CHAPTER II

BASIC CONCEPTS AND PRELIMINARIES

In this chapter, we studied some properties of the test function, the distribution, the gamma function and the elementary solutions of the partial differential operators which will be used in later chapters.

2.1 Test Functions

Let $\mathbb{R}^n$ be the n-dimensional space in which we have a Cartesian system of coordinates such that a point P is denoted by $x = (x_1, x_2, \ldots, x_n)$ and the distance r, of P from the origin, is $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Let k be a n-tuple of nonnegative integer, $k = (k_1, k_2, \ldots, k_n)$, the so-called *multi-index* of order n; then we define

$$|k| = k_1 + k_2 + \cdots + k_n, \quad x^k = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

and

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = \frac{\partial^{k_1+k_2+\cdots+k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}} = D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n},$$

where $D_j = \partial/\partial x_j, j = 1, 2, \ldots, n$. For the one-dimensional case, D^k reduces to d/dx. Furthermore, if any component of k is zero, the differentiation with respect to the corresponding variable is omitted.

Example 2.1.1 In $\mathbb{R}^3$, with $k = (3, 0, 4)$, we have

$$D^k = \partial^7/\partial x_1^3 \partial x_3^4 = D_1^3 D_3^4.$$

Definition 2.1.2 A function $f(x)$ is *locally integrable* in $\mathbb{R}^n$ if $\int_R |f(x)| dx$ exists for every bounded region R in $\mathbb{R}^n$. A function $f(x)$ is *locally integrable on a hypersurface* in $\mathbb{R}^n$ if $\int_S |f(x)| dS$ exists for every bounded region S in $\mathbb{R}^{n-1}$.

Definition 2.1.3 The *support* of a function $f(x)$ is the closure of the set of all points x such that $f(x) \neq 0$. We shall denote the support of f by supp f.

Example 2.1.4 For $f(x) = \sin x, x \in \mathbb{R}$, the support of $f(x)$ consists of the whole real line, even though $\sin x$ vanishes at $x = n\pi$.

Definition 2.1.5 ([7]). If supp f is a bounded set, then f is said to have a *compact support.*

Definition 2.1.6 The space $\mathscr{D}$ is a linear space consist of all real-valued functions $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$, such that the following hold:

(1) $\phi(x)$ is an infinitely differentiable function defined at every point of $\mathbb{R}^n$. This means that $D^k\phi$ exists for all multiindices k. Such a function is also called a C^∞ function.

(2) There exists a number A such that $\phi(x)$ vanishes for $r > A$. This means that $\phi(x)$ has a compact support.

Then $\phi(x)$ is called a *test function.*

Example 2.1.7 The support of the function

$$f(x) = \begin{cases} 0, & \text{for } -\infty < x \leq -1, \\ x+1, & \text{for } -1 < x < 0, \\ 1-x, & \text{for } 0 \leq x < 1, \\ 0, & \text{for } 1 \leq x < \infty \end{cases}$$

is $[-1, 1]$, which is compact.

Example 2.1.8 The prototype of a test function belonging to $\mathscr{D}$ is

$$\phi(x,a) = \begin{cases} \exp\left(-\frac{a^2}{a^2-r^2}\right), & \text{for } r < a, \\ 0, & \text{for } r \geq a, \end{cases} \tag{2.1.1}$$

for a is a constant and $r = |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Its support is clearly $r \leq a$.

In particular, if we consider in $\mathbb{R}$ and by taking $a = 1$, then (2.1.1) reduces to

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & \text{for } x \in (-1,1), \\ 0, & \text{for } x \in (-\infty,-1] \cup [1,\infty), \end{cases} \tag{2.1.2}$$

and the support of $\phi(x)$ is [-1, 1].

The following properties of the test functions are evident.

(1) If ϕ_1 and ϕ_2 are in $\mathscr{D}$, then so is $c_1\phi_1 + c_2\phi_2$, where c_1 and c_2 are real numbers. Thus $\mathscr{D}$ is a linear space.

(2) If $\phi \in \mathscr{D}$, then so is $D^k\phi$.

(3) For a C^∞ function $f(x)$ and $\phi \in \mathscr{D}, f\phi \in \mathscr{D}$.

(4) If $\phi(x_1, x_2, \ldots, x_m)$ is an m-dimensional test function and $\psi(x_{m+1}, x_{m+2}, \ldots, x_n)$ is an $(n-m)$-dimensional test function, then $\phi\psi$ is an n-dimensional test function in the variables $x_1, x_2, \ldots, x_n$.

Definition 2.1.9 A sequence $\{\phi_m\}, m = 1, 2, \ldots$ where $\phi_m \in \mathscr{D}$, *converges* to ϕ_0 if the following two conditions are satisfied:

(1) All ϕ_m as well as ϕ_0 vanish outside a common region.

(2) $D^k\phi_m \to D^k\phi_0$ uniformly over $\mathbb{R}^n$ as $m \to \infty$ for all multiindices k.

It is not difficult to show that $\phi_0 \in \mathscr{D}$ and hence that $\mathscr{D}$ is closed (or is complete) with respect to this definition of convergence. For the special case $\phi_0 = 0$, the sequence $\{\phi_m\}$ is called a *null sequence.*

2.2 Distributions

Definition 2.2.1 A *linear functional* f on the space $\mathscr{D}$ of test functions is an operation (or a rule) by which we assign to every test functions $\phi(x)$ a real number denoted $\langle f, \phi \rangle$, such that

$$\langle f, c_1\phi_1 + c_2\phi_2 \rangle = c_1 \langle f, \phi_1 \rangle + c_2 \langle f, \phi_2 \rangle, \tag{2.2.1}$$

for arbitrary test functions ϕ_1 and ϕ_2 and real numbers c_1 and c_2.

Definition 2.2.2 A linear functional f on $\mathscr{D}$ is *continuous* if and only if the sequence of numbers $\langle f, \phi_m \rangle$ converges to $\langle f, \phi \rangle$ when the sequence of test functions $\{\phi_m\}$ converges to the test function ϕ. Thus

$$\lim_{m\to\infty} \langle f, \phi_m \rangle = \left\langle f, \lim_{m\to\infty} \phi_m \right\rangle.$$

We now have all the tools for defining the concept of distributions.

Definition 2.2.3 A continuous linear functional on the space $\mathscr{D}$ of test functions is called a *distribution*. The space of all distributions on $\mathscr{D}$ is denoted by $\mathscr{D}'$.

The set of distributions that are most useful are those generated by locally integrable functions. Indeed, every locally integrable $f(x)$ generates a distribution through the formula

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx. \tag{2.2.2}$$

Linearity of this functional is obvious. To prove its continuity, observe that

$$|\langle f, \phi \rangle| \leq \max_{x \in \text{ supp } \phi} |\phi(x)| \int_{\text{supp } \phi} |f(x)|dx < \infty.$$

Thus, if the sequence $\{\phi_m\}$ converges to zero, then so does $\langle f, \phi_m \rangle$. Hence, it is continuous.

Definition 2.2.4 Distributions defined by (2.2.2) are called *regular*. All other distributions are called *singular*. However, we may use (2.2.2) symbolically for a singular distribution also.

Example 2.2.5 The Heaviside distribution in $\mathbb{R}^n$ is $\langle H_R, \phi \rangle = \int_R \phi(x)dx$, where

$$H_R(x) = \begin{cases} 1, & \text{for } x \in \mathbb{R}, \\ 0, & \text{for } x \notin \mathbb{R}. \end{cases} \tag{2.2.3}$$

For $\mathbb{R}$, (2.2.3) becomes

$$\langle H, \phi \rangle = \int_0^\infty \phi(x)dx. \tag{2.2.4}$$

Since $H(x)$ is a piecewise continuous function, this is a regular distribution.

In physical problems, one often encounters idealized concepts such as a force concentrated at a point ξ or an impulsive force that acts instantaneously. These forces are described by the Dirac-delta function $\delta(x-\xi)$, which has several significant properties:

$$\delta(x-\xi) = 0, \quad x \neq \xi, \tag{2.2.5}$$

$$\int_a^b \delta(x-\xi)dx = \begin{cases} 0, & \text{for} \quad a, b < \xi \quad \text{or} \quad \xi < a, b, \\ 1, & \text{for} \quad a \leq \xi \leq b, \end{cases} \tag{2.2.6}$$

and

$$\int_{-\infty}^{\infty} \delta(x-\xi)dx = 1. \tag{2.2.7}$$

Equation (2.2.7) is a special case of the general formula

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(x)dx = f(\xi), \tag{2.2.8}$$

where $f(x)$ is a sufficiently smooth function. Relation (2.2.8) is called the *shifting property* or the *reproducing property* of the delta function, and (2.2.7) is obtained from it by putting $f(x) = 1$.

Example 2.2.6 The Dirac-delta distribution in $\mathbb{R}^n$ is

$$\langle \delta(x-\xi), \phi(x) \rangle = \phi(\xi), \tag{2.2.9}$$

for ξ is a fixed point in $\mathbb{R}^n$. Linearity of this functional follows from the relation

$$\begin{aligned} \langle \delta(x-\xi), c_1\phi_1 + c_2\phi_2 \rangle &= c_1\phi_1(\xi) + c_2\phi_2(\xi) \\ &= c_1 \langle \delta(x-\xi), \phi_1 \rangle + c_2 \langle \delta(x-\xi), \phi_2 \rangle, \end{aligned}$$

where c_1 and c_2 are arbitrary real constants.

To prove continuity, we observe that $\lim_{m\to\infty} \langle \delta(x-\xi), \phi_m \rangle = \lim_{m\to\infty} \phi_m(\xi)$. However, if $\phi_m(x) \to 0$, then $\phi_m(\xi) \to 0$, and have continuity.

Thus the delta function is a distribution. We observed earlier that the delta function is not locally integrable. This distribution is therefore singular distribution.

Definition 2.2.7 The *product* of a distribution t and an infinitely differentiable function f is defined by

$$\langle ft, \phi \rangle = \langle t, f\phi \rangle, \tag{2.2.10}$$

where ϕ and $f\phi$ are element of $\mathscr{D}$.

Example 2.2.8 For an infinitely differentiable function f, $\phi \in \mathscr{D}$,

$$\langle f\delta, \phi \rangle = \langle \delta, f\phi \rangle = f(0)\phi(0) = f(0)\langle \delta, \phi \rangle = \langle f(0)\delta, \phi \rangle$$

or

$$f(x)\delta(x) = f(0)\delta(x). \tag{2.2.11}$$

It follows that in this special case it is sufficient for the function $f(x)$ to be continuous at the origin. More generally,

$$f(x)\delta(x-\xi) = f(0)\delta(x-\xi). \tag{2.2.12}$$

Definition 2.2.9 A distribution E is said to be the *fundamental solution* for the differential operator L if

$$LE = \delta, \tag{2.2.13}$$

where δ is the Dirac-delta distribution.

2.3 Gamma Function

Definition 2.3.1 The *gamma function* is denoted by Γ and is defined by

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \tag{2.3.1}$$

where z is a complex number with $Re\ z > 0$

Example 2.3.2 Show that $\Gamma(1) = 1$.

Proof. By definition 2.3.1, we obtain

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}dt \\ &= \lim_{a\to\infty}(-e^{-t}|_0^a) \\ &= 1.\end{aligned}$$

□

Proposition 2.3.3 ([1]) Let z be a complex number. Then

(1) $z\Gamma(z) = \Gamma(z+1), \quad z \neq 0, -1, -2, \ldots$

(2) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}, \quad z \neq 0, \pm 1, \pm 2, \ldots$

(3) $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \quad z \neq 0, -1, -2, \ldots.$

2.4 The Convolution

Definition 2.4.1 The *convolution* $f * g$ of two functions $f(x)$ and $g(x)$, both in $\mathbb{R}^n$, is defined as

$$f(x) * g(x) = \int_{\mathbb{R}^n} f(t)g(x-t)dt. \tag{2.4.1}$$

Example 2.4.2 Let $f(x) = e^{-ax}H(x)$ and $g(x) = e^{-bx}H(x)$ be functions in $\mathbb{R}$. Find the convolution of $f(x)$ and $g(x)$, where $H(x)$ is defined by (2.2.3), which $R = (0, \infty)$. Since

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt,$$

we obtain

$$\begin{aligned} f(x) * g(x) &= \int_{-\infty}^{\infty} f(t)g(x-t)dt \\ &= \int_0^x e^{-at}e^{-b(x-t)}dt \\ &= e^{-bx}\int_0^x e^{-(a-b)t}dt \\ &= e^{-bx}\left(\frac{e^{-(a-b)t}}{-(a-b)}\right)\Bigg|_0^x, \qquad \text{for} \quad x > 0 \\ &= \frac{1}{b-a}\left(e^{-ax} - e^{-bx}\right)H(x). \end{aligned}$$

Example 2.4.3 Let $f, g : [0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & \text{for} \quad 0 \leq x < 1, \\ 0, & \text{for} \quad x \geq 1, \end{cases}$$

and

$$g(x) = e^x, \qquad \text{for} \quad x \geq 0.$$

Thus

$$f(x) * g(x) = \int_0^x (1)e^{x-t}dt$$
$$= e^x - 1, \qquad \text{for} \quad 0 \le x < 1,$$

and

$$f(x) * g(x) = \int_0^1 (1)e^{x-t}dt + \int_1^x (0)e^{x-t}dt$$
$$= e^x - e^{x-1}, \quad \text{for} \quad x \ge 1.$$

Therefore, the convolution $f * g$ is given by

$$f(x) * g(x) = \begin{cases} e^x - 1, & \text{for} \quad 0 \le x < 1, \\ e^x - e^{x-1}, & \text{for} \quad x \ge 1. \end{cases}$$

2.5 The B-convolution

Denoted T^y by the generalized shift operator acting according to the law [6]:

$$T_x^y\varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{x_1^2 + y_1^2 - 2x_1y_1\cos\theta_1}, \dots, \sqrt{x_n^2 + y_n^2 - 2x_ny_n\cos\theta_n}\right)$$
$$\times \left(\prod_{i=1}^n \sin^{2v_i-1}\theta_i\right) d\theta_1 \cdots d\theta_n,$$

where $x, y \in \mathbb{R}_n^+, C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [6]:

$$\frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy},$$
$$U(x,0) = f(x),$$
$$U_y(x,0) = 0.$$

The convolution operator determined by the T^y is as follows:

$$(f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y)T_x^y\varphi(x)\left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \qquad (2.5.1)$$

Convolution (2.5.1) known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x > 0$ and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy.$$

(e) $(f * g)(x) = (g * f)(x)$.

2.6 The Ultra-hyperbolic Kernel

Definition 2.6.1 Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional space $\mathbb{R}^n$,

$$V = c^2(x_1^2 + x_2^2 + \cdots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.6.1}$$

where $p + q = n$, the interior of forward cone is defined by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, x_2 > 0, ..., x_n > 0 \text{ and} V > 0\}$. For any complex number α, define

$$R_{\alpha,c}^H(x) = \begin{cases} \frac{V^{\frac{\alpha-n-2|v|}{2}}}{K_{n,v}(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.6.2}$$

where

$$K_{n,v}(\alpha) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+\alpha-2|v|}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p-2|v|}{2}\right) \Gamma\left(\frac{p+2|v|-\alpha}{2}\right)}. \tag{2.6.3}$$