

Full Paper

Fibonacci p -sequences in groups

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Abstract: The Fibonacci p -sequence modulo m is studied. The Fibonacci p -orbit and basic Fibonacci p -orbit of a group are defined, and then the lengths of periods of these orbits are examined. Furthermore, the Fibonacci p -lengths and basic Fibonacci p -lengths of polyhedral groups $(2, 2, 2)$, $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ for $n \geq 3$ are obtained.

Keywords: linear recurrence sequences, Fibonacci p -sequences, Fibonacci p -orbit

INTRODUCTION

Linear recurrence sequences appear in modern research in many fields, from mathematics, physics, computer science, architecture to nature and art [1-21]. The study of recurrence sequences in groups began with the earlier work of Wall [22], who investigated the ordinary Fibonacci sequence in cyclic groups. The concept was extended to some special linear recurrence sequences by several authors [23-35]. In this paper we extend the theory to Fibonacci p -sequences.

The Fibonacci p -sequence $\{F_p(n)\}$ for any given p ($p=1, 2, 3, L$) is defined [36, 37] by the following recurrence equation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1) \quad (1)$$

for $n > p$, where $F_p(0) = 0, F_p(1) = L = F_p(p) = 1$.

The Fibonacci p -matrix Q_p was given by Stakhov [36] as follows:

$$Q_p = \begin{bmatrix} 1 & 0 & 0L & 0 & 1 \\ 1 & 0 & 0L & 0 & 0 \\ 0 & 1 & 0L & 0 & 0 \\ \mathbf{M} & \mathbf{O} & \mathbf{L} & \mathbf{M} & \mathbf{M} \\ 0L & 0 & 1 & 0 & 0 \\ 0 & 0L & 0 & 1 & 0 \end{bmatrix}_{(p+1) \times (p+1)} \quad (2)$$

Also, he showed that

$$Q_p^n = \begin{bmatrix} F_p(n+1) & F_p(n-p+1) & L & F_p(n-1) & F_p(n) \\ F_p(n) & F_p(n-p) & L & F_p(n-2) & F_p(n-1) \\ M & M & M & M & M \\ F_p(n-p+2) & F_p(n-2p+2) & L & F_p(n-p) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-2p+1) & L & F_p(n-p-1) & F_p(n-p) \end{bmatrix}_{(p+1) \times (p+1)}. \quad (3)$$

It is well known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. For example, the sequence $a, b, c, d, b, c, d, b, c, d, L$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, a, b, c, d, a, b, c, d, L$ is simply periodic with period 4.

FIBONACCI p -SEQUENCES MODULO m

Reducing the Fibonacci p -sequence $\{F_p(n)\}$ modulo m , we obtain the following repeating sequence:

$$\{F_p^{(m)}(n)\} = \{F_p^{(m)}(0), F_p^{(m)}(1), F_p^{(m)}(2), L, F_p^{(m)}(p), F_p^{(m)}(p+1), L, F_p^{(m)}(i), L\},$$

where $F_p^{(m)}(i) = F_p(i) \pmod{m}$. It has the same recurrence relation as in (1).

Theorem 1. $\{F_p^{(m)}(n)\}$ is a simple periodic sequence.

Proof. Let $S = \{(x_1, x_2, L, x_{p+1}) \mid 0 \leq x_i \leq m-1\}$. Then we have $|S| = m^{p+1}$ being finite; that is, for any $j \geq 0$, there exists $i \geq j$ such that $F_p^{(m)}(i+p) \equiv F_p^{(m)}(j+p)$, $F_p^{(m)}(i+p-1) \equiv F_p^{(m)}(j+p-1)$, L , $F_p^{(m)}(i+1) \equiv F_p^{(m)}(j+1)$ and $F_p^{(m)}(i) \equiv F_p^{(m)}(j)$. From the definition of the Fibonacci p -sequence we have $F_p(n-p-1) = F_p(n) - F_p(n-1)$. Then we can easily get that $F_p^{(m)}(i-1) \equiv F_p^{(m)}(j-1)$, $F_p^{(m)}(i-2) \equiv F_p^{(m)}(j-2)$, L , $F_p^{(m)}(i-j+1) \equiv F_p^{(m)}(1)$, and $F_p^{(m)}(i-j+1) \equiv F_p^{(m)}(0)$, which implies that $\{F_p^{(m)}(n)\}$ is a simple periodic sequence.

Let the notation $l_p(m)$ denote the length of the period of the sequence $\{F_p^{(m)}(n)\}$. For given a matrix $A = [a_{ij}]$ of integers, $A \pmod{m}$ means that the entry of A is a reduced modulo m , i.e. $A \pmod{m} = (a_{ij} \pmod{m})$. Let k be a prime and let $\langle Q_p \rangle_{k^u} = \{Q_p^i \pmod{k^u} \mid i \geq 0\}$. Since $\det Q_p = (-1)^p$, the set $\langle Q_p \rangle_{k^u}$ is a cyclic group. We denote the order of $\langle Q_p \rangle_{k^u}$ by $|\langle Q_p \rangle_{k^u}|$. From (3), it clear that $l_p(k^u) = |\langle Q_p \rangle_{k^u}|$.

Example 1. We have $\{F_2^{(5)}(n)\} = \{0, 1, 1, 2, 3, 4, 1, 4, 3, 4, 3, 1, 0, 3, 4, 4, 2, 1, 0, 2, 3, 3, 0, 3, 1, 1, 4, 0, 1, 0, 0, 1, 1, \dots\}$, and then repeat. So, we get $l_2(5) = 31$.

Theorem 2. Let t be the largest positive integer and let k be a prime such that $l_p(k) = l_p(k^t)$. Then $l_p(k^\alpha) = k^{\alpha-t} \cdot l_p(k)$ for every $\alpha \geq t$.

Proof. Let q be a positive integer. Since $Q_p^{l_p(k^{q+1})} \equiv I \pmod{k^{q+1}}$, we can write $Q_p^{l_p(k^{q+1})} \equiv I \pmod{k^q}$. Then we show that $l_p(k^{q+1})$ is divisible by $l_p(k^q)$. Also, writing $Q_p^{l_p(k^q)} = I + (a_{ij}^{(q)} \cdot k^q)$, we obtain, by the binomial expansion,

$$Q_p^{l_p(k^q)k} = \left(I + (a_{ij}^{(q)} \cdot k^q) \right)^k = \sum_{i=0}^k \binom{k}{i} (a_{ij}^{(q)} \cdot k^q)^i \equiv I \pmod{k^{q+1}}.$$

This shows that $l_p(k^{q+1})$ divides $l_p(k^q) \cdot k$. Therefore, $l_p(k^{q+1}) = l_p(k^q)$ or $l_p(k^{q+1}) = l_p(k^q) \cdot k$, and the latter holds if and only if there is an $a_{ij}^{(q)}$ which is not divisible by k . Since $l_p(k^t) \neq l_p(k^{t+1})$, there is an $a_{ij}^{(t+1)}$ which is not divisible by k ; thus, $l_p(k^{t+1}) \neq l_p(k^{t+2})$. To complete the proof, we may use an inductive method on t .

Theorem 3. If m has the prime factorisation: $m = \prod_{i=1}^t k_i^{e_i}$, ($t \geq 1$), then $l_p(m) = \text{lcm} [l_p(k_i^{e_i})]$.

Proof. The statement ' $l_p(k_i^{e_i})$ is the length of the period of $\{F_p^{(k_i^{e_i})}(n)\}$ ' implies that the sequence $\{F_p^{(k_i^{e_i})}(n)\}$ repeats only after blocks of length $u \cdot l_p(k_i^{e_i})$ ($u \in \mathbb{N}$), and the statement ' $l_p(m)$ is the length of the period $\{F_p^{(m)}(n)\}$ ' implies that $\{F_p^{(k_i^{e_i})}(n)\}$ repeats after $l_p(m)$ terms for all values i . Thus, $l_p(m)$ is of the form $u \cdot l_p(k_i^{e_i})$ for all values of i , and since any such number gives a period of $\{F_p^{(m)}(n)\}$, we easily obtain that $l_p(m) = \text{lcm} [l_p(k_i^{e_i})]$.

Definition 1. Let $l_p^{(a_0, a_1, \dots, a_p)}(m)$ denote the period of the integer-valued recurrence relation $x_n = x_{n-1} + x_{n-p+1}$ for $n > p$, $x_0 = a_0, x_1 = a_1, \dots, x_p = a_p$ when each entry is a reduced modulo m .

Theorem 4. If $a_0, a_1, \dots, a_p, b_0, b_1, \dots, b_p, m \in \mathbb{N}$ is such that $\text{gcd}(a_0, a_1, \dots, a_p, m) = 1$ and $\text{gcd}(b_0, b_1, \dots, b_p, m) = 1$, then

$$l_p^{(a_0, a_1, \dots, a_p)}(m) = l_p^{(b_0, b_1, \dots, b_p)}(m).$$

Proof. Let $X_n = \begin{bmatrix} x_{n+p} \\ x_{n+p-1} \\ \vdots \\ M \\ \vdots \\ x_{n+1} \\ x_n \end{bmatrix}$. Then it is clear that $X_n = (Q_p)^n \cdot X_0$. Since the integers modulo m form

a finite set of equivalence classes, there exist integers n and r such that $(Q_p)^{n+r}$ is congruent elementwise to $(Q_p)^r$ modulo m if p is an even integer and that $(Q_p)^{n+r+1}$ is congruent elementwise to $(Q_p)^r$ modulo m . if p and n are odd integers. Since $\det Q_p = (-1)^p$, $(Q_p)^n$ or $(Q_p)^{n+1}$ is a

$(p+1) \times (p+1)$ identity matrix. Thus, it is verified that $X_n \equiv X_0 \pmod{m}$ or $X_{n+1} \equiv X_0 \pmod{m}$. So the proof is complete.

FIBONACCI p -SEQUENCE AND BASIC FIBONACCI p -SEQUENCE IN GROUPS

Let G be a finite j -generator group and let X be the subset of G^j such that $(x_0, x_1, \dots, x_{j-1}) \in X$ if and only if G is generated by x_0, x_1, \dots, x_{j-1} . We call $(x_0, x_1, \dots, x_{j-1})$ a generating j -tuple for G .

Definition 2. For a p -tuple $(x_0, x_1, \dots, x_{p-1}) \in X$, we define Fibonacci p -orbit $F_p^{(x_0, x_1, \dots, x_{p-1})}(G) = \{a_i\}$ by

$$a_0 = x_0, a_1 = x_1, a_{p-1} = x_{p-1}, a_p = x_{p-1}, a_{n+p} = a_{n-1} \cdot a_{n+p-1}, n \geq 1.$$

The classic Fibonacci p -sequence in a cyclic group $C = \langle x \rangle$ can be written as $F_p^{(x, x, \dots, x)}(C)$.

Theorem 5. A Fibonacci p -orbit of a finite group is simply periodic.

Proof. Let n be the order of G . Since there are $n^{(p+1)}$ distinct $(p+1)$ -tuples of elements of G , at least one of the $(p+1)$ -tuples appears twice in a Fibonacci p -orbit of G . Because of the repetition, Fibonacci p -orbit is periodic.

Since the Fibonacci p -orbit is periodic, there exist natural numbers u and v , with $u > v$, such that

$$a_{u+1} = a_{v+1}, a_{u+2} = a_{v+2}, \dots, a_{u+p+1} = a_{v+p+1}.$$

By definition of the Fibonacci p -orbit, it is clear that

$$a_u = (a_{u+p+1}) \cdot (a_{u+p})^{-1} \text{ and } a_v = (a_{v+p+1}) \cdot (a_{v+p})^{-1}.$$

Then we may write $a_u = a_v$, and hence

$$a_{u-v} = a_{v-v} = a_0, a_{u-v+1} = a_{v-v+1} = a_1, \dots, a_{u-v+p} = a_{v-v+p} = a_p,$$

which implies that the Fibonacci p -orbit is simply periodic.

Let the notation $LF_p^{(x_0, x_1, \dots, x_{p-1})}(G)$ denote the length of the period of the Fibonacci p -orbit $F_p^{(x_0, x_1, \dots, x_{p-1})}(G)$. It is said to be the Fibonacci p -length with respect to the generating p -tuple $(x_0, x_1, \dots, x_{p-1})$.

To examine the concept more fully we study the action of the automorphism group $\text{Aut } G$ of G on the Fibonacci p -orbit $F_p^{(x_0, x_1, \dots, x_{p-1})}(G)$, $(x_0, x_1, \dots, x_{p-1}) \in X$. Now $\text{Aut } G$ consists of all isomorphisms $\theta: G \rightarrow G$, and if $\theta \in \text{Aut } G$ and $(x_0, x_1, \dots, x_{p-1}) \in X$, then $(x_0\theta, x_1\theta, \dots, x_{p-1}\theta) \in X$.

For a subset $A \subseteq G$ and $\theta \in \text{Aut } G$, the image of A under θ is

$$A\theta = \{a\theta : a \in A\}.$$

Lemma 1. Let $(x_0, x_1, \dots, x_{p-1}) \in X$ and let $\theta \in \text{Aut } G$. Then

$$\left(F_p^{(x_0, x_1, \dots, x_{p-1})}(G)\right)\theta = F_p^{(x_0\theta, x_1\theta, \dots, x_{p-1}\theta)}(G).$$

Proof: Let $F_p^{(x_0, x_1, L, x_{p-1})}(G) = \{a_i\}$. Since $\{a_i\}\theta = \{a_i\theta\}$ and $a_{i+p}\theta = (a_{i-1}a_{i+p-1})\theta = a_{i-1}\theta a_{i+p-1}\theta$, we have the conclusion.

Each generating p -tuple $(x_0, x_1, L, x_{p-1}) \in X$ maps to $|\text{Aut } G|$ distinct elements of X under the action of elements of $\text{Aut } G$. Hence there are

$$d_p(G) = |X|/|\text{Aut } G|$$

non-isomorphic generating p -tuples for G [27]. The notation $d_p(G)$ was introduced by Hall [38].

Suppose ω of the elements of $\text{Aut } G$ maps $F_p^{(x_0, x_1, L, x_{p-1})}(G)$ into itself. Then there are $|\text{Aut } G|/\omega$ distinct Fibonacci p -orbits $F_p^{(x_0\theta, x_1\theta, L, x_{p-1}\theta)}(G)$ for $\theta \in \text{Aut } G$.

Definition 3. For a p -tuple $(x_0, x_1, L, x_{p-1}) \in X$, the basic Fibonacci p -orbit $\overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G)$ of the basic period m is a sequence of group elements b_0, b_1, b_2, L, b_n, L for which, given an initial (seed) set $b_0 = x_0, b_1 = x_1, L, b_{p-1} = x_{p-1}, b_p = x_{p-1}$, each element is defined by

$$b_{n+p} = b_{n-1} \cdot b_{n+p-1}, \quad n \geq 1,$$

where $m \geq 1$ is the least integer, with

$$b_0 = b_m\theta, b_1 = b_{m+1}\theta, L, b_{p-1} = b_{m+p-1}\theta, b_p = b_{m+p}\theta,$$

for some $\theta \in \text{Aut } G$. Since G is a finite p -generator group and $b_m, b_{m+1}, L, b_{m+p-1}$ generates G , it follows that θ is uniquely determined. Clearly, the basic Fibonacci p -orbit $\overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G)$ contains m elements.

Let the notation $L\overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G)$ denote the basic length of the period of the basic Fibonacci p -orbit $\overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G)$. It is said to be the basic Fibonacci p -length with respect to the generating p -tuple (x_0, x_1, L, x_{p-1}) .

Theorem 6. Let G be a finite group and $(x_0, x_1, L, x_{p-1}) \in X$. If $L\overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G) = n$ and $L\overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G) = m$, then m divides n and there are n/m elements of $\text{Aut } G$ which map $F_p^{(x_0, x_1, L, x_{p-1})}(G)$ into themselves.

Proof: Since

$$F_p^{(x_0, x_1, L, x_{p-1})}(G) = \overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G) \cup \overline{F}_p^{(x_0\theta, x_1\theta, L, x_{p-1}\theta)}(G) \cup \overline{F}_p^{(x_0\theta^2, x_1\theta^2, L, x_{p-1}\theta^2)}(G) \cup L$$

and

$$L\overline{F}_p^{(x_0, x_1, L, x_{p-1})}(G) = L\overline{F}_p^{(x_0\theta, x_1\theta, L, x_{p-1}\theta)}(G),$$

we have $n = m\lambda$, where λ is the order of the automorphism $\theta \in \text{Aut } G$. Thus, it is verified that $1, \theta, \theta^2, L, \theta^{\lambda-1}$ map $F_p^{(x_0, x_1, L, x_{p-1})}(G)$ into themselves.

APPLICATIONS

For application, we obtain the lengths of the periods of the Fibonacci p -orbits and the basic Fibonacci p -orbits of polyhedral groups $(2, 2, 2)$, $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ for $n \geq 3$.

Definition 4. The polyhedral group (l, m, n) for $l, m, n > 1$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz = e \rangle$$

or

$$\langle x, y : x^l = y^m = (xy)^n = e \rangle.$$

The polyhedral group (l, m, n) is finite if and only if the number

$$\mu = lmn \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 \right) = mn + nl + lm - lmn$$

is positive, i.e. in the case $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. Its order is $2lmn/\mu$. Using Tietze transformations we may show that $(l, m, n) \cong (m, n, l) \cong (n, l, m)$. More information on these groups can be found in the work by Coxeter and Moser [39].

In this section we obtain the Fibonacci p -lengths and the basic Fibonacci p -lengths of the polyhedral groups $(2, 2, 2)$, $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ for $n \geq 3$ as applications of the results obtained.

Theorem 7. i. $LF_2^{(x,y)}((2, 2, 2)) = \overline{LF}_2^{(x,y)}((2, 2, 2)) = 7$.

ii. $LF_3^{(x,y,z)}((2, 2, 2)) = \overline{LF}_3^{(x,y,z)}((2, 2, 2)) = 15$.

Proof. i. The orbit $F_2^{(x,y)}((2, 2, 2))$ is $x, y, y, xy, x, xy, e, x, y, y, L$. So we get $LF_2^{(x,y)}((2, 2, 2)) = \overline{LF}_2^{(x,y)}((2, 2, 2)) = 7$ since $x\theta = x$ and $y\theta = y$, where θ is the identity transform.

ii. The orbit $F_3^{(x,y,z)}((2, 2, 2))$ is $x, y, z, z, y, e, z, e, y, y, x, x, z, x, e, x, y, z, z, L$. So we get $LF_3^{(x,y,z)}((2, 2, 2)) = \overline{LF}_3^{(x,y,z)}((2, 2, 2)) = 15$ since $x\theta = x$, $y\theta = y$ and $z\theta = z$, where θ is the identity transform.

Theorem 8. The lengths of the Fibonacci p -orbits and the basic lengths of the Fibonacci p -orbits of the polyhedral groups $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ for $n \geq 3$ are:

i. The Fibonacci 2-lengths and the basic Fibonacci 2-lengths in the 2-generator cases are:

i'. $LF_2^{(x,y)}((n, 2, 2)) = 7$ and $\overline{LF}_2^{(x,y)}((n, 2, 2)) = 14$.

ii'. $LF_2^{(x,y)}((2, n, 2)) = LF_2^{(x,y)}((2, 2, n)) = \begin{cases} \frac{7n}{2}, & n \equiv 0 \pmod{4}, \\ 7n, & n \equiv 2 \pmod{4}, \\ 14n, & \text{otherwise} \end{cases}$ and

$\overline{LF}_2^{(x,y)}((2, n, 2)) = \overline{LF}_2^{(x,y)}((2, 2, n)) = \begin{cases} \frac{7n}{2}, & n \equiv 0 \pmod{4}, \\ \frac{7n}{2}, & n \equiv 2 \pmod{4}, \\ 7n, & \text{otherwise.} \end{cases}$

ii. The Fibonacci 3-lengths and the basic Fibonacci 3-lengths in the 3-generator cases are:

$$LF_3^{(x,y,z)}((G_n)) = \begin{cases} \frac{15n}{2}, & n \equiv 0 \pmod{4}, \\ 15n, & n \equiv 2 \pmod{4}, \\ 30n, & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{LF}_3^{(x,y,z)}((G_n)) = \begin{cases} \frac{15n}{2}, & n \equiv 0 \pmod{4}, \\ \frac{15n}{2}, & n \equiv 2 \pmod{4}, \\ 15n, & \text{otherwise.} \end{cases}$$

Here G_n ($n \geq 3$) is one the groups $(n, 2, 2)$, $(2, n, 2)$ and $(2, 2, n)$ in the 3-generator cases.

Proof. i.i'. The orbit $F_2^{(x,y)}((n, 2, 2))$ is $x, y, y, xy, x^{-1}, xy, e, x^{-1}, x^2y, x^2y, xy, x, xy, e, x, y, y, L$.

So we get $LF_2^{(x,y)}((n, 2, 2)) = 7$ and $\overline{LF}_2^{(x,y)}((n, 2, 2)) = 14$ since $x\theta = x^{-1}$ and $y\theta = x^2y$, where θ is the inner automorphism induced by conjugation by xy .

ii'. Firstly, let us consider the group $(2, n, 2)$. The orbit $F_2^{(x,y)}((2, n, 2))$ is

$$x, y, y, xy, x, yx, y^{-2}, y^2x, y^{-1}, y^{-3}, y^5x, y^4x, yx, y^4, x, y, y^5, \\ xy^5, xy^4, yx, y^{-6}, y^2x, y^{-1}, y^{-7}, y^9x, y^8x, yx, y^8, x, y, y^9, L.$$

Using the above, the sequence becomes

$$a_0 = x, a_1 = y, a_2 = y, L, \\ a_{7+14i} = y^2x, a_{8+14i} = y^{-1}, a_{9+14i} = y^{-3-4i}, L, \\ a_{14+14i} = x, a_{15+14i} = y, a_{16+14i} = y^{5+4i}, L,$$

where $i \geq 0$. So we need the smallest $i \in N$ such that $4i = nu_1$ for $u_1 \in N$.

If $n \equiv 0 \pmod{4}$, $i = \frac{n}{4}$. Thus, $LF_2^{(x,y)}((2, n, 2)) = 14 \cdot \frac{n}{4} = \frac{7n}{2}$ and $\overline{LF}_2^{(x,y)}((2, n, 2)) = \frac{7n}{2}$ since $x\theta = x$ and $y\theta = y$, where θ is the identity transform.

If $n \equiv 2 \pmod{4}$, $i = \frac{n}{2}$. Thus, $LF_2^{(x,y)}((2, n, 2)) = 14 \cdot \frac{n}{2} = 7n$ and $\overline{LF}_2^{(x,y)}((2, n, 2)) = \frac{7n}{2}$ since $x\theta = x$ and $y\theta = y^{-1}$, where θ is the inner automorphism induced by conjugation by x .

If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$, $i = n$. Thus, $LF_2^{(x,y)}((2, n, 2)) = 14n$ and $\overline{LF}_2^{(x,y)}((2, n, 2)) = 7n$ since $x\theta = x$ and $y\theta = y^{-1}$, where θ is the inner automorphism induced by conjugation by x .

Secondly, let us consider the group $(2, 2, n)$. The orbit $F_2^{(x,y)}((2, 2, n))$ is of the following form:

$$a_0 = x, a_1 = y, a_2 = y, L, \\ a_{7+14i} = y(xy)^{4i+3}, a_{8+14i} = y(xy)^{4i+2}, a_{9+14i} = y, L, \\ a_{14+14i} = (xy)^{4i+4}x, a_{15+14i} = (xy)^{4i+3}x, a_{16+14i} = y, L,$$

where $i \geq 0$. So we need the smallest $i \in N$ such that $4i = nu_2$ for $u_2 \in N$.

If $n \equiv 0 \pmod{4}$, $i = \frac{n}{4}$. Thus, $LF_2^{(x,y)}((2, 2, n)) = 14 \cdot \frac{n}{4} = \frac{7n}{2}$ and $\overline{LF}_2^{(x,y)}((2, 2, n)) = \frac{7n}{2}$ since $x\theta = x$ and $y\theta = y$, where θ is the identity transform.

If $n \equiv 2 \pmod{4}$, $i = \frac{n}{2}$. Thus, $LF_2^{(x,y)}((2, 2, n)) = 14 \cdot \frac{n}{2} = 7n$ and $\overline{LF}_2^{(x,y)}((2, 2, n)) = \frac{7n}{2}$ since $x\theta = yxy$ and $y\theta = y$, where θ is the inner automorphism induced by conjugation by y .

If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$, $i = n$. Thus, $LF_2^{(x,y)}((2, 2, n)) = 14n$ and $\overline{LF}_2^{(x,y)}((2, 2, n)) = 7n$ since $x\theta = yxy$ and $y\theta = y$, where θ is the inner automorphism induced by conjugation by y .

ii. Firstly, let us consider the group $(n, 2, 2)$. The orbit $F_3^{(x,y,z)}((n, 2, 2))$ is

$$x, y, z, z, xz, x^{-2}, x^2z, x^{-2}, x^3z, xz, x, x^{-1}, x^4z, x^{-3}, x^{-2}, x^{-3}, x^7z, x^4z, x^2z, zx, x^8, zx^4, \\ x^6, zx^7, xz, x^{-5}, x, zx^8, x^9, x^4, x^5, zx^{13}, zx^4, z, x^5z, x^{-18}, x^{14}z, x^{-14}, x^{19}z, xz, x^{13}, x^{-1}, x^{20}z, \\ x^{-19}, x^{-6}, x^{-7}, x^{27}z, x^8z, x^2z, zx^5, x^{32}, zx^{24}, x^{26}, zx^{31}, xz, x^{-25}, x, zx^{32}, x^{33}, x^8, x^9, zx^{41}, zx^8, z, L .$$

Using the above, the sequence becomes:

$$a_0 = x, a_1 = y, a_2 = z, a_3 = z, L , \\ a_{15+30i} = x^{-3-4i}, a_{16+30i} = x^{8(i+1)^2-4(i+1)+3}, a_{17+30i} = x^{4i+4}z, a_{18+30i} = x^2z, L , \\ a_{30+30i} = x^{5+4i}, a_{31+30i} = zx^{8(i+1)^2+4(i+1)+1}, a_{32+30i} = zx^{4i+4}, a_{33+30i} = z, L ,$$

where $i \geq 0$. So we need the smallest $i \in N$ such that $4i = nu_3$ for $u_3 \in N$.

If $n \equiv 0 \pmod{4}$, $i = \frac{n}{4}$. Thus, $LF_3^{(x,y,z)}((n, 2, 2)) = 30 \cdot \frac{n}{4} = \frac{15n}{2}$ and $\overline{LF_3^{(x,y,z)}}((n, 2, 2)) = \frac{15n}{2}$ since $x\theta = x$, $y\theta = y$ and $z\theta = z$, where θ is the identity transform.

If $n \equiv 2 \pmod{4}$, $i = \frac{n}{2}$. Thus, $LF_3^{(x,y,z)}((n, 2, 2)) = 30 \cdot \frac{n}{2} = 15n$ and $\overline{LF_3^{(x,y,z)}}((n, 2, 2)) = \frac{15n}{2}$ since $x\theta = x^{-1}$, $y\theta = x^3z$ and $z\theta = x^2z$, where θ is the inner automorphism induced by conjugation by xz .

If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$, $i = n$. Thus, $LF_3^{(x,y,z)}((n, 2, 2)) = 30n$ and $\overline{LF_3^{(x,y,z)}}((n, 2, 2)) = 15n$ since $x\theta = x^{-1}$, $y\theta = x^3z$ and $z\theta = x^2z$, where θ is the inner automorphism induced by conjugation by xz .

Now, let us consider the group $(2, n, 2)$. The orbit $F_3^{(x,y,z)}((2, n, 2))$ is of the following form:

$$a_0 = x, a_1 = y, a_2 = z, a_3 = z, L , \\ a_{15+30i} = y^{16i^2+20i+4}x, a_{16+30i} = y^{-16i^2-20i-7}, a_{17+30i} = y^{16i^2+28i+9}x, a_{18+30i} = xy, L , \\ a_{30+30i} = xy^{16i^2+36i+20}, a_{31+30i} = y^{16i^2+36i+21}, a_{32+30i} = xy^{16i^2+44i+29}, a_{33+30i} = xy, L ,$$

where $i \geq 0$. So we need the smallest $i \in N$ such that $16i^2 + 36i = nu_4$ for $u_4 \in N$.

If $n \equiv 0 \pmod{4}$, $i = \frac{n}{4}$. Thus, $LF_3^{(x,y,z)}((2, n, 2)) = 30 \cdot \frac{n}{4} = \frac{15n}{2}$ and $\overline{LF_3^{(x,y,z)}}((2, n, 2)) = \frac{15n}{2}$ since $x\theta = x$, $y\theta = y$ and $z\theta = z$, where θ is the identity transform.

If $n \equiv 2 \pmod{4}$, $i = \frac{n}{2}$. Thus, $LF_3^{(x,y,z)}((2, n, 2)) = 30 \cdot \frac{n}{2} = 15n$ and $\overline{LF_3^{(x,y,z)}}((2, n, 2)) = \frac{15n}{2}$ since $x\theta = y^{-2}x$, $y\theta = y^{-1}$ and $z\theta = z$, where θ is the inner automorphism induced by conjugation by z .

If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$, $i = n$. Thus, $LF_3^{(x,y,z)}((2, n, 2)) = 30n$ and $\overline{LF_3^{(x,y,z)}}((2, n, 2)) = 15n$ since $x\theta = y^{-2}x$, $y\theta = y^{-1}$ and $z\theta = z$, where θ is the inner automorphism induced by conjugation by z .

Finally, let us consider the group $(2, 2, n)$. The orbit $F_3^{(x,y,z)}((2, 2, n))$ is of the following form:

$$a_0 = x, a_1 = y, a_2 = z, a_3 = z, L , \\ a_{15+30i} = z^{8i^2+20i+10}x, a_{16+30i} = y, a_{17+30i} = z^{8i^2+16i+5}, a_{18+30i} = z^{-1}, L , \\ a_{30+30i} = xz^{8i^2+28i+20}, a_{31+30i} = y, a_{32+30i} = z^{-8i^2-24i-15}, a_{33+30i} = z, L ,$$

where $i \geq 0$. So we need the smallest $i \in N$ such that $4i^2 + 4i = nu_5$ for $u_5 \in N$.

If $n \equiv 0 \pmod{4}$, $i = \frac{n}{4}$. Thus, $LF_3^{(x,y,z)}((2,2,n)) = 30 \cdot \frac{n}{4} = \frac{15n}{2}$ and $\overline{LF}_3^{(x,y,z)}((2,2,n)) = \frac{15n}{2}$ since $x\theta = x$, $y\theta = y$ and $z\theta = z$, where θ is the identity transform.

If $n \equiv 2 \pmod{4}$, $i = \frac{n}{2}$. Thus, $LF_3^{(x,y,z)}((2,2,n)) = 30 \cdot \frac{n}{2} = 15n$ and $\overline{LF}_3^{(x,y,z)}((2,2,n)) = \frac{15n}{2}$ since $x\theta = z^2x$, $y\theta = y$ and $z\theta = z^{-1}$, where θ is the inner automorphism induced by conjugation by y .

If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$, $i = n$. Thus, $LF_3^{(x,y,z)}((2,2,n)) = 30n$ and $\overline{LF}_3^{(x,y,z)}((2,2,n)) = 15n$ since $x\theta = z^2x$, $y\theta = y$ and $z\theta = z^{-1}$, where θ is the inner automorphism induced by conjugation by y .

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