CHAPTER I

INTRODUCTION

In mathematics, a weak solution (also called a generalized solution) to an ordinary or partial differential equation is a function for which the derivatives appearing in the equation may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. One of the most important is based on the notion of distributions. The classification of classical and generalized solutions is valid in the present case as well. However, there is one important difference. In the case of the ordinary differential equation $Lu = 0$ with constant coefficients, every solution is the classical solution. The matter is quite different for partial differential equations. The solutions in a similar situation may now include generalized functions. For instance, $\partial u/\partial x_1 = 0$, in $\mathbb{R}^2$, has among its solutions the generalized functions $\delta(x_2)$,

$$\langle \delta(x_2), \phi(x_1, x_2) \rangle = \int_{-\infty}^{\infty} \phi(x_1, 0) dx_1.$$

A function u is called *a classical solution* of the partial differential equations

$$Lu = s, \tag{1.1.1}$$

where s is a given smooth function in $\mathbb{R}^n$, if u has continuous derivatives of order p and satisfies (1.1.1) identically. If s is a distribution, then u is said to be *a generalized solution* of (1.1.1) if

$$\langle Lu, \phi \rangle = \langle s, \phi \rangle, \tag{1.1.2}$$

that is, if u satisfies (1.1.1) in the distributional sense.

Gelfand and Shilov [4] have first introduced the elementary solution of the n-dimensional classical ultra-hyperbolic operator. Trione [12] has shown that the n-dimensional ultra-hyperbolic equation has $R_{2k}(x)$ as an unique elementary solution. Later, Tellez [11] has proved that $R_{2k}(x)$ exists only for case p is odd with $p+q=n$.

Yildirim et al. [13] have introduced the Bessel ultra-hyperbolic type operator iterated k-times with $x \in \mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$,

$$\Box_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}})^k, \tag{1.1.3}$$

where $p+q=n, B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$ where $2v_i = 2\beta_i + 1, \beta_i > -\frac{1}{2}$ [6], k is nonnegative integer and n is the dimension of $\mathbb{R}_n^+$ and studied the elementary solution of this operator. Moreover, they have introduced the Bessel diamond operator and have studied the elementary solution of this operator and also the Fourier-Bessel transform of the elementary solution.

Kananthai and Nonlaopon [5] have studied the weak solution of the compound ultra-hyperbolic equation. Next, Sarikaya and Yildirim [9] have studied the weak solution of the compound Bessel ultra-hyperbolic equation. Later, Bupasiri and Nonlaopon [2] have studied the weak solution of the compound equation related to the ultra-hyperbolic operator of the form

$$\sum_{r=0}^{m} C_r \Box_c^r u(x) = f(x), \tag{1.1.4}$$

where $\Box_c^r$ is the operator which related to the ultra-hyperbolic type operator iterated r-times, defined by

$$\Box_c^r = \left(\frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^r. \tag{1.1.5}$$

Saglam et al. [8] have developed the operator of (1.1.3), defind by

$$\Box_{B,c}^k = \left[\frac{1}{c^2}(B_{x_1} + B_{x_2} + \cdots + B_{x_p}) - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})\right]^k \tag{1.1.6}$$

and is called the ultra-hyperbolic Bessel operator iterated k-times. Moreover, they studied the product of the ultra-hyperbolic Bessel operator related to elastic waves.

In this thesis, we will consider the equation

$$\square_{B,c}^{k} u(x) = f(x), \tag{1.1.7}$$

where $u(x)$ and $f(x)$ are some generalized function, and

we will develop the equation (1.1.7) to the form

$$\sum_{k=0}^{m} C_k \square_{B,c}^{k} u(x) = f(x), \tag{1.1.8}$$

which is called the compound ultra-hyperbolic Bessel equation and by convention $\square_{B,c}^{0} u(x) = u(x)$. In finding the solutions of (1.1.8), we use the properties of convolutions for the generalized functions.