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Full Paper

Integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex and other convex functions

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Abstract: Some new integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex functions and other convex functions such as the *P*-convex, quasi-convex, *m*-convex, (*α,m*)-convex and *s*-convex functions have been established.

Keywords: integral inequality, Hermite-Hadamard type inequality, strongly logarithmically convex function, Hölder inequality, *P*-convex function, quasi-convex function, *m*-convex function, (*α,m*)-convex function, *s*-convex function

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INTRODUCTION

Throughout this paper we use the following notation:

$$R=(-\infty,\infty),\quad R_0=[0,\infty)\quad \text{and}\quad R_+=(0,\infty).$$

The following definitions are well known in the literature.

Definition 1. A function $f:I\subseteq R\to R$ is said to be convex on I if

$$f\left(\lambda x+(1-\lambda)y\right)\le \lambda f(x)+(1-\lambda)f(y)$$

holds for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 2. A function $f:I\subseteq R\to R_+$ is said to be logarithmically convex if

$$f\left(\lambda x+(1-\lambda)y\right)\le [f(x)]^{\lambda}[f(y)]^{1-\lambda}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 3 [1]**.** A function $f: I \subseteq R \to R_0$ is said to be *P*-convex if

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 4 [1]**.** A function $f: I \subseteq R \to R_0$ is said to be quasi-convex if

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 5 [2]**.** For $f:[0,b] \to R$ and $m \in (0,1]$, if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an m-convex function on $[0,b]$.

Definition 6 [3]**.** For $f:[0,b] \to R$ and $(\alpha, m) \in (0,1]^2$, if

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

is valid for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an (α, m)-convex function on $[0,b]$.

Definition 7 [4]**.** Let $s \in (0,1]$. A function $f:[a,b] \subseteq R_0 \to R_0$ is said to be s-convex (in the second sense) if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 8 [5]**.** A function $f:[a,b] \to R$ is said to be strongly convex with modulus $c \ge 0$ if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

is valid for all $x, y \in [a,b]$ and $t \in [0,1]$.

Definition 9 [6]**.** A function $f: I \subseteq R \to R_+$ is said to be strongly logarithmically convex with modulus $c \ge 0$ if

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t} - ct(1-t)(x-y)^2$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Remark 1. If a function $f: I \subseteq R \to R_+$ is strongly logarithmically convex, then it is also a strongly convex function, and so it is integrable on I.

Remark 2. For all $x \in [0,1]$ and $0 \le c \le e^{-1}$, it is easy to obtain that $1 - cx \ge e^{-x}$. For all $x, y \in [0,1]$ and $0 \le c \le e^{-1}$, we acquire

$$e^{[tx+(1-t)y]^2 - [tx^2 + (1-t)y^2]} \le 1 - ct(1-t)(x-y)^2 \le 1 - \frac{ct(1-t)(x-y)^2}{e^{tx^2+(1-t)y^2}}$$

for all $t \in [0,1]$. As a result, when $0 \le c \le e^{-1}$, the function $f(x) = e^{x^2}$ is a strongly logarithmically convex functions on $[0,1]$.

In recent decades, a number of inequalities of Hermite-Hadamard type for different kinds of convex functions have been established. Some of them may be reformulated as follows.

Theorem 1 [1]. Let $f: I^{o} \subseteq R \to R_0$ be a *P*-convex mapping on I, $a,b \in I$ with $a<b$, and $f \in L([a,b])$. Then

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le f(a)+f(b).$$

Theorem 2 [7]. Let $f,g:[a,b]\to R_0$ be a convex functions on $[a,b]\subseteq R$ with $a<b$. Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b) \tag{1}$$

and

$$N(a,b)=f(a)g(b)+f(b)g(a). \tag{2}$$

Theorem 3 [8]. Let $f,g:[a,b]\to R_0$ and $fg\in L([a,b])$ with $0\le a<b<\infty$. If f is convex and nonnegative on $[a,b]$ and g is s-convex on $[a,b]$ for some fixed $s\in(0,1]$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b),$$

where $M(a,b)$ and $N(a,b)$ are defined by (1) and (2).

In recent years, some inequalities of Hermite-Hadamard type for other kinds of convex functions were established [9, 10, 11, 12, 13, 14, 15, 16, and closely related references therein]. It was reported that some kinds of Hermite-Hadamard type inequalities have applications in statistics and other mathematical sciences [17]. The goal of this paper is to establish some new integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex functions and other convex functions such as *P*-convex, quasi-convex, *m*-convex, (α,m)-convex and *s*-convex functions.

INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE

Now we start out to establish some new integral inequalities of Hermite-Hadamard type for the product of strongly logarithmically convex functions and other convex functions.

Theorem 4. For $-\infty<a<b<\infty$, let $f,g:[a,b]\to R_+$. If f and g^q are strongly logarithmically convex functions on $[a,b]$ with modulus $c\ge 0$ for $q\ge 1$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \left[f(b)G\left(\frac{f(a)}{f(b)}\right)-\frac{c(b-a)^2}{6}\right]^{1-1/q}\left\{f(b)g^q(b)G\left(\frac{f(a)}{f(b)}\left[\frac{g(a)}{g(b)}\right]^q\right)\right.$$
$$\left.-c(b-a)^2\left[f(b)F\left(\frac{f(a)}{f(b)}\right)+g^q(b)F\left(\left[\frac{g(a)}{g(b)}\right]^q\right)\right]+\frac{c^2(b-a)^4}{30}\right\}^{1/q},$$

where

$$G(u)=\begin{cases}1, & u=1,\\ \dfrac{u-1}{\ln u}, & u\ne 1\end{cases} \quad\text{and}\quad F(u)=\begin{cases}\dfrac{1}{6}, & u=1,\\ \dfrac{(u+1)\ln u-2(u-1)}{(\ln u)^3}, & u\ne 1.\end{cases} \tag{3}$$

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Proof. Taking $x = ta + (1-t)b$ for $t \in [0,1]$, $v_f = \frac{f(a)}{f(b)}$ and $u = \frac{f(a)}{f(b)}\left[\frac{g(a)}{g(b)}\right]^q$, and using Hölder's inequality and the strongly logarithmic convexity of the functions f and g^q, the following is figured out:

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x = \int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t$$

$$\le \left[\int_0^1 f\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 f\left(ta+(1-t)b\right)g^q\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1/q}$$

$$\le \left\{\int_0^1\left[f^t(a)f^{1-t}(b) - ct(1-t)(b-a)^2\right]\mathrm{d}t\right\}^{1-1/q}$$

$$\times\left\{\int_0^1\left[f^t(a)f^{1-t}(b) - ct(1-t)(b-a)^2\right]\left[g^{qt}(a)g^{q(1-t)}(b) - ct(1-t)(b-a)^2\right]\mathrm{d}t\right\}^{1/q}$$

$$= \left[f(b)\int_0^1 v^t{}_f dt - c(b-a)^2\int_0^1 t(1-t)\mathrm{d}t\right]^{1-1/q}\left\{f(b)g^q(b)\int_0^1 u^t\mathrm{d}t\right.$$

$$\left. -c(b-a)^2\int_0^1\left[f^t(a)f^{1-t}(b) + g^{qt}(a)g^{q(1-t)}(b)\right]t(1-t)\mathrm{d}t + c^2(b-a)^4\int_0^1 t^2(1-t)^2\mathrm{d}t\right\}^{1/q}$$

$$= \left[f(b)G(v_f) - \frac{c(b-a)^2}{6}\right]^{1-1/q}\left\{f(b)g^q(b)G(u) - c(b-a)^2\left[f(b)F(v_f) + g^q(b)F(v^q{}_g)\right] + \frac{c^2(b-a)^4}{30}\right\}^{1/q}.$$

Theorem 4 is thus proved.

Corollary 1. Under the conditions of Theorem 4,

1) when $c = 0$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le f(b)g(b)G^{1-1/q}\left(\frac{f(a)}{f(b)}\right)G^{1/q}\left(\frac{f(a)}{f(b)}\left[\frac{g(a)}{g(b)}\right]^q\right);$$

2) when $c = 0$ and $q = 1$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le f(b)g(b)G\left(\frac{f(a)g(a)}{f(b)g(b)}\right).$$

Theorem 5. For $a,b \in R_0$ with $a < b$, let $f,g:[a,b] \to R_+$. If f is an s-convex function on $[a,b]$ for some $s \in (0,1]$ and g^q is a strongly logarithmically convex function on $[a,b]$ with modulus $c \ge 0$ for $q \ge 1$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \left[\frac{f(a)+f(b)}{s+1}\right]^{1-1/q}\left\{\left[\frac{g^q(a)-g^q(b)}{s+2} + \frac{g^q(b)}{s+1}\right]f(a)\right.$$

$$\left. + \left[\frac{g^q(a)-g^q(b)}{(s+1)(s+2)} + \frac{g^q(b)}{s+1}\right]f(b) - \frac{c(b-a)^2}{(s+2)(s+3)}\left[f(a)+f(b)\right]\right\}^{1/q}.$$

Proof. Employing the conditions that f is s-convex and g^q is strongly logarithmically convex on $[a,b]$, let $x = ta + (1-t)b$ for $t \in [0,1]$ and $u = \left[\frac{g(a)}{g(b)}\right]^q$. Then making use of Hölder's inequality the following is generated:

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x = \int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t$$

$$\le\left[\int_0^1 f\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 f\left(ta+(1-t)b\right)g^q\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1/q}$$

$$\le\left\{\int_0^1\left[t^s f(a)+(1-t)^s f(b)\right]\mathrm{d}t\right\}^{1-1/q}\left\{\int_0^1\left[t^s f(a)+(1-t)^s f(b)\right]\left[g^{qt}(a)g^{q(1-t)}(b)\right.\right.$$

$$\left.\left.-ct(1-t)(b-a)^2\right]\mathrm{d}t\right\}^{1/q}$$

$$=\left[\frac{1}{s+1}[f(a)+f(b)]\right]^{1-1/q}\left\{g^q(b)\int_0^1\left[t^s u^t f(a)+(1-t)^s u^t f(b)\right]\mathrm{d}t\right.$$

$$\left.-\frac{c(b-a)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{1/q}.$$

Since $u^t\le(u-1)t+1$ for all $0\le t\le 1$, we obtain

$$\int_0^1 t^s u^t\mathrm{d}t\le\int_0^1 t^s[(u-1)t+1]\mathrm{d}t=\frac{u-1}{s+2}+\frac{1}{s+1}$$

and

$$\int_0^1(1-t)^s u^t\mathrm{d}t\le\int_0^1(1-t)^s[(u-1)t+1]\mathrm{d}t=\frac{u-1}{(s+1)(s+2)}+\frac{1}{s+1}.$$

Accordingly,

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\left[\frac{f(a)+f(b)}{s+1}\right]^{1-1/q}\left\{g^q(b)\int_0^1\left[t^s u^t f(a)+(1-t)^s u^t f(b)\right]\mathrm{d}t\right.$$

$$\left.-\frac{c(b-a)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{1/q}$$

$$\le\left[\frac{f(a)+f(b)}{s+1}\right]^{1-1/q}\left\{g^q(b)\left[\left(\frac{u-1}{s+2}+\frac{1}{s+1}\right)f(a)\right.\right.$$

$$\left.\left.+\left(\frac{u-1}{(s+1)(s+2)}+\frac{1}{s+1}\right)f(b)\right]-\frac{c(b-a)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{1/q}$$

$$=\left[\frac{f(a)+f(b)}{s+1}\right]^{1-1/q}\left\{\left[\frac{g^q(a)-g^q(b)}{s+2}+\frac{g^q(b)}{s+1}\right]f(a)\right.$$

$$\left.+\left[\frac{g^q(a)-g^q(b)}{(s+1)(s+2)}+\frac{g^q(b)}{s+1}\right]f(b)-\frac{c(b-a)^2}{(s+2)(s+3)}[f(a)+f(b)]\right\}^{1/q}.$$

The proof of Theorem 5 is thus complete.

Corollary 2. Under the conditions of Theorem 5,

1) if $c=0$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\frac{[f(a)+f(b)]^{1-1/q}}{(s+1)(s+2)^{1/q}}$$

$$\times[(s+1)f(a)g^q(a)+f(a)g^q(b)+f(b)g^q(a)+(s+1)f(b)g^q(b)]^{1/q};$$

2) if $c=0$ and $q=1$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\le\frac{(s+1)f(a)g(a)+f(a)g(b)+f(b)g(a)+(s+1)f(b)g(b)}{(s+1)(s+2)};$$

3) if $c=0$, $q=1$ and $s=1$, then

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$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{2f(a)g(a)+f(a)g(b)+f(b)g(a)+2f(b)g(b)}{6}.$$

Theorem 6. For $a,b\in R_0$ with $a<b$, let $f,g:R_0\to R_+$. If f is an (α,m)-convex function on $[0,b/m]$ for $(\alpha,m)\in(0,1]^2$ and g^q is a strongly logarithmically convex function on $[a,b]$ with modulus $c\ge 0$ for $q\ge 1$, then

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \left\{\frac{1}{\alpha+1}\left[f(a)+m\alpha f\left(\frac{b}{m}\right)\right]\right\}^{1-1/q} &\left\{\left[\frac{g^q(a)-g^q(b)}{\alpha+2}+\frac{g^q(b)}{\alpha+1}\right]f(a)\right.\\ &+m\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}\left(3+\alpha+(1+\alpha)\left[\frac{g(a)}{g(b)}\right]^q\right)\right]f\left(\frac{b}{m}\right)g^q(b)\\ &\left.-\frac{c(b-a)^2}{(\alpha+2)(\alpha+3)}\left[f(a)+\frac{m\alpha(\alpha+5)}{6}f\left(\frac{b}{m}\right)\right]\right\}^{1/q}.\end{aligned}$$

Proof. Let $x=ta+(1-t)b$ for $t\in[0,1]$ and $u=\left[\frac{g(a)}{g(b)}\right]^q$. By making use of the (α,m)-convexity of f and the strongly logarithmic convexity of g^q, and applying Hölder's inequality, the following is brought out:

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x &= \int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t\\ &\le \left[\int_0^1 f\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 f\left(ta+(1-t)b\right)g^q\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1/q}\\ &\le \left\{\int_0^1\left[t^\alpha f(a)+m(1-t^\alpha)f\left(\frac{b}{m}\right)\right]\mathrm{d}t\right\}^{1-1/q}\\ &\quad\times\left\{\int_0^1\left[t^\alpha f(a)+m(1-t^\alpha)f\left(\frac{b}{m}\right)\right]\left[g^{qt}(a)g^{q(1-t)}(b)-ct(1-t)(b-a)^2\right]\mathrm{d}t\right\}^{1/q}\\ &= \left\{\frac{1}{\alpha+1}\left[f(a)+m\alpha f\left(\frac{b}{m}\right)\right]\right\}^{1-1/q}\left\{g^q(b)\int_0^1\left[t^\alpha f(a)+m(1-t^\alpha)f\left(\frac{b}{m}\right)\right]u^t dt\right.\\ &\quad\left.-\frac{c(b-a)^2}{(\alpha+2)(\alpha+3)}\left[f(a)+\frac{m\alpha(\alpha+5)}{6}f(\frac{b}{m})\right]\right\}^{1/q}.\end{aligned}$$

By virtue of the inequality $u^t\le(u-1)t+1$ for all $0\le t\le 1$, we have

$$\begin{aligned}g^q(b)\int_0^1\left[t^\alpha f(a)+m(1-t^\alpha)f\left(\frac{b}{m}\right)\right]u^t\mathrm{d}t &\le g^q(b)\int_0^1\left[t^\alpha f(a)+m(1-t^\alpha)f\left(\frac{b}{m}\right)\right][(u-1)t+1]\mathrm{d}t\\ &= g^q(b)\left\{\left(\frac{u-1}{\alpha+2}+\frac{1}{\alpha+1}\right)f(a)+m\left[\frac{\alpha(3+u+\alpha+\alpha u)}{2(\alpha+1)(\alpha+2)}\right]f\left(\frac{b}{m}\right)\right\}\\ &= \left[\frac{g^q(a)-g^q(b)}{\alpha+2}+\frac{g^q(b)}{\alpha+1}\right]f(a)+m\left[\frac{\alpha(3+u+\alpha+\alpha u)}{2(\alpha+1)(\alpha+2)}\right]f\left(\frac{b}{m}\right)g^q(b).\end{aligned}$$

Theorem 6 is thus proved.

Corollary 3. Under the conditions of Theorem 6,

1) when $c=0$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{1}{2^{1/q}(\alpha+1)(\alpha+2)^{1/q}}\left[f(a)+\alpha m f\left(\frac{b}{m}\right)\right]^{1-1/q}$$

$$\times\left\{2[(\alpha+1)g^q(a)+g^q(b)]f(a)+\alpha m[(\alpha+3)g^q(b)+(\alpha+1)g^q(a)]f\left(\frac{b}{m}\right)\right\}^{1/q};$$

2) when $c=0$ and $q=1$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{2[(\alpha+1)g(a)+g(b)]f(a)+\alpha m[(\alpha+3)g(b)+(\alpha+1)g(a)]f(b/m)}{2(\alpha+1)(\alpha+2)};$$

3) when $c=0$, $q=1$ and $m=1$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{2[(\alpha+1)g(a)+g(b)]f(a)+\alpha[(\alpha+3)g(b)+(\alpha+1)g(a)]f(b)}{2(\alpha+1)(\alpha+2)};$$

4) when $c=0$, $q=1$ and $m=\alpha=1$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{[2g(a)+g(b)]f(a)+[2g(b)+g(a)]f(b)}{6}.$$

Theorem 7. For $a,b\in R$ with $a<b$, let $f,g:R\to R_+$. If f is a *P*-convex function on $[a,b]$ and g^q is a logarithmically convex function on $[a,b]$ for $q\ge 1$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \le [f(a)+f(b)]\left[g^q(b)G\left(\left[\frac{g(a)}{g(b)}\right]^q\right)-\frac{c(b-a)^2}{6}\right]^{1/q},$$

where $G(u)$ is defined by (3).

Proof. Taking $x=ta+(1-t)b$ for $t\in[0,1]$ and denoting $u=\left[\frac{g(a)}{g(b)}\right]^q$ lead to

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x = \int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t$$

$$\le\left[\int_0^1 f\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 f\left(ta+(1-t)b\right)g^q\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1/q}$$

$$\le[f(a)+f(b)]^{1-1/q}\left\{\int_0^1[f(a)+f(b)][g^{qt}(a)g^{q(1-t)}(b)-ct(1-t)(b-a)^2]\mathrm{d}t\right\}^{1/q}$$

$$\le[f(a)+f(b)]\left[g^q(b)\int_0^1 u^t dt-\frac{c(b-a)^2}{6}\right]^{1/q}$$

$$=[f(a)+f(b)]\left[g^q(b)G(u)-\frac{c(b-a)^2}{6}\right]^{1/q}.$$

Theorem 7 is thus proved.

Corollary 4. Under the conditions of Theorem 7,

1) when $c=0$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le [f(a)+f(b)]g(b)G^{1/q}\left(\left[\frac{g(a)}{g(b)}\right]^q\right);$$

2) when $c=0$ and $q=1$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le [f(a)+f(b)]g(b)G\left(\frac{g(a)}{g(b)}\right).$$

Theorem 8. For $a,b\in R$ with $a<b$, let $f,g:R\to R_+$. If f is a quasi-convex function on $[a,b]$ and g^q is a logarithmically convex function on $[a,b]$ for $q\geq 1$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq \max\{f(a),f(b)\}\left[g^q(b)G\left(\left[\frac{g(a)}{g(b)}\right]^q\right)-\frac{c(b-a)^2}{6}\right]^{1/q},$$

where $G(u)$ is defined by (3).

Proof. Let $x=ta+(1-t)b$ for $t\in[0,1]$ and $u=\left[\frac{g(a)}{g(b)}\right]^q$. This gives

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x=\int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)\mathrm{d}t$$

$$\leq\left[\int_0^1 f\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 f\left(ta+(1-t)b\right)g^q\left(ta+(1-t)b\right)\mathrm{d}t\right]^{1/q}$$

$$\leq[\max\{f(a),f(b)\}]^{1-1/q}\left[\int_0^1\max\{f(a),f(b)\}[g^{qt}(a)g^{q(1-t)}(b)-ct(1-t)(b-a)^2]\mathrm{d}t\right]^{1/q}$$

$$\leq\max\{f(a),f(b)\}\left[g^q(b)G(u)-\frac{c(b-a)^2}{6}\right]^{1/q}.$$

Theorem 8 is thus proved.

Corollary 5. Under conditions of Theorem 8,

1) when $c=0$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\leq\max\{f(a),f(b)\}g(b)G^{1/q}\left(\left[\frac{g(a)}{g(b)}\right]^q\right);$$

2) when $c=0$ and $q=1$,

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\leq\max\{f(a),f(b)\}g(b)G\left(\frac{g(a)}{g(b)}\right).$$

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