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Full Paper

Doubly censored mixture of two Rayleigh distributions: Properties and applications

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Abstract: The Bayes estimation of the parameters of single and mixture of Rayleigh distributions under double censoring is discussed. The informative and non-informative priors under squared error loss function and k-loss function are assumed for the posterior estimation. The posterior risks, associated with each estimator are used to compare the performance of different estimators using the simulated and real life data sets.

Keywords: inverse transformation method, mixture model, double censoring, loss functions, Bayes estimator

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INTRODUCTION

In survival analysis data are always subject to censoring. The right censoring is the most common type of the censoring. In right censoring the survival time is smaller than the observed right censoring time. However, in some cases the data are subjected to left as well as right censoring. In case of left censoring, an analyst can only have the information that the survival time is larger than or equal to the observed left censoring time. A more difficult censoring scheme is established when both the initial and final times are interval-censored. Such a situation is referred as double censoring and the data with both left and right censored observations are known as doubly censored data.

The analysis of the doubly censored data for simple (single) distribution has been performed by many authors. Fernandez [1] studied the maximum likelihood prediction based on type-II doubly censored samples from exponential distribution. Fernandez [2] investigated Bayesian estimation based on censored samples from Pareto populations. Khan et al. [3] discussed predictive inference from a two-parameter Rayleigh life model using doubly censored samples. Kim and Song [4] considered Bayesian estimation of parameters of generalised exponential distribution using doubly censored samples. Khan et al. [5] performed the sensitivity analysis of predictive modelling for responses from three-parameter Weibull model using doubly censored sample of cancer patients. Pak et al. [6] proposed an estimation of Rayleigh scale parameter under doubly type-II censoring from imprecise data.

In statistics a mixed model is signified as a convex fusion of other probability distributions. It can be used to model a full statistical population with subpopulations, where the components of mixture probability densities are the probability densities of the subpopulations. The mixed model may appropriately be used to the model data set, where the subsets of the whole data set own different properties that can best be modelled separately. They can be more mathematically manageable as the individual mixture components are more easily dealt with than the overall mixture density. The families of mixture distributions have many applications in different fields such as fisheries, botany, agriculture, economics, psychology, medicine, finance, electrophoresis, geology, communication theory and zoology.

Soliman [7] obtained estimators for the finite mixture of Rayleigh model on the basis of progressively censored data. Sultan et al. [8] analysed some properties of the mixture of two inverse Weibull distributions. Saleem and Aslam [9] presented a comparison of the maximum likelihood estimates with the Bayesian estimates under uniform and Jeffreys' priors for the parameters of the Rayleigh mixture. Kundu and Howalder [10] discussed the Bayesian estimation and prediction of the inverse Weibull distribution for type-II censored data. Saleem et al. [11] investigated the Bayesian properties of the mixture of Power function distribution using the complete and censored samples. Shi and Yan [12] studied the two-parameter exponential distribution under type-I censoring to obtain empirical Bayes estimates. Eluebaly and Bouguila [13] discussed a Bayesian approach to explore the finite generalised Gaussian mixed models which include several standard mixtures extensively used in signal and image processing applications such as Gaussian and Laplace. Sultan and Moisheer [14] developed an approximate Bayes estimation of the parameters and reliability function of the two-component mixture of inverse Weibull distributions under type-II censoring. The other contributions regarding Bayesian analysis of the mixed models can be seen from the work of Kazmi et al. [15], Ali et al. [16], Ali et al. [17], Feroze and Aslam [18], Feroze and Aslam [19], Feroze and Aslam [20], Sindhu et al. [21] and Sindhu et al. [22].

METHODS

This section contains the introduction of the model and the likelihood function, along with the derivation of posterior distributions, Bayes estimators and posterior risks. The prior elicitation is also discussed.

Proposed Mixed Model and Likelihood Function

The probability density function of the Rayleigh distribution with rate parameter λ_i is:

$$f_i\left(x_{ij}\right)=2x_{ij}\lambda_i^2\exp\left(-x_{ij}^2\lambda_i^2\right),\ 0<x_{ij}<\infty,\ \ \lambda_i^2>0,\ i=1,2 \text{ and } j=1,2,\ldots,n_i\ . \tag{1}$$

This function can be obtained by putting $\lambda^2=1/\theta^2$ in the probability density function used by Saleem and Aslam [9].

The cumulative distribution function of the distribution is

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$$F_i\left(x_{ij}\right)=1-\exp\left(-\lambda_i^2 x_{ij}^2\right), \qquad 0<x_{ij}<\infty, \quad \lambda_i^2>0, \; i=1,2 \text{ and } j=1,2,\ldots,n_i . \tag{2}$$

Saleem and Aslam [9] used the distribution function for a mixture of two-component densities. The same kind of mixture distribution with mixing weights (p_1, 1- p_1) can be written as

$$f\left(x\right)=p_1 f_1\left(x\right)+\left(1-p_1\right)f_2\left(x\right), \qquad 0<p_1<1. \tag{3}$$

Again considering the cumulative distribution function for the mixed model used by Saleem and Aslam [9], the cumulative distribution function for mixture distribution can be written as

$$F\left(x\right)=p_1 F_1\left(x\right)+\left(1-p_1\right)F_2\left(x\right). \tag{4}$$

Identifiability is a necessary condition for a model to produce precise inferences. Teicher [23] pioneered the study of identifiability of finite mixture distributions and showed that the class of scale parameters of the mixed models is identifiable. As we also use the scale parameters of the mixed model with a special case of Rayleigh distribution, the model is identifiable and we can use it for analysis.

The graphs of single and mixture of Raleigh models under different parametric values are presented in Figures 1 and 2 respectively. They tend to be more peaked for larger values of the parameters.

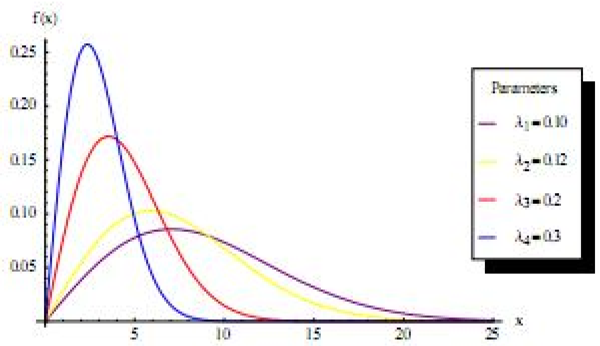

Figure 1. Graphs of single Raleigh distribution under different parametric values

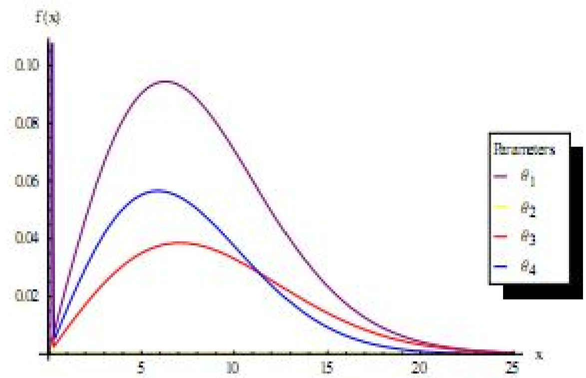

Figure 2. Graphs of mixture of Raleigh distributions under different parametric values
[θ_1 = (p1 = 0.45, λ_1 = 0.1, λ_2 = 0.12),
θ_2 = (p1 = 0.45, λ_1 = 10, λ_2 = 12),
θ_3 = (p1 = 0.45, λ_1 = 0.1, λ_2 = 12),
θ_4 = (p1 = 0.45, λ_1 = 10, λ_2 = 0.12)]

Likelihood function under doubly censored samples using single Rayleigh distribution

Consider a random sample of size 'n' from a Rayleigh distribution, and let x_r,..., x_s be the ordered observations that can only be observed. The remaining 'r – 1' smallest observations and the 'n – s' largest observations are assumed to be censored. Then the likelihood function for type-II doubly censored sample x = (x_r,..., x_s) as used by Feroze and Aslam [24] can be written as

$$L\left(\lambda|\text{x}\right)=\frac{n!}{\left(r-1\right)!\left(n-s\right)!}\left[F\left(x_r|\lambda\right)\right]^{r-1}\left[1-F\left(x_s|\lambda\right)\right]^{n-s}\prod_{i=r}^{s}f\left(x_i|\lambda\right)$$

$$L\left(\lambda|\text{x}\right)\propto\left[1-e^{-\lambda^2 x_r^2}\right]^{r-1}\left[e^{-\lambda^2 x_s^2}\right]^{n-s}\prod_{i=r}^{s}\lambda^2 e^{-\lambda^2 x^2} .$$

After simplifications, it becomes

$$L(\lambda|\mathrm{x}) \propto \sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\lambda^{2m}e^{-\lambda^2\left(\sum_{i=r}^{s}x_i^2+(n-s)x_s^2+jx_r^2\right)}, \tag{5}$$

where $m = n - s - r + 1$.

Likelihood function under doubly censored samples using mixed Rayleigh distributions

Consider a random sample of size 'n' from the Rayleigh distribution, and let $x_r, x_{r+1}, ..., x_s$ be the ordered observations that can only be observed. The remaining '$r-1$' smallest observations and the '$n-s$' largest observations are assumed to be censored. Now, based on causes of failure, the failed items are assumed to come either from subpopulation 1 or from subpopulation 2, so the $x_{1r_1}, ..., x_{1s_1}$ and $x_{2r_2}, ..., x_{2s_2}$ failed items come from the first and second subpopulations respectively. The rest of the observations which are less than x_r and greater than x_s are assumed to be censored from each component, where $x_s = \max\left(x_{1,s_1}, x_{2,s_2}\right)$ and $x_r = \min\left(x_{1,r_1}, x_{2,r_2}\right)$. Therefore, the numbers of failed items, $m_1 = s_1 - r_1 + 1$ and $m_2 = s_2 - r_2 + 1$, can be observed from the first and second subpopulations respectively. The remaining $n-(s-r+2)$ items are assumed to be censored observations and $s-r+2$ are the uncensored items, where $r = r_1 + r_2$, $s = s_1 + s_2$ and $m = m_1 + m_2$. Then assuming the causes of failure of the left censored items are identified, the likelihood function for type-II doubly censored sample, $\mathrm{x} = \left\{\left(x_{1r_1}, ..., x_{1s_1}\right), \left(x_{2r_2}, ..., x_{2s_2}\right)\right\}$, as done by Feroze and Aslam [20], can be written as

$$L(\lambda_1,\lambda_2,p_1|\mathrm{x}) \propto p_1^{s_1}(1-p_1)^{s_2}\left\{F_1\left(x_{(r_1)},\lambda_1\right)\right\}^{r_1-1}\left\{F\left(x_{(r_2)},\lambda_2\right)\right\}^{r_2-1}\left\{1-F\left(x_s,\lambda_1,\lambda_2\right)\right\}^{n-s}\left\{\prod_{i=r_1}^{s_1}f_1\left(x_{1(i)},\lambda_1\right)\right\}\left\{\prod_{i=r_2}^{s_2}f_2\left(x_{2(i)},\lambda_2\right)\right\}, \tag{6}$$

$$L(\lambda_1,\lambda_2,p_1|\mathrm{x}) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}p_1^{\,n-s-k_3+s_1}(1-p_1)^{s_2+k_3}$$
$$\times\lambda_1^{2m_1}\lambda_2^{2m_2}\exp\left\{-\lambda_1^2\left(\Omega\left(x_{1j}\right)\right)\right\}\exp\left\{-\lambda_2^2\left(\Omega\left(x_{2j}\right)\right)\right\}, \tag{7}$$

where $\Omega\left(x_{1j}\right) = \sum_{i=r_1}^{s_1}x^2_{1(i)} + \left(n-s-k_3\right)x^2_{(s)} + kx^2_{(r_1)}$, $\Omega\left(x_{2j}\right) = \sum_{i=r_2}^{s_2}x^2_{2(i)} + k_3x^2_{(s)} + kx^2_{(r_2)}$,

$m_1 = s_1 - r_1 + 1$ and $m_2 = s_2 - r_2 + 1$.

Bayes Estimation

This section covers the Bayesian analysis of single and mixture of Rayleigh distributions under uniform and Nakagami priors.

Bayesian estimation for single model using uniform and Nakagami priors

The uniform prior proposed by Laplace [25] for the parameter of the Raleigh distribution is

$$f(\lambda) \propto 1,\ \lambda > 0. \tag{8}$$

The posterior distribution under uniform prior using the likelihood function in equation (5) can be obtained as

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$$\pi(\lambda|\mathrm{x}) \propto \sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \lambda^{2m} e^{-\lambda^2\left(\sum_{i=r}^{s} x_i^2+(n-s)x_s^2+jx_r^2\right)} \quad (9)$$

The Nakagami distribution proposed by Nakagami [26] is used as a prior distribution for the rate parameter λ, with the hyper-parameters a and b given by

$$f(\lambda)=\frac{2a^a}{\Gamma(a)b^a}\lambda^{2a-1}\exp\left(\frac{-\lambda^2 a}{b}\right), \quad a,b>0,\lambda>0 . \quad (10)$$

The posterior distribution under Nakagami prior using the likelihood function in equation (5) can be obtained as

$$\pi(\lambda|\mathrm{x}) \propto \sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j} \lambda^{2m+2a-1} e^{-\lambda^2\left(\sum_{i=r}^{s} x_i^2+(n-s)x_s^2+jx_r^2+a/b\right)} \quad (11)$$

The graphs of the posterior distribution under uniform and Nakagami priors for doubly censored single Rayleigh distribution under different parametric values using a simulated sample of size n = 40 are presented in Figures 3 and 4 respectively. Again, the graphs for posterior distributions tend to be more peaked for larger values of the parameters. Similarly, the curves of posterior distribution under Nakagami prior are more peaked than those under uniform prior. In the case of Nakagami prior, the elicited values of hyper-parameters are used in the graphs.

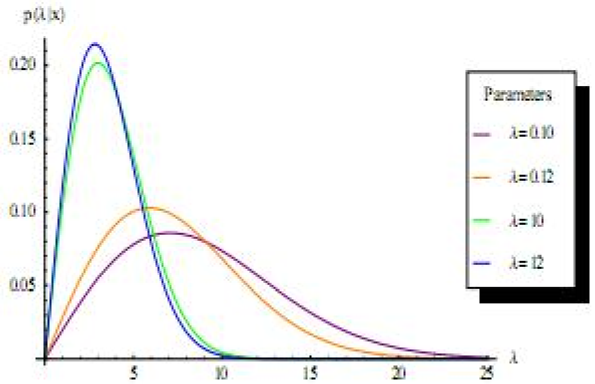

Figure 3. Graphs of posterior distribution under uniform prior

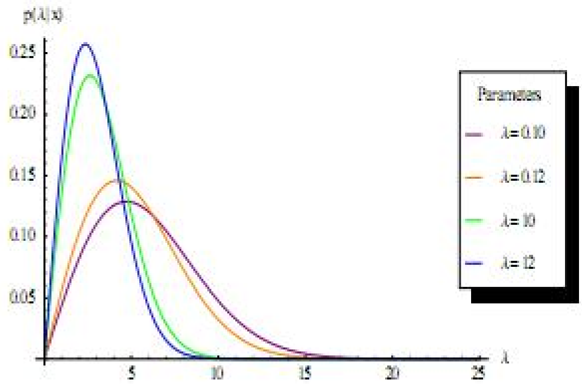

Figure 4. Graphs of posterior distribution under Nakagami prior

Bayesian estimation for mixed model using uniform and Nakagami priors

The uniform prior for the vector $\Theta = (\lambda_1, \lambda_2, p_1)$ of the mixed model can be assumed as

$$g(\Theta) \propto 1 . \quad (12)$$

By multiplying equation (12) with equation (7), the joint posterior density for the vector Θ, given the data, becomes

$$\pi(\Theta\,|\,\mathrm{x}) \propto \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\prod_{i=1}^{2}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}p_1^{\,n-s-k_3+s_1} \times(1-p_1)^{s_2+k_3}\lambda_i^{2m_i}\exp\left\{-\lambda_i^2\Omega(x_{ij})\right\} \quad (13)$$

For the Bayes estimation using Nakagami prior, let us assume that the parameters λ_i (i = 1, 2) and p_1 are independent random variables, and then we consider the following priors for different parameters. The prior for the rate parameters λ_i for i = 1, 2 is assumed as the Nakagami distribution with the hyper-parameters a_i and b_i given by

$$f_{\lambda_i}(\lambda_i) = \frac{2a_i^{a_i}}{\Gamma(a_i)b_i^{a_i}}\lambda_i^{2ai-1}\exp\left(\frac{-\lambda_i^2 a_i}{b_i}\right), \quad a_i, b_i > 0 \ . \tag{14}$$

The prior for p_1 is assumed to be the beta distribution, whose density is given by

$$f_p(p_1) = \frac{\Gamma(c_1+d_1)}{\Gamma(c_1)\Gamma(d_1)}p_1^{c_1-1}(1-p_1)^{d_1-1}, \quad c_1, d_1 > 0 \ . \tag{15}$$

From equations (14) and (15), we propose the following joint prior density of the vector $\Theta = (\lambda_1, \lambda_2, p_1)$:

$$g(\Theta) \propto \lambda_i^{2ai-1}\exp\left(\frac{-\lambda_i^2 a_i}{b_i}\right)p_1^{c_1-1}(1-p_1)^{d_1-1}, 0 < p_1 < 1,\ a_i > 0,\ b_i > 0,\ c_1 > 0,\ d_1 > 0 \tag{16}$$

By multiplying equation (16) with equation (7), the joint posterior density for the vector Θ, given the data, becomes

$$\begin{aligned}\pi(\Theta|\,\mathrm{x}) \propto &\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}\prod_{i=1}^{2}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}p_1^{n-s-k_3+s_1+c_1-1}(1-p_1)^{s_2+k_3+d_1-1} \\ &\times\lambda_i^{2(a_i+m_i)-1}\exp\left\{-\lambda_i^2\left(\frac{a_i}{b_i}+\Omega(x_{ij})\right)\right\}\end{aligned} \ . \tag{17}$$

The marginal distributions of λ_i (i = 1, 2) and p_1 can be obtained by integrating the nuisance parameters.

Figures 5 and 6 show the graphs of the marginal posterior distributions for both components under uniform prior, using a simulated sample of size n = 40. As observed in the case of single posterior models, the marginal posterior distributions tend to be more peaked for larger values of the parameters. Similarly, the curves of marginal posterior distributions under Nakagami prior are more peaked than those under uniform prior.

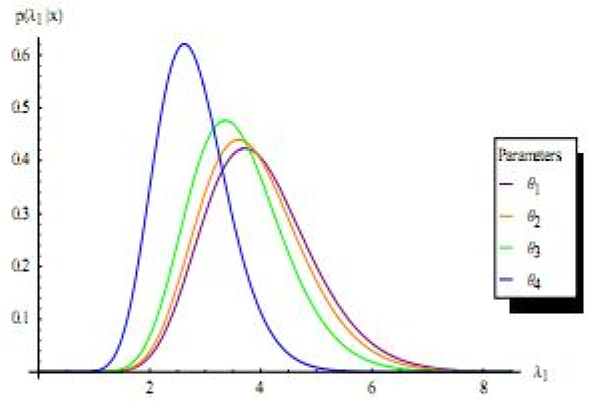

Figure 5. Graphs of marginal posterior distribution for first component under uniform prior

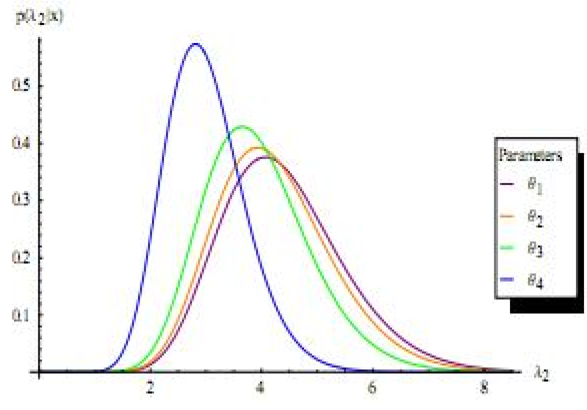

Figure 6. Graphs of marginal posterior distribution for second component under uniform prior

[θ_1 = (p1 = 0.45, λ_1 = 0.1, λ_2 = 0.12); θ_2 = (p1 = 0.45, λ_1 = 10, λ_2 = 12); θ_3 = (p1 = 0.45, λ_1 = 0.1, λ_2 = 12); θ_4 = (p1 = 0.45, λ_1 = 10, λ_2 = 0.12)]

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Bayes Estimation of Vector of Parameters Θ

The Bayesian point estimation is linked to a loss function in general, signifying the loss occurring when the estimate $\hat{\theta}$ differs from true parameter θ. As there is no specific rule of thumb that helps us to decide the appropriate loss function to be used, the squared error loss function (SELF) is used in this paper as it serves as standard loss. It is well known that under the SELF, the Bayes estimator of a function of the parameters is the posterior mean of the loss function and the risk is the posterior variance. It is defined as $l\left(\hat{\theta},\theta\right)=\left(\theta-\hat{\theta}\right)^2$. It was initially used in estimation problems when the unbiased estimator of θ was being considered. Another reason for its attractiveness is its relationship to the least squares theory. The use of SELF makes the calculations simpler.

The K-loss function (KLF), proposed by Wasan [27] and defined as $l\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2/\hat{\theta}\theta$, is well fitted for a measure of inaccuracy for an estimator of a scale parameter of the distribution defined as $R^+=\left(0,\infty\right)$. Under KLF, the Bayes estimates and posterior risks are defined as $\hat{\theta}=\sqrt{E(\theta\,|\,\mathbf{x})/E(\theta^{-1}\,|\,\mathbf{x})}$ and $\rho\left(\hat{\theta}\right)=2\left\{\sqrt{E(\theta\,|\,\mathbf{x})E(\theta^{-1}\,|\,\mathbf{x})}-1\right\}$ respectively. Recently Ali [28] used this loss function and also suggested a modified KLF.

In the Bayesian estimation in the case of single Rayleigh model, the Bayes estimator and posterior risk using SELF under uniform prior are respectively presented as

$$\hat{\lambda}_{(SELF)}=\frac{\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\Gamma\left(m+1\right)\left(\sum_{i=r}^{s}x_i^2+\left(n-s\right)x_s^2+jx_r^2\right)^{-(m+1)}}{\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\Gamma\left(m+1/2\right)\left(\sum_{i=r}^{s}x_i^2+\left(n-s\right)x_s^2+jx_r^2\right)^{-(m+1/2)}}$$

$$\rho\left(\hat{\lambda}_{(SELF)}\right)=\frac{\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\Gamma\left(m+3/2\right)\left(\sum_{i=r}^{s}x_i^2+\left(n-s\right)x_s^2+jx_r^2\right)^{-(m+3/2)}}{\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\Gamma\left(m+1/2\right)\left(\sum_{i=r}^{s}x_i^2+\left(n-s\right)x_s^2+jx_r^2\right)^{-(m+1/2)}}-\left(\hat{\lambda}_{(SELF)}\right)^2.$$

The Bayes estimators and posterior risks under Nakagami prior and KLF can be obtained with little modifications.

In the Bayesian estimation in the case of mixture of Rayleigh models, the respective marginal distribution of each parameter is used to derive the Bayes estimators and posterior risks for λ_1, λ_2 and p_1 under SELF and KLF. The Bayes estimators of λ_1, λ_2 and p_1 under SELF, assuming Nakagami prior, are given as

$$\hat{\lambda}_{1(SELF)}=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B\left(A_1,A_2\right)\Gamma\left(a_1+m_1+1/2\right)\Gamma\left(a_2+m_2\right)}{2\left\{a_1/b_1+\Omega\left(x_{1j}\right)\right\}^{(a_1+m_1+1/2)}2\left\{a_2/b_2+\Omega\left(x_{2j}\right)\right\}^{(a_2+m_2)}},$$

$$\hat{\lambda}_{2(SELF)}=N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B\left(A_1,A_2\right)\Gamma\left(a_1+m_1\right)\Gamma\left(a_2+m_2+1/2\right)}{2\left\{a_1/b_1+\Omega\left(x_{1j}\right)\right\}^{(a_1+m_1)}2\left\{a_2/b_2+\Omega\left(x_{2j}\right)\right\}^{(a_2+m_2+1/2)}},$$

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$$\hat{p}_{1(SELF)} = N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1+1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\{a_1/b_1+\Omega(x_{1j})\}^{(a_1+m_1)}2\{a_2/b_2+\Omega(x_{2j})\}^{(a_2+m_2)}}.$$

The posterior risks of λ_1, λ_2 and p_1 are given as

$$\rho(\hat{\lambda}_{1(SELF)}) = N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1+1)\Gamma(a_2+m_2)}{2\{a_1/b_1+\Omega(x_{1j})\}^{(a_1+m_1+1)}2\{a_2/b_2+\Omega(x_{2j})\}^{(a_2+m_2)}}-(\hat{\lambda}_{1(SELF)})^2,$$

$$\rho(\hat{\lambda}_{2(SELF)}) = N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2+1)}{2\{a_1/b_1+\Omega(x_{1j})\}^{(a_1+m_1)}2\{a_2/b_2+\Omega(x_{2j})\}^{(a_2+m_2+1)}}-(\hat{\lambda}_{2(SELF)})^2,$$

$$\rho(\hat{p}_{1(SELF)}) = N^{-1}\sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1+2,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\{a_1/b_1+\Omega(x_{1j})\}^{(a_1+m_1)}2\{a_2/b_2+\Omega(x_{2j})\}^{(a_2+m_2)}}-(\hat{p}_{1(SELF)})^2,$$

where N^{-1} is formulised as

$$N^{-1} = \sum_{k_1=0}^{r_1-1}\sum_{k_2=0}^{r_2-1}\sum_{k_3=0}^{n-s}(-1)^{k_1+k_2}\binom{r_1-1}{k_1}\binom{r_2-1}{k_2}\binom{n-s}{k_3}\frac{B(A_1,A_2)\Gamma(a_1+m_1)\Gamma(a_2+m_2)}{2\{a_1/b_1+\Omega(x_{1j})\}^{(a_1+m_1)}2\{a_2/b_2+\Omega(x_{2j})\}^{(a_2+m_2)}},$$

$A_1 = n-s-k_3+s_1+c_1$ and $A_2 = s_2+k_3+d_1$.

Similarly, expressions for Bayes estimators and their posterior risks under KLF and uniform prior can be obtained with little modifications.

Elicitation

In Bayesian analysis the elicitation of opinion is an important step. It helps us to easily understand the expert's opinions. In statistical inference the hyper-parameters of a prior distribution are determined by the characteristics of a certain predictive distribution proposed by an expert. In this study we focus on a method of elicitation based on prior predictive distribution. The elicitation of the hyper-parameters from the prior $p(\lambda)$ is a complex task. The prior predictive distribution is used for the elicitation of the hyper-parameters, which is compared with the experts' judgement about this distribution, and then the hyper-parameters are chosen in such a way so as to make the judgment agree as closely as possible with the given distribution. More detail on this may be obtained from the work of Grimshaw et al. [29], O'Hagan et al. [30], Jenkinson [31] and Leon et al. [32]. Aslam [33] suggested the method of elicitation that compares the prior predictive distribution with the expert's assessments about the distribution involved and then chooses the hyper-parameters that make the assessment agree closely with the member of the family. This method has also been used by Kazmi et al. [15]. The prior predictive distribution is derived using the following formula:

$$p(y) = \int_{\Theta} p(y|\Theta)p(\Theta)d\Theta$$

For the elicitation for single Rayleigh model under Nakagami distribution, the prior predictive distribution using Nakagami prior is

$$p(y) = 2(ab^{-1})^a \frac{ya}{(y^2+ab^{-1})^{(a+1)}},\ y>0 \qquad (18)$$

For the elicitation of the two hyper-parameters, two intervals are considered. From equation (18), the experts' probabilities/assessments are supposed to be 0.10 for each case. The two integrals for equation (18) are considered with the following limits of values of random variable 'Y': (0, 10) and (10, 20) respectively. For the elicitation of the hyper-parameters a and b, these two equations are solved simultaneously through a computer program developed in *SAS* package using the command of PROC SYSLIN. Thus, the values of the hyper-parameters obtained by applying this methodology are 0.010348 and 0.735261 respectively.

For the elicitation for mixture of Rayleigh models under Nakagami distribution, the prior predictive distribution using Nakagami prior is

$$p(y) = 2\left(a_1 b_1^{-1}\right)^{a_1} \frac{y a_1 c_1}{\left(c_1+d_1\right)\left(y^2+a_1 b_1^{-1}\right)^{(a_1+1)}} + 2\left(a_2 b_2^{-1}\right)^{a_2} \frac{y a_2 d_1}{\left(c_1+d_1\right)\left(y^2+a_2 b_2^{-1}\right)^{(a_2+1)}},\ y>0 \tag{19}$$

For the elicitation of the six hyper-parameters, six different intervals are considered. From equation (19), the expert's probabilities/assessments are supposed to be 0.10 for each case. The six integrals for equation (19) are considered with the following limits of values of random variable 'Y': (0, 10), (10, 20), (20, 30), (30, 40), (40, 50) and (50, 60) respectively. For the elicitation of the hyper-parameters a_1, a_2, b_1, b_2, c_1 and d_1, these six equations are solved simultaneously through a computer program developed in *SAS* package using the command of PROC SYSLIN. Thus, the values of the hyper-parameters obtained by applying this methodology are 0.000231, 0.012109, 0.52114, 4.99325, 0.52130 and 0.14790 respectively.

RESULTS AND DISCUSSION

A simulation study was carried out to investigate the performance of Bayes estimates under a tenfold choice of parametric values, different sample sizes, and different values of mixing parameter. We took random samples of sizes n = 20, 40 and 80 from single and two-component mixture of Rayleigh distributions with tenfold choice of the parameters. The choice of censoring time was made in such a way that the censoring rate in the resultant sample was approximately 20%. The processes for the generation of data from the single and mixed models are discussed in the following sections.

Simulation

In the case of single Rayleigh model the parametric space is considered as $\lambda \in \left(0.1,10\right)$. The data are generated using the following steps.

Step 1: Draw samples of size 'n' from Rayleigh model using inverse transformation technique by taking the generator $x=\sqrt{-\lambda^{-2}\ln\left(1-u\right)}$, where 'u' is a uniform random variable.

Step 2: Determine the test termination points on the left and right, i.e. the values of x_r (test termination point from left) and x_s (test termination point from right).

Step 3: The observations which are less than x_r and greater than x_s are considered to be censored.

Step 4: Use the observations which are greater than or equal to x_r and less than or equal to x_s for the analysis.

Step 5: Repeat steps 1- 4 ten thousand times and calculate the average of the estimates.

It should be noted that the values of x_r and x_s are assumed to be such that an equal number of values are censored from left and right, i.e 10% from each side.

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In the case of mixture of Rayleigh models, the parametric space is considered as $(\lambda_1, \lambda_2) \in \{(0.1,\ 0.12), (10,\ 12), (0.1,\ 12), (10,\ 0.12)\}$, $p_1 = 0.45$. To generate the mixture data we make use of probabilistic mixing with probabilities p_1 and (1- p_1). A uniform number u is generated n times and if $u < p_1$, the observation is taken randomly from F_1 (Rayleigh distribution with parameter λ_1), otherwise from F_2 (Rayleigh distribution with parameter λ_2). To implement censored samplings, we consider that the $x_{1r_1},...,x_{1s_1}$ and $x_{2r_2},...,x_{2s_2}$ failed items come from the first and second subpopulations respectively. The values of x_r and x_s are assumed to be such that an equal number of values are censored from left and right, i.e. 10% from each side. The simulated data sets are obtained using the following steps:

Step 1: Draw samples of size 'n' from each component of the mixture model using inverse transformation technique by taking the generator $x = \sqrt{-\lambda^{-2}\ln(1-u)}$, where 'u' is a uniform random variable.

Step 2: Generate a uniform random number u, corresponding to each observation.

Step 3: If $u \le p_1$, take the observation from the first subpopulation; if $u > p_1$, take the observation from the second subpopulation.

Step 4: Determine the test termination points on the left and right, i.e. the values of x_r (test termination point from left) and x_s (test termination point from right).

Step 5: The observations which are less than x_r and greater than x_s are considered to be censored from each component.

Step 6: Use the observations which are greater than or equal to x_r and less than or equal to x_s for the analysis.

Step 7: Repeat steps 1 - 6 ten thousand times and calculate the average of the estimates.

Table 1 represents the Bayes estimates and corresponding posterior risks for doubly censored single Rayleigh model. From this table, it can be seen that estimated values of the parameters converge to the true values of the parameters, and the amount of the corresponding posterior risk decreases with increase in sample size. As the amount of posterior risks associated with the estimates under Nakagami prior is smaller than that under uniform prior, so the performance of Nakagami prior is better than uniform prior. On the other hand, the performance of SELF is better than KLF when the parametric values are small, while in the case of larger values of the parameter, the performance of KLF is better than SELF.

Numerical results of the simulation study, presented in Tables 2-5, reveal interesting properties of the proposed Bayes estimators for the mixture of Rayleigh models. The estimated values of the parameters converge to the true values of the parameters, and the amounts of posterior risks tend to decrease by increasing the sample size. Another interesting point concerning the posterior risks of the estimates of λ_1 and λ_2 is that increasing (or decreasing) the proportion of component in the mixture decreases (or increases) the amount of posterior risks for the estimates of λ_1.

The Bayes estimates of the lifetime parameters are either over- or underestimated. The estimates of the mixing parameter (p_1) also have mixed behaviour: sometimes overestimated and other times underestimated. The performance of Nakagami prior seems better than uniform prior as the associated magnitude of posterior risks is smaller in the case of Nakagami prior. In comparing the loss functions, it is assessed that the magnitude of posterior risks under SELF is smaller than that under KLF for a smaller choice of true parametric values, i.e. for $(\lambda_1, \lambda_2) = (0.1, 0.12)$. On the other hand, for quite larger values of parameters, i.e. for $(\lambda_1, \lambda_2) = (10, 12)$ and for significantly

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different values of parameters, i.e. for (λ_1, λ_2) = (0.1, 12) and (10, 0.12), the KLF produces better results. It should also be mentioned here that the SELF in the majority of cases produces better convergence than the KLF.

Table 1. Bayes estimators and posterior risks (in brackets) under uniform prior and Nakagami prior for single model using $\lambda \in (0.1, 10)$

n	Uniform prior				Nakagami prior			
	SELF		KLF		SELF		KLF	
	$\lambda = 0.10$	$\lambda = 10$	$\lambda = 0.10$	$\lambda = 10$	$\lambda = 0.10$	$\lambda = 10$	$\lambda = 0.10$	$\lambda = 10$
20	0.10664 (0.00051)	11.41053 (4.47964)	0.10540 (0.08900)	10.72593 (0.08795)	0.10550 (0.00050)	10.88324 (4.27545)	0.10427 (0.08647)	10.23027 (0.08394)
40	0.10855 (0.00023)	10.93152 (2.48177)	0.10440 (0.04732)	10.66980 (0.04397)	0.10375 (0.00023)	10.42546 (2.45224)	0.10394 (0.04572)	10.17586 (0.04344)
80	0.10329 (0.00012)	10.47666 (1.10186)	0.10252 (0.02175)	10.44836 (0.02387)	0.10221 (0.00012)	10.10949 (1.07332)	0.10145 (0.02151)	10.08218 (0.02325)

Table 2. Bayes estimators and posterior risks (in brackets) under uniform prior for mixed model using $(\lambda_1, \lambda_2, p_1) = $ (0.1, 0.12, 0.45) and (10, 12, 0.45)

n	SELF					
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
20	0.107209 (0.000503)	0.132703 (0.000564)	0.515202 (0.013700)	11.086576 (4.321071)	13.549166 (4.923963)	0.517763 (0.013560)
40	0.100505 (0.000230)	0.130256 (0.000314)	0.509777 (0.007576)	10.567887 (2.488090)	13.461262 (2.797394)	0.502870 (0.007513)
80	0.103622 (0.000118)	0.127356 (0.000167)	0.502084 (0.003912)	9.912993 (1.088862)	13.026748 (1.624447)	0.484931 (0.003864)
KLF						
20	0.102959 (0.087594)	0.129129 (0.072430)	0.497539 (0.133359)	10.221412 (0.084839)	12.865358 (0.070344)	0.502526 (0.131133)
40	0.106241 (0.046633)	0.128014 (0.038191)	0.487087 (0.071148)	10.014874 (0.044080)	12.850151 (0.038230)	0.493996 (0.071735)
80	0.094853 (0.021888)	0.123316 (0.020362)	0.484133 (0.034902)	10.285132 (0.023584)	12.557065 (0.020691)	0.475643 (0.037357)

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Table 3. Bayes estimators and posterior risks (in brackets) under uniform prior for mixed model using $\left(\lambda_1, \lambda_2, p_1\right) = $ (0.10, 12, 0.45) and (10, 0.12, 0.45)

n	SELF					
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
20	0.098497 (0.000316)	14.214270 (4.921202)	0.552917 (0.012464)	12.212382 (4.343465)	0.117154 (0.000331)	0.420005 (0.012162)
40	0.091638 (0.000135)	13.945293 (2.401793)	0.543338 (0.006765)	12.073530 (2.219332)	0.119216 (0.000160)	0.432426 (0.006444)
80	0.095115 (0.000067)	13.649684 (1.193893)	0.534044 (0.003386)	11.518744 (1.119705)	0.121828 (0.000077)	0.450936 (0.003253)
KLF						
20	0.092387 (0.068477)	13.492645 (0.055819)	0.541278 (0.100292)	11.771346 (0.068494)	0.108699 (0.054418)	0.436242 (0.152270)
40	0.096671 (0.033686)	13.246342 (0.026147)	0.528118 (0.052829)	11.638419 (0.033122)	0.121915 (0.026645)	0.440904 (0.083827)
80	0.098805 (0.016265)	12.535256 (0.013083)	0.520105 (0.027307)	11.569858 (0.016581)	0.124054 (0.013209)	0.445349 (0.042834)

Table 4. Bayes estimators and posterior risks (in brackets) under uniform prior for mixed model using $\left(\lambda_1, \lambda_2, p_1\right) = $ (0.1, 0.12, 0.45) and (10, 12, 0.45)

n	SELF					
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
20	0.104076 (0.000479)	0.127713 (0.000558)	0.498425 (0.013229)	10.82959 (4.256710)	12.9605 (4.8388)	0.497084 (0.013244)
40	0.099427 (0.000223)	0.12652 (0.000306)	0.48622 (0.007231)	10.12890 (2.39442)	12.88640 (2.71684)	0.479166 (0.007305)
80	0.099036 (0.000114)	0.125807 (0.000161)	0.478841 (0.003865)	9.61493 (1.05203)	12.67810 (1.58376)	0.462094 (0.003820)
KLF						
20	0.101884 (0.086648)	0.123181 (0.069120)	0.480102 (0.129905)	10.11040 (0.083902)	12.73760 (0.068839)	0.481255 (0.12979)
40	0.101669 (0.045012)	0.123008 (0.037679)	0.471869 (0.070063)	9.85076 (0.043278)	12.45580 (0.037003)	0.474996 (0.069297)
80	0.090768 (0.021446)	0.121778 (0.019760)	0.470942 (0.034345)	9.91883 (0.022678)	12.11990 (0.019891)	0.468498 (0.036650)

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Table 5. Bayes estimators and posterior risks (in brackets) under uniform prior for mixed model using $(\lambda_1, \lambda_2, p_1) =$ (0.10, 12, 0.45) and (10, 0.12, 0.45)

n	SELF					
	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
20	0.095619 (0.000301)	13.67980 (4.868910)	0.534912 (0.012036)	11.92930 (4.278770)	0.112064 (0.000325)	0.403231 (0.011879)
40	0.090655 (0.000131)	13.54530 (2.343030)	0.51823 (0.006457)	11.5720 (2.13578)	0.114125 (0.000155)	0.412042 (0.006265)
80	0.090905 (0.000065)	13.48370 (1.148460)	0.509322 (0.003346)	11.17240 (1.08183)	0.118567 (0.000075)	0.42970 (0.003216)
KLF						
20	0.0914225 (0.067737)	12.87110 (0.053268)	0.522308 (0.097694)	11.64350 (0.067737)	0.10762 (0.053254)	0.417776 (0.15071)
40	0.092511 (0.032515)	12.72830 (0.025796)	0.511618 (0.052023)	11.44770 (0.032519)	0.118174 (0.025790)	0.423946 (0.080978)
80	0.09455 (0.015937)	12.37891 (0.012696)	0.505934 (0.026871)	11.15780 (0.015944)	0.119735 (0.012698)	0.438659 (0.042023)

Real Data Analysis

We analysed a real data set to illustrate the methodology discussed in the previous section. In order to show the usefulness of the proposed mixture distribution, we applied the results to the survival times (in years) of a set of cancer patients given chemotherapy treatment. The details of this data can be seen from Bekker et al. [34] and the references cited therein. We used the Kolmogorov-Smirnov and chi-square tests to see whether the data follow the Rayleigh distribution. These tests, with p-values of 0.2170 and 0.2681 respectively, indicate that the data follow the Rayleigh distribution at 5% level of significance. The original data consisted of 46 values regarding survival times (in years) of cancer patients given chemotherapy treatment. A uniform number 'u' was generated for each of the 46 values. If $u < p_1$, the observation was allotted to F_1 (the Rayleigh distribution with parameter λ_1, i.e. the first component of the mixture); otherwise to F_2 (from the Rayleigh distribution with parameter λ_2, i.e. the second component of the mixture). The observations allotted to the first and second components of the mixture were considered as population-I and population-II respectively. The observations which were less than 0.047 (i.e. x_r) and greater than 3.978 (i.e. x_s) were assumed to be censored from left and right respectively from each population. The remaining doubly censored data from population-I and population-II are presented in Table 6. The values of x_r and x_s were assumed to be such that an equal number of values were censored from left and right, i.e. 10% from each side. The Bayes estimates were obtained assuming non-informative and informative priors under SELF and KLF.

Tables 7-8 contain the Bayesian estimation of parameters of the single and mixture of Rayleigh distributions under real-life data set. The findings from the real-life analysis are in close agreement with those from the simulation study. It should be noted that the estimates under SELF and Nakagami prior are associated with smaller amounts of posterior risks.

Table 6. Doubly-censored, real-life data of the mixture regarding survival times (in years) of cancer patients given chemotherapy treatment

Population-I	Population-II
0.197, 0.534, 0.115, 0.296, 0.121, 0.466, 0.529, 1.447, 0.863, 0.132, 0.395, 0.696, 2.825, 3.658, 3.978, 3.743, 2.343, 2.178, 0.540, 4.003, 1.553, 1.485, 2.83, 2.416	0.260, 1.099, 0.501, 0.458, 0.641, 0.334, 0.570, 0.164, 0.203, 0.282, 0.047, 1.271, 1.589, 1.326, 0.841, 2.444

Table 7. Bayes estimators and posterior risks (in brackets) for single Rayleigh model using real data set

Uniform prior		Nakagami prior	
SELF	KLF	SELF	KLF
0.386117 (0.002481)	0.383239 (0.035318)	0.382462 (0.002451)	0.377970 (0.035057)

Table 8. Bayes estimators and posterior risks (in brackets) for mixed Rayleigh models using real data set

Prior	SELF			KLF		
$p_1 = 0.45$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{p}_1$
Uniform prior	0.381184 (0.002447)	0.756889 (0.007935)	0.535297 (0.006711)	0.378104 (0.034814)	0.723677 (0.031677)	0.532335 (0.053736)
Nakagami prior	0.377204 (0.002420)	0.722023 (0.007572)	0.516536 (0.006537)	0.373998 (0.034429)	0.716491 (0.030999)	0.509802 (0.053186)

CONCLUSIONS

In this article the Bayesian inference of the single and mixture of Rayleigh models under type-II double censoring has been considered assuming informative and non-informative priors. The simulation study has displayed some interesting properties of the Bayes estimates. It is noted in each case that the posterior risks of estimates of lifetime parameters are reduced as the sample size increases. The performance of the Nakagami prior in each case (single or mixed model) is found to be better than the uniform prior. On the other hand, the performance of SELF is better for a smaller choice of parametric values, while for larger values of parameter the performance of KLF is better. This property is also the same in the case of single and mixed models. A real-life example further strengthens the findings from the simulation study. The study can further be extended by considering some other censoring techniques and by using some more flexible probability distributions.

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