CHAPTER I

INTRODUCTION

In mathematical analysis, distributions (or generalized functions) are objects that generalize functions. Distributions make it possible to differentiate functions whose derivative does not exist in the classical sense. In particular, any locally integrable function has a distributional derivative. Distributions are widely used to formulate generalized solutions of partial differential equations. Where a classical solution may not exist or be very difficult to establish, a distribution solution to a differential equation is often much easier. Distributions are also important in physics and engineering where many problems naturally lead to differential equations whose solutions or initial conditions are distributions, such as the Dirac delta distribution.

Distributions have close relation to the solutions of differential equations. In the case of the ordinary differential equation $Lu=0$ with constant coefficients, every solution is the classical solution. The matter is quite different for partial differential equations. The solutions in a similar situation may now include generalized functions.

In the mathematical study of heat conduction and diffusion, a heat kernel is the fundamental solution to the heat equation on a particular domain with appropriate boundary conditions. It is also one of the main tools in the study of the spectrum of the Laplace operator, and is thus of some auxiliary importance throughout mathematical physics. The heat kernel represents the evolution of temperature in a region whose boundary is held fixed at a particular temperature (typically zero), such that an initial unit of heat energy is placed at a point at time $t = 0$.

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{1.1.1}$$

with the initial condition $u(x,0) = f(x)$, where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

denotes the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$. The solution of (1.1.1) can be expressed in the form

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} f(x-y) e^{-|y|^2/4c^2 t} dy$$

or the solution in the classical convolution form $u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} e^{-|x|^2/4c^2 t} \tag{1.1.2}$$

and the symbol $*$ designates as the classical convolution and $E(x,t)$, which is so called the heat kernel.

Nonlaopon and Kananthai have studied the ultra-hyperbolic heat equation [10, 11, 12]

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box^k u(x,t) \tag{1.1.3}$$

with the initial condition $u(x,0) = f(x)$, where $\Box^k$ is the ultra-hyperbolic operator iterated k-times, and is defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, $f(x)$ is the given generalized function and c is a positive constant. The solution of (1.1.3) can be expressed in the form

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) \exp\left(c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{j=1}^{p} \xi_j^2 \right)^k + i(\xi, y) \right) d\xi dy \tag{1.1.4}$$

or the solution in the classical convolution form $u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{\Omega} \exp\left(c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{j=1}^{p} \xi_j^2 \right)^k + i(\xi, x) \right) d\xi, \tag{1.1.5}$$

which is so called *ultra-hyperbolic heat kernel* and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed, $t > 0$.

Saglam et al. have studied Bessel diamond heat equation [13]

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Diamond_B^k u(x,t) \tag{1.1.6}$$

with the initial condition $u(x,0) = f(x)$, for $x \in \mathbb{R}_n^+ = \{x : x = (x_1, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$. The operator $\Diamond_B^k$ is first introduced by Yildirim et al. and named the Bessel diamond operator iterated k-times [16] defined by

$$\Diamond_B^k = [(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^2 - (B_{p+1} + \cdots + B_{x_{p+q}})^2]^k,$$

where $p + q = n$, $B_{x_i} = \dfrac{\partial^2}{\partial x_i^2} + \dfrac{2v_i}{x_i}\dfrac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1, \alpha_i > -\dfrac{1}{2}, x_i > 0, i = 1, 2, \ldots, n$ and n is the dimension of the $\mathbb{R}_n^+$, k is a positive integer, $u(x,t)$ is an unknown function of the form $(x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}_n^+ \times (0, \infty)$, $f(x)$ is the given generalized function and c is a positive constant. The solution of (1.1.6) can be expressed in the form

$$u(x,t) = \int_{\mathbb{R}_n^+} \left[C_v \int_{\mathbb{R}_n^+} e^{c^2 t V^k(z)} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(y_i z_i) z_i^{2v_i} dz \right] T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) dy, \tag{1.1.7}$$

where $J_{v_i - \frac{1}{2}} x_i y_i$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator, $V(z) = (z_1^2 + z_2^2 + \cdots + z_p^2)^2 - (z_{p+1}^2 + z_{p+2}^2 + \cdots + z_{p+q}^2)^2$ and

$$C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \right)^{-1}.$$

Or the solution in the B-convolution form $u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = C_v \int_{\Omega+} e^{c^2 t [y_1^2 + \cdots + y_p^2)^2 - (y_{p+1}^2 + \cdots + y_{p+q}^2)^2]^k} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} dy, \tag{1.1.8}$$

which is so called *Bessel diamond heat kernel* and $\Omega^+ \subset \mathbb{R}_n^+$ is the spectrum of $E(x,t)$ for any fixed, $t > 0$.

Saglam et al. have studied Bessel ultra-hyperbolic heat equation [14]

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box_B^k u(x,t) \tag{1.1.9}$$

with the initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}_n^+$. The operator $\Box_B^k$ is named the Bessel ultra-hyperbolic operator iterated k-times, and is defined by

$$\Box_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}})^k,$$

where $p+q=n$ is the dimension of the $\mathbb{R}_n^+$. The solution of (1.1.9) can be written in the B-convolution form $u(x,t)=E(x,t)*f(x)$, where

$$E(x,t)=C_v\int_{\Omega}e^{(-1)^k c^2 t V^k(y)}\left(\prod_{i=1}^{n}J_{v_i-\frac{1}{2}}(x_i y_i)y_i^{2v_i}\right)dy, \tag{1.1.10}$$

which is so called *Bessel ultra-hyperbolic heat kernel* and $\Omega\subset\mathbb{R}_n^+$ is the spectrum of $E(x,t)$ for any fixed, $t>0$.

Kananthai has studied diamond heat equation [1]

$$\frac{\partial}{\partial t}u(x,t)=c^2\Diamond u(x,t) \tag{1.1.11}$$

with the initial condition $u(x,0)=f(x)$ for $x\in\mathbb{R}^n$ of the n-dimensional Euclidean space. The operator $\Diamond$ is first introduced by Kananthai and named the diamond operator [2] defined by

$$\Diamond=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}\right)^2-\left(\frac{\partial^2}{\partial x_{p+1}^2}+\frac{\partial^2}{\partial x_{p+2}^2}+\cdots+\frac{\partial^2}{\partial x_{p+q}^2}\right)^2,$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$. The solution of (1.1.11) can be expressed in the classical convolution form $u(x,t)=E(x,t)*f(x)$, where

$$E(x,t)=\frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)+i(\xi,x)\right]d\xi, \tag{1.1.12}$$

which is so called *diamond heat kernel* and $\Omega\subset\mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed, $t>0$.

Furthermore, Lunnaree and Nonlaopon have studied generalized diamond heat equation [9]

$$\frac{\partial}{\partial t}u(x,t)=c^2\Diamond^k u(x,t) \tag{1.1.13}$$

with the initial condition $u(x,0)=f(x)$, where $\Diamond^k$ is the diamond operator iterated k-times and is defined by

$$\Diamond^k=\left[\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_p^2}\right)^2-\left(\frac{\partial^2}{\partial x_{p+1}^2}+\cdots+\frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k,$$

k is a positive integer, $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$. The solution of (1.1.13) can be expressed in the classical convolution form

$u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k + i(\xi, x) \right] d\xi, \tag{1.1.14}$$

and $\Omega \in \mathbb{R}^n$ is spectrum of the $E(x,t)$ for any fixed $t > 0$.

The operator $\oplus^k$ has been studied first by Kananthai et al. and is named the oplus operator iterated k-times [7] given by

$$\oplus^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k, \tag{1.1.15}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$ and k is a positive integer. Next, Kananthai et al. have studied the fundamental solution of the operator $\oplus^k$ related to wave equation and Laplacian [6]. Kananthai and Suantai have studied the convolution product, Fourier transform and inversion of the distributional kernel $K_{\alpha,\beta,\gamma,\nu}$ related to the operator $\oplus^k$ [3, 4, 5]. Moreover, Tariboon and Kananthai have studied the Green function of the operator $(\oplus + m^2)^k$ [15].

In this work, we study the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \oplus^k u(x,t) \tag{1.1.16}$$

with the initial condition $u(x,0) = f(x)$, for $x \in \mathbb{R}^n$, where the operator $\oplus^k$ is named the oplus operator iterated k-times, and is defined by

$$\oplus^k = \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k, \tag{1.1.17}$$

$p + q = n$ is the dimension of space $\mathbb{R}^n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function and c is a positive constant. We obtain $u(x,t) = E(x,t) * f(x)$ as a solution of (1.1.16), where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^4 - \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^4 \right)^k + i(\xi, x) \right] d\xi \tag{1.1.18}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$. The function $E(x,t)$ is called *the oplus heat kernel* or the fundamental solution of (1.1.16).