



CHAPTER III

ON THE OPLUS HEAT KERNEL RELATED TO THE SPECTRUM

In this chapter, we study the oplus heat kernel, which is related to the spectrum and the oplus heat equation. Moreover, we also study the interesting properties of the oplus heat kernel.

3.1 The Fundamental Solution

First, we shall consider the fundamental solution of the partial differential operators.

Lemma 3.1.1 *Let L be the operator defined by*

$$L = \frac{\partial}{\partial t} - c^2 \oplus^k, \quad (3.1.1)$$

where \oplus^k is oplus operator iterated k -times, and is given by

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k,$$

$p + q = n$ is the dimension of \mathbb{R}^n , k is a positive integer, $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$ and c is a positive constant. Then we obtain

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right) + i(\xi, x) \right] d\xi \quad (3.1.2)$$

as a fundamental solution of (3.1.1) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.

Proof. Let $E(x, t)$ be the kernel or the fundamental solution of operator L and δ be the Dirac-delta distribution. Thus, we have

$$\frac{\partial}{\partial t} E(x, t) - c^2 \oplus^k E(x, t) = \delta(x) \delta(t).$$

Applying the Fourier transform, which is defined by (2.4.1), to the both sides of the above equation, considering $\widehat{\delta}(x) = 1/(2\pi)^{n/2}$, we obtain

$$\frac{\partial}{\partial t} \widehat{E}(\xi, t) - c^2 \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \widehat{E}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \delta(t).$$

Thus, we get

$$\widehat{E}(\xi, t) = \frac{H(t)}{(2\pi)^{n/2}} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \right]$$

where $H(t)$ is the Heaviside function, because $H(t) = 1$ holds for $t > 0$.

Therefore,

$$\widehat{E}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k \right]$$

which has been already by (2.4.5). Thus from (2.4.2), we have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi.$$

Thus, we have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi,$$

where Ω is the spectrum of $E(x, t)$ and $t > 0$. □

3.2 Main Results

In this section, we study the the oplus heat kernel and its properties.

Theorem 3.2.1 *Let us consider the equation*

$$\frac{\partial}{\partial t} u(x, t) = c^2 \oplus^k u(x, t) \quad (3.2.1)$$

with the initial condition

$$u(x, 0) = f(x), \quad (3.2.2)$$

where \oplus^k is oplus operator iterated k -times, and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k,$$

$p + q = n$ is the dimension of the space \mathbb{R}^n , k is a positive integer, $u(x, t)$ is an unknown function for $(x, t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then

$$u(x, t) = E(x, t) * f(x)$$

is a solution of (3.2.1), which satisfies (3.2.2), where $E(x, t)$ is given by (3.1.2).

Proof. Taking the Fourier transform, which is defined by (2.4.1), to both sides of (3.2.1), we obtain

$$\frac{\partial}{\partial t} \widehat{u}(\xi, t) = c^2 \left[\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right]^k \widehat{u}(\xi, t).$$

Thus, we get

$$\widehat{u}(\xi, t) = K(\xi) \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right) \right]^k \quad (3.2.3)$$

where $K(\xi)$ is constant and $\widehat{u}(\xi, 0) = K(\xi)$.

Now, by (3.2.2) we have

$$K(\xi) = \widehat{u}(\xi, 0) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi, x)} f(x) dx \quad (3.2.4)$$

and by the inversion in (2.4.2), (3.2.3) and (3.2.4) we obtain

$$\begin{aligned} u(x, t) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi, x)} \widehat{u}(\xi, t) d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x - y) \right] f(y) dy d\xi. \end{aligned} \quad (3.2.5)$$

Set

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi. \quad (3.2.6)$$

Since the integral of (3.2.6) is divergent, therefore we choose $\Omega \subset \mathbb{R}^n$ to be the spectrum of $E(x, t)$ and by (3.1.2), we have

$$E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi. \quad (3.2.7)$$

Thus (3.2.6) can be written in the convolution form

$$u(x, t) = E(x, t) * f(x).$$

Since $E(x, t)$ exists and from (2.4.4), then

$$\begin{aligned} \lim_{t \rightarrow 0} E(x, t) &= \frac{1}{(2\pi)^n} \int_{\Omega} e^{i(\xi, x)} d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\xi, x)} d\xi \\ &= \delta(x), \quad \text{for } x \in \mathbb{R}^n. \end{aligned} \quad (3.2.8)$$

Thus for the solution $u(x, t) = E(x, t) * f(x)$ of (3.2.1), then

$$\lim_{t \rightarrow 0} u(x, t) = u(x, 0) = \delta * f(x) = f(x)$$

which satisfies (3.2.2). □

Theorem 3.2.2 *The kernel $E(x, t)$ defined by (3.2.7) has the following properties :*

(1) $E(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ the space of continuous functions with infinitely differentiable.

(2) $\left(\frac{\partial}{\partial t} - c^2 \oplus^k\right) E(x, t) = 0$, for $t > 0$.

(3) $\lim_{t \rightarrow 0} E(x, t) = \delta$.

Proof.

(1) From (3.2.7), and

$$\frac{\partial^n}{\partial x^n} E(x, t) = \frac{1}{(2\pi)^n} \int_{\Omega} \frac{\partial^n}{\partial x^n} \exp \left[c^2 t \left(\left(\sum_{i=1}^p \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi,$$

we have $E(x, t) \in C^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.

(2) From $u(x, t) = E(x, t) * f(x)$, we have the following equality for $f(x) = \delta(x)$ by Fourier transformation

$$u(x, t) = E(x, t).$$

Then by direct computation, we obtain

$$\left(\frac{\partial}{\partial t} - c^2 \oplus^k\right) E(x, t) = 0.$$

(3) This case is obvious by (3.2.8). □