

(b) Since $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ are tempered distribution by Lemma 3.1.1, choose $\text{supp } R_{2k,c}^H(x) = K \subset \bar{\Gamma}_+$, where K is a compact set and $\bar{\Gamma}_+$ designates of Γ_+ closure. Then $R_{2k,c}^H(x)$ is a tempered distribution with compact support. By Donoghue [3], $R_{2k,c}^H(x) * R_{2m,c}^H(x)$ exists and is a tempered distribution.

(c) From equation $\square_{B,c}^{k+m} u(x) = \delta(x)$, by Lemma 3.1.2 we obtain $u(x) = R_{2k+2m,c}^H(x)$. For any m is a nonnegative integer, we write

$$\square_{B,c}^{k+m} u(x) = \square_{B,c}^k \square_{B,c}^m u(x) = \delta(x),$$

then by Lemma 3.1.2, we have the following equality

$$\square_{B,c}^m u(x) = R_{2k,c}^H(x).$$

B -convolving both sides by $R_{2m,c}^H(x)$, we obtain

$$R_{2m,c}^H(x) * \square_{B,c}^m u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x),$$

or

$$\square_{B,c}^m R_{2m,c}^H(x) * u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x).$$

Then from Lemma 3.1.2, we have the following equality

$$\delta(x) * u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x).$$

It follows that

$$u(x) = R_{2k,c}^H(x) * R_{2m,c}^H(x).$$

From the fact that $u(x) = R_{2k+2m,c}^H(x)$, we obtain

$$R_{2k,c}^H(x) * R_{2m,c}^H(x) = R_{2k+2m,c}^H(x).$$

- (d) Since $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ are tempered distributions with compact supports, thus $R_{2k,c}^H(x)$ and $R_{2m,c}^H(x)$ are the elements of space of B -convolution algebra u' of distribution. Now $R_{2k,c}^H(x) * R_{2m,c}^H(x) = \delta(x)$ then by Zemanian [14], show that $R_{2k,c}^H(x) = R_{2m,c}^{H*-1}(x)$ is an unique inverse. \square

Lemma 3.1.4 Let $R_{2k,c}^H(x)$ and $K_{n,v}(2k)$ be defined by (2.6.2) and (2.6.3), respectively. Then

- (a) $K_{n,v}(2k+2) = 2k(2k+2-n-2|v|)K_{n,v}(2k)$,
- (b) $\square_{B,c}^k R_{2m,c}^H(x) = R_{2m-2k,c}^H(x)$, where k and m are nonnegative integer,
- (c) $R_{-2k,c}^H(x) = \square_{B,c}^k \delta(x)$, where k is a nonnegative integer.

Proof.

- (a) From (2.6.3), we have

$$\begin{aligned} K_{n,v}(2k+2) &= \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+2k+2-n-2|v|}{2}\right) \Gamma\left(\frac{1-2k-2}{2}\right) \Gamma(2k+2)}{\Gamma\left(\frac{2+2k+2-p-2|v|}{2}\right) \Gamma\left(\frac{p+2|v|-2k-2}{2}\right)} \\ &= \frac{\pi^{\frac{n+2|v|-1}{2}} \frac{(2k+2-n-2|v|)}{2} \Gamma\left(\frac{2k+2-n-2|v|}{2}\right)}{\frac{2k+2-p-2|v|}{2} \Gamma\left(\frac{2k+2-p-2|v|}{2}\right)} \\ &\quad \cdot \frac{\left(\frac{-2}{1+2k}\right) \Gamma\left(\frac{1-2k}{2}\right) (2k)(2k+1) \Gamma(2k)}{\frac{2}{p+2|v|-2k-2} \Gamma\left(\frac{p+2|v|-2k}{2}\right)} \\ &= 2k(2k+2-n-2|v|)K_{n,v}(2k). \end{aligned}$$



(b) By Lemma 3.1.3(c), we have

$$\begin{aligned}\delta(x) * R_{2m,c}^H(x) &= R_{2k,c}^H(x) * R_{2m-2k,c}^H(x) \\ \square_{B,c}^k R_{2k,c}^H(x) * R_{2m,c}^H(x) &= R_{2k,c}^H(x) * R_{2m-2k,c}^H(x) \\ R_{2k,c}^H(x) * \square_{B,c}^k R_{2m,c}^H(x) &= R_{2k,c}^H(x) * R_{2m-2k,c}^H(x),\end{aligned}$$

and so,

$$\square_{B,c}^k R_{2m,c}^H(x) = R_{2m-2k,c}^H(x).$$

(c) For $m = k$, by Lemma 3.1.4(b), we have

$$\square_{B,c}^m R_{2m,c}^H(x) = R_{0,c}^H(x), \quad R_{0,c}^H(x) = \delta(x).$$

For $m = 0$, by Lemma 3.1.4(b), we have

$$\square_{B,c}^k R_{0,c}^H(x) = R_{-2k,c}^H(x) \quad \text{or} \quad \square_{B,c}^k \delta(x) = R_{-2k,c}^H(x).$$

□

3.2 Main Result

Theorem 3.2.1 *Given the compound equation related to the ultra-hyperbolic Bessel operator of the form*

$$\sum_{r=0}^m C_r \square_{B,c}^r u(x) = f(x), \quad (3.2.1)$$

where $\square_{B,c}^r$ is the ultra-hyperbolic Bessel operator iterated r -times defined by (1.1.6), f is a given generalized function, u is an unknown function, $x \in \mathbb{R}_n^+$, n is odd and C_r is a constant. Then (3.2.1) has an unique weak solution

$$u(x) = f(x) * R_{2m,c}^H(x) * (C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x))^{*-1}, \quad (3.2.2)$$

where

$$\begin{aligned}w(x) &= C_{m-1} + C_{m-2} \frac{V}{2(4-n-2|v|)} + C_{m-3} \frac{V^2}{2 \cdot 4(4-n-2|v|)(6-n-2|v|)} + \dots \\ &+ C_0 \frac{V^{m-1}}{2 \cdot 4 \cdot 6 \cdots 2(m-1)(4-n-2|v|)(6-n-2|v|) \cdots (2m-n-2|v|)}\end{aligned} \quad (3.2.3)$$

and V defined by (2.6.1) and $(C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x))^{*-1}$ is an inverse of $C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x)$.

Proof. By Lemma 3.1.3(a), equation (3.2.1) can be written as

$$(C_m \square_{B,c}^m \delta(x) + C_{m-1} \square_{B,c}^{m-1} \delta(x) + \cdots + C_1 \square_{B,c} \delta(x) + C_0 \delta(x)) * u(x) = f(x).$$

B -convolving both sides by $R_{2m,c}^H(x)$, defined by (2.6.2), we obtain

$$\begin{aligned} & [C_m \square_{B,c}^m R_{2m,c}^H(x) + C_{m-1} \square_{B,c}^{m-1} R_{2m,c}^H(x) + \cdots + C_1 \square_{B,c} R_{2m,c}^H(x) + C_0 R_{2m,c}^H(x)] * u(x) \\ &= f(x) * R_{2m,c}^H(x). \end{aligned}$$

By Lemma 3.1.2 and Lemma 3.1.4(b), we obtain

$$\begin{aligned} & [C_m \delta(x) + C_{m-1} R_{2,c}^H(x) + C_{m-2} R_{4,c}^H(x) + \cdots + C_1 R_{2(m-1),c}^H(x) + C_0 R_{2m,c}^H(x)] * u(x) \\ &= f(x) * R_{2m,c}^H(x). \end{aligned} \tag{3.2.4}$$

By Lemma 3.1.4(a), we obtain

$$R_{4,c}^H(x) = \frac{V^{\frac{4-n-2|v|}{2}}}{K_n(4)} = R_{2,c}^H(x) \cdot \frac{V}{2(4-n-2|v|)K_n(2)}.$$

Similarly,

$$\begin{aligned} R_{6,c}^H(x) &= R_{2,c}^H(x) \cdot \frac{V^2}{2 \cdot 4(4-n-2|v|)(6-n-2|v|)} \\ R_{8,c}^H(x) &= R_{2,c}^H(x) \cdot \frac{V^3}{2 \cdot 4 \cdot 6(4-n-2|v|)(6-n-2|v|)(8-n-2|v|)} \\ &\vdots \\ R_{2m,c}^H(x) &= R_{2,c}^H(x) \cdot \frac{V^{m-1}}{2 \cdot 4 \cdot 6 \cdots 2(m-1)(4-n-2|v|)(6-n-2|v|)} \\ &\quad \frac{1}{(8-n-2|v|) \cdots (2m-n-2|v|)}. \end{aligned}$$

Thus we obtain the function $w(x)$ of (3.2.3). Now $w(x)$ is continuous and infinitely differentiable in classical sense for n is odd. Since $R_{2,c}^H(x)$ is a tempered distribution with compact support, hence $w(x) R_{2,c}^H(x)$ also is tempered distribution with compact support and so $C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x)$. By Lemma 3.1.3(d), $C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x)$ has an unique inverse denoted by

$$(C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x))^{*-1}.$$

Now (3.2.4) can be written as

$$(C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x)) * u(x) = f(x) * R_{2m,c}^H(x), \text{ where } R_{0,c}^H(x) = \delta(x).$$

B-convolving both sides by $(C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x))^{*-1}$, we have

$$u(x) = f(x) * R_{2m,c}^H(x) * (C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x))^{*-1}.$$

Since $R_{2m,c}^H(x)$ is an unique by Lemma 3.1.2 and $(C_m R_{0,c}^H(x) + w(x) R_{2,c}^H(x))^{*-1}$ also is an unique by Lemma 3.1.3(d), it follows that $u(x)$ is an unique weak solution of (3.2.1) with odd-dimensional n . This completes the proof. \square