

Appendix

- A1** Kittikorn Nakprasit, Weerayuth Nilsrakoo and **Satit Saejung**, *Weak and strong convergence theorems of an implicit iteration process for a countable family of nonexpansive mappings*, Fixed Point Theory and Applications, Volume 2008 (2008), Article ID 732193, 18 pages doi:10.1155/2008/732193 (impact factor(2007): 0.562)
- A2** Weerayuth Nilsrakoo and **Satit Saejung**, *Weak convergence theorems for a countable family of Lipschitzian mappings*, Journal of Computational and Applied Mathematics 230(2), (2009) 451–462. (impact factor(2008): 1.048)
- A3** Weerayuth Nilsrakoo and **Satit Saejung**, *Strong convergence theorems for a countable family of quasi-Lipschitzian mappings and its applications*, Journal of Mathematical Analysis and Applications 356(2009) 154–167. (impact factor(2008): 1.046)
- A4** Weerayuth Nilsrakoo and **Satit Saejung**, *Convergence theorems for a countable family of Lipschitzian mappings*, Applied Mathematics and Computation 214 (2009) 595–606. (impact factor(2008): 0.961)
- A5** Weerayuth Nilsrakoo and **Satit Saejung**, *An implicit iteration for a countable family of nonexpansive mappings in Banach spaces*, Numerical Functional Analysis and Optimizations 30 (5-6) (2009) 591–612. (impact factor(2008): 0.568)

- A6** Daruni Boonchari and **Satit Saejung**, *Weak and strong convergence theorems of an implicit iteration for a countable family of continuous pseudocontractive mappings*, Journal of Computational and Applied Mathematics, 233(4)(2009) 1108–1116. (impact factor(2008): 1.048)
- A7** Daruni Boonchari and **Satit Saejung**, *Approximation of common fixed points of a countable family of relatively nonexpansive mappings*, Fixed Point Theory and Applications, Volume 2010, Article ID 407651, 26 pages. doi:10.1155/2010/407651 (impact factor(2008): 0.728)

- A1. Kittikorn Nakprasit, Weerayuth Nilsrakoo and **Satit Saejung**, *Weak and strong convergence theorems of an implicit iteration process for a countable family of nonexpansive mappings*, Fixed Point Theory and Applications, Volume 2008 (2008), Article ID 732193, 18 pages doi:10.1155/2008/732193 (impact factor(2007): 0.562)

Research Article

Weak and Strong Convergence Theorems of an Implicit Iteration Process for a Countable Family of Nonexpansive Mappings

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Using the implicit iteration and the hybrid method in mathematical programming, we prove weak and strong convergence theorems for finding common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Our results include many convergence theorems by Xu and Ori (2001) and Zhang and Su (2007) as special cases. We also apply our method to find a common element to the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem. Finally, we propose an iteration to obtain convergence theorems for a continuous monotone mapping.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a nonempty subset of H . A mapping $T : C \rightarrow H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C. \quad (1.1)$$

We denote by $F(T)$ the set of all fixed points of T . If C is bounded closed convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [1]). We write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x . There are many methods for approximating fixed points of a nonexpansive mapping. Xu and Ori [2] introduced the following implicit iteration process to approximate a common fixed point of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$: an initial point $x_0 \in C$,

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1} \\
&\vdots
\end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. The iteration above can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1, \tag{1.3}$$

where $T_n \equiv T_{n \bmod N}$, here the mod N function takes values in $\{1, 2, \dots, N\}$. They proved that this process converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$. Recently, to obtain a strong convergence theorem, Zhang and Su [3] modify iteration processes (1.3) by the implicit hybrid method for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$: an initial point $x_0 \in C$,

$$\begin{aligned}
&x_0 \in C \text{ is arbitrary,} \\
&y_n = \alpha_n x_n + (1 - \alpha_n) T_n z_n, \\
&z_n = \beta_n y_n + (1 - \beta_n) T_n y_n, \\
&C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots,
\end{aligned} \tag{1.4}$$

where $T_n \equiv T_{n \bmod N}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1]$ with $\alpha_n < 1$.

In this paper, we establish weak and strong convergence theorems for finding common fixed points of a countable family of nonexpansive mappings in a real Hilbert space. Our results include many convergence theorems by [2, Theorems 2] and [3, Theorems 2.4] as special cases. The new iteration introduced in this paper is applied to find a common element to the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem. We also propose an iteration to obtain convergence theorems for a continuous monotone mapping.

2. Preliminaries

Let H be a real Hilbert space. Then,

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfies the following.

(1) Opial's condition [4], that is, for any sequence $\{x_n\}$ with $x_n \rightarrow x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\| \quad (2.3)$$

holds for every $y \in H$ with $y \neq x$.

(2) The Kadec-Klee property [1], that is, for any sequence $\{x_n\}$ with $x_n \rightarrow x$ and $\|x_n\| \rightarrow \|x\|$ together implies $\|x_n - x\| \rightarrow 0$.

Let C be a nonempty closed convex subset of H . Then, for any $x \in H$, there exists the nearest point $P_C x$ in C such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C. \quad (2.4)$$

Such a mapping, P_C is called the metric projection of H onto C . We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{iff} \quad \langle x - z, z - y \rangle \geq 0 \quad \forall y \in C. \quad (2.5)$$

Lemma 2.1 (see [5, Lemma 1]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n \quad \forall n \geq 1, \quad (2.6)$$

and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.2 (see [6, Lemma 2.2]). *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$. Then, $\liminf_{n \rightarrow \infty} b_n = 0$.*

Lemma 2.3 (see [7, Lemma 3.2]). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{x_n\}$ be a sequence in H such that*

$$\|x_{n+1} - y\| \leq \|x_n - y\| \quad \forall y \in C, n \in \mathbb{N}. \quad (2.7)$$

Then, the sequence $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

To deal with a family of mappings, the following conditions are introduced. Let C be a subset of a Banach space, let $\{T_n\}$ and \mathcal{T} be families of mappings of C with $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, where $F(\mathcal{T})$ is the set of all common fixed points of all mappings in \mathcal{T} .

(a) $\{T_n\}$ is said to satisfy the AKTT-condition [8] if for each bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup \{ \|T_{n+1}z - T_n z\| : z \in B \} < \infty. \quad (2.8)$$

(b) $\{T_n\}$ is said to satisfy the NST-condition (I) with \mathcal{T} [9] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \quad \text{implies} \quad \lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0 \quad \forall T \in \mathcal{T}. \quad (2.9)$$

In particular, if $\mathcal{T} = \{T\}$, that is, \mathcal{T} consists of one mapping T , then $\{T_n\}$ is said to satisfy the NST-condition (I) with T .

(c) $\{T_n\}$ is said to satisfy the NST-condition (II) [9] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0 \text{ implies } \lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0 \quad \forall m \in \mathbb{N}. \quad (2.10)$$

Inspired by conditions above, we introduce the following one.

(d) $\{T_n\}$ is said to satisfy the NST*-condition with \mathcal{T} if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0, \quad \lim_{n \rightarrow \infty} \|z_n - z_{n+1}\| = 0 \quad (2.11)$$

imply that $\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0$ for all $T \in \mathcal{T}$. In particular, if $\mathcal{T} = \{T\}$, then we simply say that $\{T_n\}$ satisfies the NST*-condition with T .

Remark 2.4. (i) If $\{T_n\}$ satisfies the NST-condition (I) with \mathcal{T} , then $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} .

(ii) If $\{T_n\}$ satisfies the NST-condition (II), then $\{T_n\}$ satisfies the NST*-condition with $\{T_n\}$.

Lemma 2.5 (see [8, Lemma 3.2]). *Let C be a nonempty closed subset of a Banach space, and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition, then there exists a mapping $T : C \rightarrow C$ such that*

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in C, \quad (2.12)$$

and $\lim_{n \rightarrow \infty} \sup \{\|Tz - T_n z\| : z \in B\} = 0$ for each bounded subset B of C .

Lemma 2.6. *Let C be a nonempty closed subset of a Banach space, and let $\{T_n\}$ be a family of mappings of C into itself which satisfies AKTT-condition and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let T be the mapping from C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then, $\{T_n\}$ satisfies the NST-condition (I) with T . This implies that $\{T_n\}$ satisfies the NST*-condition with T .*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$. We apply Lemma 2.5 to get

$$\begin{aligned} \|z_n - Tz_n\| &\leq \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \\ &\leq \|z_n - T_n z_n\| + \sup \{\|T_n z - Tz\| : z \in \{z_n\}\} \longrightarrow 0. \end{aligned} \quad (2.13)$$

Hence, we obtain that $\{T_n\}$ satisfies the NST-condition (I) with T . This completes the proof. \square

Lemma 2.7. *Let C be a nonempty subset of a Banach space, and let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Then, $\{T_n\}$ satisfies NST*-condition with $\mathcal{T} = \{T_1, T_2, \dots, T_N\}$, where $T_n \equiv T_{n \bmod N}$.*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0, \quad \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0. \quad (2.14)$$

Obviously, it is easy to see that $\lim_{n \rightarrow \infty} \|z_{n+i} - z_n\| = 0$ for each $i = 1, 2, \dots, N$. Consequently,

$$\begin{aligned} \|z_n - T_{n+i}z_n\| &\leq \|z_n - z_{n+i}\| + \|z_{n+i} - T_{n+i}z_{n+i}\| + \|T_{n+i}z_{n+i} - T_{n+i}z_n\| \\ &\leq 2\|z_n - z_{n+i}\| + \|z_{n+i} - T_{n+i}z_{n+i}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (2.15)$$

This implies that $\lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0$ for each $m = 1, 2, \dots, N$. This completes the proof. \square

Remark 2.8. There are families of mappings $\{T_n\}$ and \mathcal{T} such that

- (1) $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} ;
- (2) $\{T_n\}$ fails the NST-condition (I) with \mathcal{T} and the NST-condition (II).

The following example shows that the NST*-condition with \mathcal{T} is strictly weaker than NST-condition (I) with \mathcal{T} and the NST-condition (II).

Example 2.9. Let $H := \mathbb{R}^2$ and $C := [0, 1] \times [0, 1]$. Define $T_1, T_2 : C \rightarrow C$ as follows:

$$T_1(x, y) = (x, 1 - y), \quad T_2(x, y) = (1 - x, y) \quad (2.16)$$

for all $(x, y) \in C$. Hence, T_1 and T_2 are nonexpansive mappings with

$$F(T_1) \cap F(T_2) = \left([0, 1] \times \left\{ \frac{1}{2} \right\} \right) \cap \left(\left\{ \frac{1}{2} \right\} \times [0, 1] \right) = \left\{ \left(\frac{1}{2}, \frac{1}{2} \right) \right\} \neq \emptyset. \quad (2.17)$$

Let $T_n = T_{n \pmod{2}}$. By Lemma 2.7, we have $\{T_n\}$ satisfies NST*-condition with $\{T_1, T_2\}$.

- (a) $\{T_n\}$ fails the NST-condition (I) with $\mathcal{T} = \{T_1, T_2\}$. In fact, let $z_{2n-1} = (1, 1/2)$ and $z_{2n} = (1/2, 1)$ for all $n \in \mathbb{N}$. Then, $z_{2n-1} \in F(T_{2n-1}) = F(T_1)$ and $z_{2n} \in F(T_{2n}) = F(T_2)$. In particular, $\|z_n - T_n z_n\| \equiv 0$. Clearly,

$$\|z_n - T_1 z_n\| \rightarrow 0, \quad \|z_n - T_2 z_n\| \rightarrow 0. \quad (2.18)$$

Hence, $\{T_n\}$ fails the NST-condition (I) with $\{T_1, T_2\}$.

- (b) $\{T_n\}$ fails the NST-condition (II). To this end, let $z_{4n-3} = (1/4, 1/4)$, $z_{4n-2} = (1/4, 3/4)$, $z_{4n-1} = (3/4, 3/4)$, and $z_{4n} = (3/4, 1/4)$ for all $n \in \mathbb{N}$. Then, $\|z_{n+1} - T_n z_n\| \equiv 0$. But,

$$\|z_n - T_1 z_n\| \rightarrow 0, \quad \|z_n - T_2 z_n\| \rightarrow 0. \quad (2.19)$$

Hence, $\{T_n\}$ fails the NST-condition (II).

Lemma 2.10 (see [10]). *Let C be a nonempty closed convex subset of a strictly convex Banach space, S and T be two nonexpansive mappings of C into itself with a common fixed point, and $0 < \beta < 1$. Let U be a mapping defined by*

$$U = T(\beta I + (1 - \beta)S), \quad (2.20)$$

where I is the identity mapping. Then, U is a nonexpansive mapping from C into itself and $F(U) = F(T) \cap F(S)$.

Lemma 2.11. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . Let $\{U_n\}$ be a family of nonexpansive mappings from C into itself defined by*

$$U_n = T_n(\beta_n I + (1 - \beta_n)T_n) \quad (2.21)$$

for all $n \in \mathbb{N}$, where I is the identity mapping, and $\{\beta_n\}$ is a sequence in $[a, 1]$ for some $a \in (0, 1]$. Then, $\{U_n\}$ satisfies the NST*-condition with \mathcal{T} .

Proof. By Lemma 2.10, we have $F(U_n) = F(T_n)$ for all $n \in \mathbb{N}$ and so,

$$\bigcap_{n=1}^{\infty} F(U_n) = F(\mathcal{T}) \neq \emptyset. \quad (2.22)$$

Let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n \rightarrow \infty} \|z_n - U_n z_n\| = 0, \quad \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0. \quad (2.23)$$

Since

$$\begin{aligned} \|z_n - T_n z_n\| &\leq \|z_n - U_n z_n\| + \|T_n(\beta_n z_n + (1 - \beta_n)T_n z_n) - T_n z_n\| \\ &\leq \|z_n - U_n z_n\| + (1 - \beta_n) \|z_n - T_n z_n\| \\ &\leq \|z_n - U_n z_n\| + (1 - a) \|z_n - T_n z_n\|, \end{aligned} \quad (2.24)$$

it follows that

$$\|z_n - T_n z_n\| \leq \frac{1}{a} \|z_n - U_n z_n\| \rightarrow 0. \quad (2.25)$$

Since $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} , we have

$$\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0 \quad \forall T \in \mathcal{T}. \quad (2.26)$$

Hence, we obtain that $\{U_n\}$ satisfies the NST*-condition with \mathcal{T} . This completes the proof. \square

3. Weak convergence theorems

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of nonexpansive mappings from C into itself with a common fixed point. Let $\{x_n\}$ be a sequence in C defined by $x_0 \in C$ and*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n \quad (3.1)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Then,

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for each $p \in \bigcap_{n=1}^{\infty} F(T_n)$;
- (ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 < \infty$.

Proof. Observe that if C is a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow C$ is a nonexpansive mapping, then for every $u \in C$, $\alpha \in (0, 1]$, the mapping $S = S_{(\alpha, T)} : C \rightarrow C$ defined by

$$Sx = \alpha u + (1 - \alpha)Tx \quad (x \in C) \quad (3.2)$$

is a $(1 - \alpha)$ -contraction, that is, for all $x, y \in C$,

$$\|Sx - Sy\| = (1 - \alpha)\|Tx - Ty\| \leq (1 - \alpha)\|x - y\|. \quad (3.3)$$

Consequently, S has a unique fixed point $x^* \in C$. Thus, there exists a unique $x^* \in C$, that is,

$$x^* = \alpha u + (1 - \alpha)Tx^*. \quad (3.4)$$

This implies that the implicit iteration scheme (3.1) is well defined. To see (i), we let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. It follows from (2.2) that

$$\begin{aligned} \|x_n - p\|^2 &= \|\alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T_n x_n - p)\|^2 \\ &= \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) \|T_n x_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 \\ &\leq \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2. \end{aligned} \quad (3.5)$$

Since $\alpha_n > 0$, we have

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|^2 - (1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2. \quad (3.6)$$

In particular,

$$\|x_n - p\| \leq \|x_{n-1} - p\|. \quad (3.7)$$

So, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Furthermore, from (3.6), we have

$$(1 - \alpha_n) \|x_{n-1} - T_n x_n\|^2 \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2. \quad (3.8)$$

Summing from 1 to m and tending to infinity for m , we have (ii). This completes the proof. \square

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . Then, the sequence $\{x_n\}$ in C defined by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, b]$ for some $b \in (0, 1)$, converges weakly to $w \in F(\mathcal{T})$. Moreover, $\lim_{n \rightarrow \infty} P_{F(\mathcal{T})} x_n = w$.*

Proof. It follows from Lemma 3.1(i) that $\{x_n\}$ is bounded. By Lemma 3.1(ii) and $\alpha_n \leq b$, we have

$$\sum_{n=1}^{\infty} \|x_{n-1} - T_n x_n\|^2 < \infty. \quad (3.9)$$

It follows that $\lim_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0$. From (3.1), we immediately have

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = \lim_{n \rightarrow \infty} \alpha_n \|x_{n-1} - T_n x_n\| = 0, \quad (3.10)$$

and so,

$$\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0. \quad (3.11)$$

Since $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} , we have

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0 \quad \forall T \in \mathcal{T}. \quad (3.12)$$

We now extract a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. So, by the demiclosedness principle, $w \in F(\mathcal{T})$. To prove that $x_n \rightharpoonup w$, suppose that there exists another subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $x_{m_j} \rightharpoonup w' \neq w$. So, we have $w' \in F(\mathcal{T})$. It follows from Lemma 3.1(i) and Opial's condition that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - w\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - w\| < \lim_{i \rightarrow \infty} \|x_{n_i} - w'\| \\ &= \lim_{j \rightarrow \infty} \|x_{m_j} - w'\| < \lim_{j \rightarrow \infty} \|x_{m_j} - w\| \\ &= \lim_{n \rightarrow \infty} \|x_n - w\|, \end{aligned} \quad (3.13)$$

arriving at a contradiction. Hence, $x_n \rightharpoonup w \in F(\mathcal{T})$. Finally, we prove that $\lim_{n \rightarrow \infty} z_n = w$, where $z_n = P_{F(\mathcal{T})} x_n$ for each $n \in \mathbb{N}$. By (3.7) and Lemma 2.3, there is $w_0 \in F(\mathcal{T})$ such that $z_n \rightarrow w_0$. From $z_n = P_{F(\mathcal{T})} x_n$ and $w \in F(\mathcal{T})$, we have

$$\langle x_n - z_n, z_n - w \rangle \geq 0 \quad \forall n \in \mathbb{N}. \quad (3.14)$$

It follows from $z_n \rightarrow w_0$ and $x_n \rightharpoonup w$ that

$$\langle w - w_0, w_0 - w \rangle \geq 0, \quad (3.15)$$

and then $w_0 = w$. This completes the proof. \square

Using Theorem 3.2 and Lemma 2.7, we have the following result.

Corollary 3.3 (see [2, Theorem 2]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Then, the sequence $\{x_n\}$ in C defined by (1.3), where $\{\alpha_n\}$ is a sequence in $(0, b]$ for some $b \in (0, 1)$, converges weakly to $w = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^N F(T_n)} x_n$.*

In the presence of the stronger condition than NST*-condition with \mathcal{T} , we are able to weaken the restriction on $\{\alpha_n\}$.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $\{T_n\}$ be a family of nonexpansive mappings of C into itself which satisfies the AKTT-condition and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let T be the mapping from C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$, and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then, the sequence in C defined by (3.1), where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, converges weakly to $w = \lim_{n \rightarrow \infty} P_{F(T)} x_n$.*

Proof. By Lemmas 2.2 and 3.1(ii) and $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, we have

$$\liminf_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0, \quad (3.16)$$

and hence,

$$\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = \liminf_{n \rightarrow \infty} \alpha_n \|x_{n-1} - T_n x_n\| = 0. \quad (3.17)$$

Next, we prove that the limit $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ exists. Since $\{x_n\}$ is bounded, it follows from AKTT-condition that

$$\sum_{n=1}^{\infty} \sup \{ \|T_n z - T_{n-1} z\| : z \in \{x_n\} \} < \infty. \quad (3.18)$$

Notice that

$$\begin{aligned} \|x_n - x_{n-1}\| &= (1 - \alpha_n) \|x_{n-1} - T_n x_n\| \\ &\leq (1 - \alpha_n) (\|x_{n-1} - T_{n-1} x_{n-1}\| + \|T_{n-1} x_{n-1} - T_{n-1} x_n\| + \|T_{n-1} x_n - T_n x_n\|) \\ &\leq (1 - \alpha_n) \|x_{n-1} - T_{n-1} x_{n-1}\| + (1 - \alpha_n) \|x_{n-1} - x_n\| \\ &\quad + (1 - \alpha_n) \sup \{ \|T_n z - T_{n-1} z\| : z \in \{x_n\} \}, \end{aligned} \quad (3.19)$$

so we have

$$\alpha_n \|x_n - x_{n-1}\| \leq (1 - \alpha_n) \|x_{n-1} - T_{n-1} x_{n-1}\| + (1 - \alpha_n) \sup \{ \|T_n z - T_{n-1} z\| : z \in \{x_n\} \}. \quad (3.20)$$

It follows that

$$\begin{aligned} \|x_n - T_n x_n\| &= \frac{\alpha_n}{1 - \alpha_n} \|x_n - x_{n-1}\| \\ &\leq \|x_{n-1} - T_{n-1} x_{n-1}\| + \sup \{ \|T_n z - T_{n-1} z\| : z \in \{x_n\} \}. \end{aligned} \quad (3.21)$$

By Lemma 2.1 and (3.18), we have $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ exists. Thus, we have

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0. \quad (3.22)$$

From the definition of T , we have T is nonexpansive. By Lemma 2.6, we have $\{T_n\}$ satisfies the NST*-condition with T . As in the proof of Theorem 3.2, $\{x_n\}$ converges weakly to $w = \lim_{n \rightarrow \infty} P_{F(T)} x_n$. \square

Remark 3.5. Since the NST*-condition is implied by the AKTT-condition, Theorem 3.4 still holds under the same condition of $\{\alpha_n\}$ as in Theorem 3.2.

As in [8, Theorem 4.1], we can generate a family $\{T_n\}$ of nonexpansive mappings satisfying the AKTT-condition by using convex combination of a general family $\{S_k\}$ of nonexpansive mappings with a common fixed point.

Corollary 3.6. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$. Let $\{\beta_n^k\}$ be a family of positive real numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that

- (i) $\sum_{k=1}^n \beta_n^k = 1$ for every $n \in \mathbb{N}$;
- (ii) $\lim_{n \rightarrow \infty} \beta_n^k > 0$ for every $k \in \mathbb{N}$;
- (iii) $\sum_{n=1}^{\infty} \sum_{k=1}^n |\beta_{n+1}^k - \beta_n^k| < \infty$.

Let $\{S_k\}$ be a family of nonexpansive mappings from C into itself with a common fixed point. Then, the sequence $\{x_n\}$ in C defined by (3.1), where $T_n \equiv \sum_{k=1}^n \beta_n^k S_k$, converges weakly to $w = \lim_{n \rightarrow \infty} P_{\bigcap_{k=1}^{\infty} F(S_k)} x_n$.

4. Strong convergence theorems

We next use the hybrid method from mathematical programming to obtain several strong convergence theorems.

Theorem 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . Let $\{x_n\}$ be a sequence in C defined as follows:

$$\begin{aligned} x_0 &\in C \text{ is arbitrary,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T_n y_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots, \end{aligned} \tag{4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})} x_0$.

Proof. We first prove that C_n and Q_n are closed and convex for each $n \in \mathbb{N} \cup \{0\}$. From the definitions of C_n and Q_n , it is obvious that C_n is closed and Q_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. We prove that C_n is convex. Since $\|y_n - z\| \leq \|x_n - z\|$ is equivalent to

$$\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z \rangle \leq 0, \tag{4.2}$$

(by (2.1)) it follows that C_n is convex. Next, we show that

$$F(\mathcal{T}) \subset C_n \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{4.3}$$

Let $p \in F(\mathcal{T})$ and $n \in \mathbb{N} \cup \{0\}$. Since

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n x_n + (1 - \alpha_n) T_n y_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|T_n y_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|y_n - p\|, \end{aligned} \tag{4.4}$$

it follows that

$$\|y_n - p\| \leq \|x_n - p\|, \quad (4.5)$$

and hence, $p \in C_n$. Therefore, we obtain (4.3). Now, we show that

$$F(\mathcal{T}) \subset Q_n \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (4.6)$$

We prove this by induction. For $n = 0$, we have $F(\mathcal{T}) \subset C = Q_0$. Suppose that $F(\mathcal{T}) \subset Q_n$. Then, $\emptyset \neq F(\mathcal{T}) \subset C_n \cap Q_n$ and there exists a unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n} x_0$. Then,

$$\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geq 0 \quad (4.7)$$

for each $z \in C_n \cap Q_n$. In particular,

$$\langle x_{n+1} - p, x_0 - x_{n+1} \rangle \geq 0 \quad (4.8)$$

for each $p \in F(\mathcal{T})$. It follows that $F(\mathcal{T}) \subset Q_{n+1}$, and hence (4.6) holds. Therefore,

$$F(\mathcal{T}) \subset C_n \cap Q_n \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (4.9)$$

This implies that $\{x_n\}$ is well defined. It follows from the definition of Q_n that $x_n = P_{Q_n} x_0$, that is,

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall z \in Q_n \text{ and all } n \in \mathbb{N} \cup \{0\}. \quad (4.10)$$

In particular,

$$\|x_n - x_0\| \leq \|z - x_0\| \quad \forall z \in F(\mathcal{T}) \text{ and all } n \in \mathbb{N} \cup \{0\}. \quad (4.11)$$

On the other hand, from $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\| \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (4.12)$$

Therefore, $\{\|x_n - x_0\|\}$ is nondecreasing and bounded. So, $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists. This implies that $\{x_n\}$ is bounded. Since $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\langle x_n - x_{n+1}, x_0 - x_n \rangle \geq 0. \quad (4.13)$$

It follows from (2.1) that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \end{aligned} \quad (4.14)$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (4.15)$$

Since $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \\ &\leq 2\|x_n - x_{n+1}\| \longrightarrow 0. \end{aligned} \quad (4.16)$$

It follows from $\alpha_n \leq b < 1$ that

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - T_n y_n\| + \|T_n y_n - T_n x_n\| \\ &\leq \|x_n - T_n y_n\| + \|y_n - x_n\| \\ &= \frac{1}{1 - \alpha_n} \|y_n - x_n\| + \|y_n - x_n\| \\ &\leq \frac{1}{1 - b} \|y_n - x_n\| + \|y_n - x_n\| \longrightarrow 0. \end{aligned} \quad (4.17)$$

Since $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} , we have

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0 \quad \forall T \in \mathcal{T}. \quad (4.18)$$

Finally, we show that $x_n \rightarrow w$, where $w = P_{F(\mathcal{T})}x_0$. Since $\{x_n\}$ is bounded, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightarrow w'$. Since $I - T$ is demiclosed and by using (4.18), we have $w' \in F(\mathcal{T})$. By (4.11), we have

$$\|x_n - x_0\| \leq \|w - x_0\|. \quad (4.19)$$

It follows from $w = P_{F(\mathcal{T})}x_0$ and the lower semicontinuity of the norm that

$$\|w - x_0\| \leq \|w' - x_0\| \leq \liminf_{k \rightarrow \infty} \|x_{n_k} - x_0\| \leq \limsup_{k \rightarrow \infty} \|x_{n_k} - x_0\| \leq \|w - x_0\|. \quad (4.20)$$

Thus, we obtain that $\lim_{k \rightarrow \infty} \|x_{n_k} - x_0\| = \|w' - x_0\| = \|w - x_0\|$. Using the Kadec-Klee property of H , we obtain that $\lim_{k \rightarrow \infty} x_{n_k} = w' = w$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that the whole sequence $\{x_n\}$ converges strongly to $P_{F(\mathcal{T})}x_0$. \square

Using Theorem 4.1 and Lemmas 2.7 and 2.11, we have the following result.

Corollary 4.2 (see [3, Theorem 2.4]). *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Then, the sequence $\{x_n\}$ in C defined by (1.4), where $\{\alpha_n\}$ is a sequence in $(0, a]$ for some $a \in (0, 1)$, and $\{\beta_n\}$ is a sequence in $[b, 1]$ for some $b \in (0, 1]$, converges strongly to $P_{\bigcap_{n=1}^N F(T_n)}x_0$.*

5. Applications

5.1. Equilibrium problems

Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction of $C \times C$ into \mathbb{R} , where \mathbb{R} is the set of real numbers. The equilibrium problem for $f : C \times C \rightarrow \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \quad \forall y \in C. \quad (5.1)$$

The set of solutions of (5.1) is denoted by $EP(f)$. Numerous problems in physics, optimization, and economics are reduced to find a solution of (5.1). Some methods have been proposed to solve the equilibrium problem [11–17]. In 2005, Combettes and Hirstoaga [12] introduced an iterative scheme of finding the best approximation to the initial data when $EP(f)$ is nonempty, and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (see [11]).

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for any $x, y \in C$;

(A3) f is upper-hemicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} f(tz + (1-t)x, y) \leq f(x, y); \quad (5.2)$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma is shown in [11, Corollary 1] and [12, Lemma 2.12].

Lemma 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfies (A1)–(A4), and let $r > 0$ and $x \in H$. Then, there exists a unique $x^* \in C$ such that*

$$f(x^*, y) + \frac{1}{r} \langle y - x^*, x^* - x \rangle \geq 0 \quad \forall y \in C. \quad (5.3)$$

Moreover, let T_r be a mapping of H into C defined by

$$T_r(x) = x^* \quad \forall x \in H. \quad (5.4)$$

Then, the following conditions hold:

(i) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|T_r x - x - (T_r y - y)\|^2; \quad (5.5)$$

(ii) $F(T_r) = EP(f)$;

(iii) $EP(f)$ is closed and convex.

Lemma 5.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let S be a nonexpansive mapping of C into H , and let T be a firmly nonexpansive mapping from H into C such that $F(S) \cap F(T) \neq \emptyset$. Then, ST is a nonexpansive mapping from H into itself and*

$$F(ST) = F(S) \cap F(T). \quad (5.6)$$

Proof. Since T is firmly nonexpansive, there exists a nonexpansive mapping U such that $T = (1/2)(I + U)$ and $F(U) = F(T)$. As in the proof of Lemma 2.10, the conclusion holds. \square

Motivated by Tada and Takahashi [16] and S. Takahashi and W. Takahashi [17], we prove weak and strong convergence theorems for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space. Using Theorem 3.4 and Lemmas 5.1 and 5.2, we have Theorem 5.3.

Theorem 5.3. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4), and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be two sequences generated by $x_0 \in H$ and

$$\begin{aligned} f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0 \quad \forall y \in C, \\ x_n &= \alpha_n x_{n-1} + (1 - \alpha_n) S u_n \end{aligned} \quad (5.7)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(f)$. Moreover, $\lim_{n \rightarrow \infty} P_{F(S) \cap EP(f)} x_n = w$.

Proof. It is noted that the iteration scheme is well defined. As in the proof of [14, Theorem 16], it follows from $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ that

$$\sum_{n=1}^{\infty} \sup \{ \|T_{r_{n+1}} z - T_{r_n} z\| : z \in B \} < \infty \quad (5.8)$$

for any bounded subset B of H . Moreover, by Lemma 2.5, the mapping T defined by

$$Tx = \lim_{n \rightarrow \infty} T_{r_n} x \quad \forall x \in H \quad (5.9)$$

satisfies

$$F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n}) = EP(f). \quad (5.10)$$

It is easy to see that T is a firmly nonexpansive mapping of H into C . Write $T_n \equiv ST_{r_n}$ then, by Lemma 5.2, we have T_n is a nonexpansive mapping from H into itself, and

$$F(T_n) = F(ST_{r_n}) = F(S) \cap F(T_{r_n}) = F(S) \cap EP(f) = F(ST) \quad (5.11)$$

for all $n \in \mathbb{N}$ and so,

$$\bigcap_{n=1}^{\infty} F(T_n) = F(ST) = F(S) \cap EP(f). \quad (5.12)$$

Since S is nonexpansive, (5.8) and (5.9), we have

$$\sum_{n=1}^{\infty} \sup \{ \|T_{n+1} z - T_n z\| : z \in B \} < \infty \quad (5.13)$$

for any bounded subset B of H , and

$$STx = S \left(\lim_{n \rightarrow \infty} T_{r_n} x \right) = \lim_{n \rightarrow \infty} ST_{r_n} x = \lim_{n \rightarrow \infty} T_n x \quad \forall x \in H. \quad (5.14)$$

Applying Theorem 3.4, $\{x_n\}$ converges weakly to $w = \lim_{n \rightarrow \infty} P_{F(S) \cap EP(f)} x_n$. \square

Similarly, we have the following strong convergence theorem. We safely suppress the proof.

Theorem 5.4. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4), and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be two sequences generated by $x_0 \in H$ and*

$$\begin{aligned} f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle &\geq 0 \quad \forall y \in C, \\ y_n &= \alpha_n x_{n-1} + (1 - \alpha_n) S u_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots, \end{aligned} \tag{5.15}$$

where $\{\alpha_n\}$ is a sequence in $(0, a)$ for some $a \in (0, 1)$, and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(S) \cap EP(f)} x_0$.

5.2. Convergence theorem for monotone mappings

Let H be a real Hilbert space, and C be a nonempty closed convex subset of H . Let $A : C \rightarrow H$ be a mapping. The classical variational inequality is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \quad \forall y \in C. \tag{5.16}$$

The set of solutions of classical variational inequality is denoted by $VIP(C, A)$. The variational inequality has been extensively studied in the literatures (see [7, 18–23] and the references therein). We recall that a mapping $A : C \rightarrow H$ is said to be

(a) monotone if

$$\langle Au - Av, u - v \rangle \geq 0 \quad \forall u, v \in C; \tag{5.17}$$

(b) α -inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2 \quad \forall u, v \in C; \tag{5.18}$$

(c) r -strongly monotone if there exists a constant $r > 0$ such that

$$\langle Au - Av, u - v \rangle \geq r \|u - v\|^2 \quad \forall u, v \in C; \tag{5.19}$$

(d) relaxed (γ, r) -cocoercive if there exist constants $\gamma, r > 0$ such that

$$\langle Au - Av, u - v \rangle \geq -\gamma \|Au - Av\|^2 + r \|u - v\|^2 \quad \forall u, v \in C; \tag{5.20}$$

(e) μ -Lipschitzian if there exists a constant $\mu > 0$ such that

$$\|Au - Av\| \leq \mu \|u - v\| \quad \forall u, v \in C. \tag{5.21}$$

Remark 5.5. (1) Every α -inverse-strongly monotone mapping is monotone and $1/\alpha$ -Lipschitzian.

(2) Every r -strongly monotone is monotone.

(3) Every relaxed (γ, r) -cocoercive and μ -Lipschitzian mapping with $\gamma\mu^2 \leq r$ is monotone.

Lemma 5.6. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let A be a continuous monotone mapping of C into H . Define a bifunction $f : C \times C \rightarrow \mathbb{R}$ as follows:*

$$f(x, y) = \langle Ax, y - x \rangle \quad \forall x, y \in C. \quad (5.22)$$

Then,

(i) [14, Lemma 19] f satisfies (A1)–(A4) and $VIP(C, A) = EP(f)$;

(ii) [14, Lemma 20] If $x \in H$, $u \in C$, and $r > 0$, then

$$f(u, y) + \frac{1}{r} \langle y - u, u - x \rangle \geq 0 \quad \forall y \in C \iff u = P_C(x - rAu). \quad (5.23)$$

Using Theorem 5.3 and Lemma 5.6, we have Theorem 5.7.

Theorem 5.7. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let A be a continuous monotone mapping of C , and let S be a nonexpansive mapping of C into H such that $F(S) \cap VIP(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$ and*

$$\begin{aligned} u_n &= P_C(x_n - r_n Au_n), \\ x_n &= \alpha_n x_{n-1} + (1 - \alpha_n) S u_n \end{aligned} \quad (5.24)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges weakly to $w \in F(S) \cap VIP(C, A)$. Moreover, $\lim_{n \rightarrow \infty} P_{F(S) \cap VIP(C, A)} x_n = w$.

Using Theorem 5.4 and Lemma 5.6, we also have Theorem 5.8.

Theorem 5.8. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let A be a continuous monotone mapping of C , and let S be a nonexpansive mapping of C into H such that $F(S) \cap VIP(C, A) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in H$ and*

$$\begin{aligned} u_n &= P_C(y_n - r_n Au_n), \\ y_n &= \alpha_n x_{n-1} + (1 - \alpha_n) S u_n, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots, \end{aligned} \quad (5.25)$$

where $\{\alpha_n\}$ is a sequence in $(0, a]$ for some $a \in (0, 1)$, and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n \rightarrow \infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(S) \cap VIP(C, A)} x_0$.

Remark 5.9. (1) By Remark 5.5, we obtain a strong convergence theorem for α -inverse-strongly monotone mappings, r -strongly monotone and continuous mappings and relaxed (γ, r) -cocoercive and μ -Lipschitzian mappings with $\gamma\mu^2 \leq r$.

(2) Some weak and strong convergence theorems for monotone Lipschitzian mappings were established by several authors [7, 18–23]. However, there is a monotone continuous mapping which is not Lipschitzian (see [14, Remark 23]). Therefore, Theorems 5.7 and 5.8 provide a new convergence theorem for a wider class of mappings.

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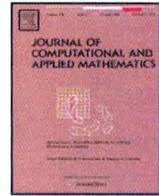
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Weak convergence theorems for a countable family of Lipschitzian mappings

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ABSTRACT

This paper is concerned with convergence of an approximating common fixed point sequence of countable Lipschitzian mappings in a uniformly convex Banach space. We also establish weak convergence theorems for finding a common element of the set of fixed points, the set of solutions of an equilibrium problem, and the set of solutions of a variational inequality. With an appropriate setting, we obtain and improve the corresponding results recently proved by Moudafi [A. Moudafi, Weak convergence theorems for nonexpansive mappings and equilibrium problems. *J. Nonlinear Convex Anal.* 9 (2008) 37–43], Tada–Takahashi [A. Tada and W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem. *J. Optim. Theory Appl.* 133 (2007) 359–370], and Plubtieng–Kumam [S. Plubtieng and P. Kumam, Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings. *J. Comput. Appl. Math.* (2008) doi:10.1016/j.cam.2008.05.045]. Some of our results are established with weaker assumptions.

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1. Introduction

Let C be a subset of a real Banach space E . A mapping $T : C \rightarrow E$ is said to be *Lipschitzian* if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be L -Lipschitzian. Clearly, if T is L_1 -Lipschitzian and $L_1 < L_2$, then T is L_2 -Lipschitzian. Throughout the paper, we assume that every Lipschitzian mapping is L -Lipschitzian with $L \geq 1$. If $L = 1$, then T is known as a nonexpansive mapping. We denote by $F(T)$ the set of fixed points of T . If C is a nonempty bounded closed convex subset of a uniformly convex Banach space and T is a nonexpansive self-mapping of C , then $F(T) \neq \emptyset$ (see [1]). There are many methods for approximating fixed points of a mapping. In 1953, Mann [2] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, Aoyama [3] extended iteration (1.1) to obtain weak convergence to a common fixed point of a countable family of nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ by the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_n x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

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where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, the present authors [4] proved that the iteration (1.2) converges weakly to a common fixed point of a countable family of Lipschitzian mappings in a real Hilbert space.

In this paper, we establish weak convergence theorem for finding common fixed points of a countable family of Lipschitzian mappings in a uniformly convex Banach space. We also establish weak convergence of iterative sequences for finding a common element of the set of fixed points, the set of solutions of an equilibrium problem, and the set of solutions of a variational inequality. With an appropriate setting, we obtain the corresponding results due to Moudafi [5], Tada–Takahashi [6], and Plubtieng–Kumam [7]. Some of our results are established with weaker assumptions.

2. Preliminaries

A real Banach space E is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \text{ imply } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

We need some facts and tools in a uniformly convex Banach space which are listed as lemmas below.

Lemma 2.1 ([8]). *Let E be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq d$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq d$ and $\lim_{n \rightarrow \infty} \|\lambda_n x_n + (1 - \lambda_n)y_n\| = d$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 2.2 ([9], Theorem 2). *Let E be a uniformly convex Banach space and $B_r := \{x \in E : \|x\| \leq r\}$, where $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|), \tag{2.1}$$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$.

We write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x .

Lemma 2.3 ([10], Lemma 1.1). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E . Then there is a strictly increasing and continuous convex function $\gamma : [0, \infty) \rightarrow [0, \infty)$ with $\gamma(0) = 0$ such that, for every L -Lipschitzian mapping $T : C \rightarrow E$ and, the following inequality holds*

$$\gamma \left(\frac{1}{L} \|tTx + (1 - t)Ty - T(tx + (1 - t)y)\| \right) \leq \|x - y\| - \frac{1}{L} \|Tx - Ty\|,$$

for all $x, y \in C$ and $t \in [0, 1]$.

We also need the following lemma (see [11], Lemma 1).

Lemma 2.4. *Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Recall that a mapping $T : C \rightarrow E$ is *demi-closed* at y , if $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$.

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a real Banach space E , let $\{T_n\}$ be a family of mappings of C into E with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\omega_w\{z_n\}$ denotes the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C . $\{T_n\}$ is said to satisfy

(a) the *AKTT-condition (I)* [12] if for each bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty.$$

(b) the *AKTT-condition (II)* [3] if for each bounded subset B of C and each increasing sequence $\{n_i\}$ of \mathbb{N} , there exists a mapping $T : C \rightarrow E$ with $I - T$ is demi-closed at 0 and a subsequence $\{n_{ij}\}$ of $\{n_i\}$ such that

$$\lim_{j \rightarrow \infty} \sup\{\|T_{n_{ij}}z - Tz\| : z \in B\} = 0 \text{ and } F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

(c) the *NST-condition* [13] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_nz_n\| = 0 \text{ implies } \omega_w\{z_n\} \subset \bigcap_{n=1}^{\infty} F(T_n).$$

Remark 2.5. If $\{T_n\}$ satisfies the NST-condition and $T_nz \rightarrow z \in C$, then $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.6 ([12], Lemma 3.2). Let C be a nonempty closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into E which satisfies the AKTT-condition (I), then the mapping $T : C \rightarrow E$ defined by

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in C \tag{2.2}$$

satisfies

$$\lim_{n \rightarrow \infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of C . In particular, if $I - T$ is demi-closed at 0 and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, then $\{T_n\}$ satisfies the AKTT-condition (II).

From now on, we will write $(\{T_n\}, T)$ satisfies AKTT-condition (I) if $\{T_n\}$ satisfies AKTT-condition (I) and T is defined by (2.2).

Lemma 2.7 ([14], Theorem 10.4). Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Then $I - T$ is demi-closed at 0.

Lemma 2.8. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition (I) and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{T_n\}$ satisfies the AKTT-condition (II).

Proof. It follows from the definition of T and Lemma 2.7 that T is nonexpansive and $I - T$ is demiclosed at 0, respectively. Applying Lemma 2.6, $\{T_n\}$ satisfies the AKTT-condition (II). \square

Lemma 2.9. Let C be a nonempty closed convex subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. If $\{T_n\}$ satisfies AKTT-condition (II), then $\{T_n\}$ satisfies the NST-condition.

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$ and $z_0 \in \omega_w\{z_n\}$. Then $z_{n_i} \rightarrow z_0$ for some subsequence $\{n_i\}$ of $\{n\}$. Since $\{T_n\}$ satisfies AKTT-condition (II), there exists a mapping $T : C \rightarrow E$ with $I - T$ is demi-closed at 0 and a subsequence $\{n_i\}$ of $\{n_i\}$ such that

$$\lim_{j \rightarrow \infty} \sup\{\|T_{n_{i_j}} z - Tz\| : z \in \{z_n\}\} = 0 \quad \text{and} \quad F(T) = \bigcap_{n=1}^{\infty} F(T_n).$$

Then $z_{n_{i_j}} \rightarrow z_0$ and

$$\begin{aligned} \|z_{n_{i_j}} - T_{n_{i_j}} z_{n_{i_j}}\| &\leq \|z_{n_{i_j}} - T_{n_{i_j}} z_{n_{i_j}}\| + \|T_{n_{i_j}} z_{n_{i_j}} - Tz_{n_{i_j}}\| \\ &\leq \|z_{n_{i_j}} - T_{n_{i_j}} z_{n_{i_j}}\| + \sup\{\|T_{n_{i_j}} z - Tz\| : z \in \{z_n\}\} \rightarrow 0, \end{aligned}$$

as $j \rightarrow \infty$. Since $I - T$ is demi-closed at 0, $z_0 \in F(T)$ and hence $\omega_w\{z_n\} \subset F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. This implies that $\{T_n\}$ satisfies the NST-condition. \square

Lemma 2.10. Let C and K be nonempty closed convex subsets of a real Banach space E . Let $\mathcal{S} := \{S_n\}$ be a family of L_n -Lipschitzian mappings of C into K with $L_n \rightarrow 1$ and $F(\mathcal{S}) := \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $\mathcal{W} := \{W_n\}$ be a family of nonexpansive mappings of K into C with $F(\mathcal{W}) := \bigcap_{n=1}^{\infty} F(W_n) \neq \emptyset$ and

$$\|W_n x - u\|^2 \leq \|x - u\|^2 - a_n \|W_n x - x\|^2 \quad \text{for all } x \in K, u \in F(\mathcal{W}) \text{ and } n \in \mathbb{N},$$

where $\{a_n\}$ is a sequence in $[a, \infty) \subset (0, \infty)$. Let $\{T_n\}$ be a family of mappings defined by

$$T_n = S_n W_n \quad \text{for all } n \in \mathbb{N}.$$

If $\{S_n\}$ and $\{W_n\}$ satisfy NST-condition and $F(\mathcal{S}) \cap F(\mathcal{W}) \neq \emptyset$, then $\{T_n\}$ is a family of L_n -Lipschitzian mappings of K into itself satisfies NST-condition and $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{S}) \cap F(\mathcal{W})$.

Proof. It is easy to see that $\{T_n\}$ is a family of L_n -Lipschitzian mappings of K into itself. To show that $\{T_n\}$ satisfies NST-condition, let $\{z_n\}$ be a bounded sequence in K such that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0.$$

Let $u \in F(\mathcal{S}) \cap F(\mathcal{W})$ be given. Then

$$\begin{aligned} \|T_n z_n - u\|^2 &= \|S_n W_n z_n - u\|^2 \\ &\leq L_n^2 \|W_n z_n - u\|^2 \\ &\leq L_n^2 \|z_n - u\|^2 - a_n L_n^2 \|W_n z_n - z_n\|^2 \\ &\leq L_n^2 \|z_n - u\|^2 - a \|W_n z_n - z_n\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} a\|W_n z_n - z_n\|^2 &\leq (L_n^2 - 1)\|z_n - u\|^2 + (\|z_n - u\|^2 - \|T_n z_n - u\|^2) \\ &= (L_n^2 - 1)\|z_n - u\|^2 + (\|z_n - u\| - \|T_n z_n - u\|)(\|z_n - u\| + \|T_n z_n - u\|) \\ &\leq (L_n^2 - 1)\|z_n - u\|^2 + \|z_n - T_n z_n\|(\|z_n - u\| + \|T_n z_n - u\|). \end{aligned}$$

Since $\{z_n - u\}$ and $\{T_n z_n - u\}$ are bounded,

$$\lim_{n \rightarrow \infty} \|W_n z_n - z_n\| = 0.$$

So, we get

$$\begin{aligned} \|S_n W_n z_n - W_n z_n\| &\leq \|S_n W_n z_n - z_n\| + \|W_n z_n - z_n\| \\ &= \|T_n z_n - z_n\| + \|W_n z_n - z_n\| \rightarrow 0. \end{aligned}$$

Since $\{S_n\}$ and $\{W_n\}$ satisfy the NST-condition, we have $\omega_w\{z_n\} \subset F(\mathcal{S}) \cap F(\mathcal{W})$. It easy to see that $F(\mathcal{S}) \cap F(\mathcal{W}) \subset \bigcap_{n=1}^\infty F(T_n)$. To see the reverse inclusion, let $z \in \bigcap_{n=1}^\infty F(T_n)$. Follow the first part of the proof above but now let $z_n \equiv z$. Then $z \in F(\mathcal{S}) \cap F(\mathcal{W})$. This implies that $\{T_n\}$ satisfies the NST-condition. This completes the proof. \square

Lemma 2.11. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. If $\{T_n\}$ satisfies the NST-condition, then $\bigcap_{n=1}^\infty F(T_n)$ is closed and convex,*

Proof. It follows from the continuity of T_n that $F(T_n)$ is closed and so is $\bigcap_{n=1}^\infty F(T_n)$. Now we prove that $\bigcap_{n=1}^\infty F(T_n)$ is convex. To this end, let $x, y \in \bigcap_{n=1}^\infty F(T_n)$. Put $z = tx + (1 - t)y$, where $t \in (0, 1)$. Let $\tilde{x}_n = t(T_n z - x)$ and $\tilde{y}_n = (1 - t)(y - T_n z)$. Then

$$\begin{aligned} \|\tilde{x}_n\| &= t\|T_n z - x\| \leq tL_n\|z - x\| = t(1 - t)L_n\|x - y\|, \\ \|\tilde{y}_n\| &= (1 - t)\|y - T_n z\| \leq (1 - t)L_n\|y - z\| = t(1 - t)L_n\|x - y\|, \end{aligned}$$

and

$$\|(1 - t)\tilde{x}_n + t\tilde{y}_n\| = \|(1 - t)t(T_n z - x) + t(1 - t)(y - T_n z)\| = t(1 - t)\|x - y\|.$$

This together with $L_n \rightarrow 1$ gives

$$\limsup_{n \rightarrow \infty} \|\tilde{x}_n\| \leq t(1 - t)\|x - y\|, \quad \limsup_{n \rightarrow \infty} \|\tilde{y}_n\| \leq t(1 - t)\|x - y\|$$

and

$$\lim_{n \rightarrow \infty} \|(1 - t)\tilde{x}_n + t\tilde{y}_n\| = t(1 - t)\|x - y\|.$$

By Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \|T_n z - z\| = \lim_{n \rightarrow \infty} \|\tilde{x}_n - \tilde{y}_n\| = 0.$$

Then $z \in \bigcap_{n=1}^\infty F(T_n)$ and hence $\bigcap_{n=1}^\infty F(T_n)$ is convex. This completes the proof. \square

3. Main results

In this section, we prove several weak convergence theorems. We first prove a weak convergence theorem of the sequence $\{x_n\}$ which defined by (1.2) in a uniformly convex Banach space satisfying Opial's condition. Recall that E satisfies Opial's condition [15] if $x_n \rightharpoonup x$ and $x \neq y$ imply that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

It is well known [15] that all Hilbert spaces and ℓ^p spaces, $1 \leq p < \infty$, have this property, while all L^p spaces do not have this property unless $p = 2$.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Banach space E and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^\infty (L_n - 1) < \infty$ and $\bigcap_{n=1}^\infty F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by (1.2). Then*

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for each $p \in \bigcap_{n=1}^\infty F(T_n)$,
- (ii) there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that $\sum_{n=1}^\infty \alpha_n(1 - \alpha_n)g(\|x_n - T_n x_n\|) < \infty$.

Proof. To see (i), let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Then

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|T_n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) L_n \|x_n - p\| \\ &= (1 + (1 - \alpha_n)(L_n - 1)) \|x_n - p\| \\ &\leq (1 + (L_n - 1)) \|x_n - p\|, \end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$. From $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and Lemma 2.4, we have (i). Note that $\{x_n - p\}$ and $\{T_n x_n - p\}$ are bounded. We may assume that such sequences belong to B_r where $r > 0$. By Lemma 2.2, there exists $g : [0, \infty) \rightarrow [0, \infty)$ is a continuous strictly increasing and convex function with $g(0) = 0$ such that (2.1) is satisfied. Then

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|T_n x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|) \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) L_n^2 \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|) \\ &\leq (1 + (1 - \alpha_n)(L_n^2 - 1)) \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|). \end{aligned}$$

That is,

$$\alpha_n (1 - \alpha_n) g(\|x_n - T_n x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1 - \alpha_n)(L_n^2 - 1)M,$$

where $M = \sup\{\|x_n - p\|^2 : n \in \mathbb{N}\}$. Summing from 1 to m and tending to infinity for m , we have (ii). This completes the proof. \square

Theorem 3.2. Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $\{T_n\}$ satisfies the NST-condition, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\}$ is a sequence in $[a, b] \subset (0, 1)$, converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. It follows from Lemma 3.1(i) that $\{x_n\}$ is bounded. By Lemma 3.1(ii) and $\alpha_n \in [a, b] \subset (0, 1)$, we have

$$\sum_{n=1}^{\infty} g(\|x_n - T_n x_n\|) < \infty.$$

In particular, $\lim_{n \rightarrow \infty} g(\|x_n - T_n x_n\|) = 0$ and so

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

Since $\{T_n\}$ satisfies the NST-condition, we have $\omega_\omega \{x_n\} \subset \bigcap_{n=1}^{\infty} F(T_n)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ for some $z \in C$. Then $z \in \bigcap_{n=1}^{\infty} F(T_n)$. It follows from Opial's condition and $\lim_{n \rightarrow \infty} \|x_n - p\|$ for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$ that $x_n \rightharpoonup z \in \bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. \square

In the presence of the stronger condition than NST-condition, a variable control sequence $\{\alpha_n\}$ is taken into consideration.

Theorem 3.3. Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $(\{T_n\}, T)$ satisfies AKTT-condition (I) and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. By Lemma 3.1(ii) and $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, we have

$$\liminf_{n \rightarrow \infty} g(\|x_n - T_n x_n\|) = 0.$$

This implies that

$$\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

We next prove that the limit $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ actually exists. Since $\{x_n\}$ is bounded, it follows that

$$\sum_{n=1}^{\infty} \sup\{\|T_n z - T_{n+1} z\| : z \in \{x_n\}\} < \infty. \tag{3.2}$$

Notice that

$$\begin{aligned} \|x_{n+1} - T_{n+1} x_{n+1}\| &= \|\alpha_n (x_n - T_n x_n) + (T_n x_n - T_{n+1} x_{n+1})\| \\ &\leq \alpha_n \|x_n - T_n x_n\| + \|T_n x_n - T_{n+1} x_{n+1}\| \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_n \|x_n - T_n x_n\| + \|T_n x_n - T_n x_{n+1}\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
 &\leq \alpha_n \|x_n - T_n x_n\| + L_n \|x_n - x_{n+1}\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
 &= (1 + (1 - \alpha_n)(L_n - 1)) \|x_n - T_n x_n\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
 &\leq (1 + (1 - \alpha_n)(L_n - 1)) \|x_n - T_n x_n\| + \sup\{\|T_n z - T_{n+1} z\| : z \in \{x_n\}\}.
 \end{aligned}$$

By Lemma 2.4 and (3.2), the limit $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ exists. Thus

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

By Lemmas 2.8 and 2.9, $\{T_n\}$ satisfies the NST-condition. As in the proof of Theorem 3.2, $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^\infty F(T_n)$. This completes the proof. \square

Secondly, we prove a weak convergence of the iteration (1.2) in a uniformly convex Banach space E whose dual E^* has the Kadec–Klee property. Most of the weak convergence theorems are proved in a uniformly convex Banach space and the presence of Opial’s condition or the Fréchet differentiability of the norm (see e.g., [3, 11]). Before going on, let us recall some additional geometric properties. A Banach space E has

- (a) the *Kadec–Klee property* [14] if for every $\{x_n\}$ in E , $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ together imply $x_n \rightarrow x$.
- (b) a *Fréchet differentiable norm* if, for any $x \in S_E$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is approached uniformly as y varies over S_E , where S_E denotes the unit sphere of E .

It is known that if E is reflexive and has a Fréchet differentiable norm, then E^* has the Kadec–Klee property (see also [16, Lemma 1]). However, there exist uniformly convex Banach spaces which have neither Opial’s condition nor a Fréchet differentiable norm but their duals do have the Kadec–Klee property (see also [17, 14]).

The following lemma is our main tool for proving the weak convergence theorem.

Lemma 3.4 ([17], Lemma 3.2). *Let E be a uniformly convex Banach space such that its dual E^* has the Kadec–Klee property. Suppose $\{x_n\}$ is a bounded sequence in E such that*

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$. Then $\omega_w\{x_n\}$ is a singleton.

Lemma 3.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^\infty (L_n - 1) < \infty$ and $\bigcap_{n=1}^\infty F(T_n)$ is nonempty and let $\{x_n\}$ be a sequence in C defined by (1.2). Then the limit*

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \bigcap_{n=1}^\infty F(T_n)$.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded, so we let $R > 0$ be such that $\{x_n\} \subset K := B_R \cap C$. Then K is a nonempty closed convex bounded subset of E . Let $p, q \in \bigcap_{n=1}^\infty F(T_n)$ and set

$$a_n(t) := \|tx_n + (1 - t)p - q\|.$$

Then $\lim_{n \rightarrow \infty} a_n(0) = \|p - q\|$, and from Lemma 3.1, $\lim_{n \rightarrow \infty} a_n(1) = \lim_n \|x_n - q\|$ exists. Now, we consider the case $t \in (0, 1)$. Hence, we can define $S_n : C \rightarrow C$ by

$$S_n x = \alpha_n x + (1 - \alpha_n) T_n x, \quad x \in C.$$

Then S_n is an L_n -Lipschitzian mapping. In fact, for all $x, y \in C$,

$$\begin{aligned}
 \|S_n x - S_n y\| &\leq \alpha_n \|x - y\| + (1 - \alpha_n) \|T_n x - T_n y\| \\
 &\leq \alpha_n \|x - y\| + (1 - \alpha_n) L_n \|x - y\| \\
 &\leq \alpha_n L_n \|x - y\| + (1 - \alpha_n) L_n \|x - y\| = L_n \|x - y\|.
 \end{aligned}$$

Moreover, $x_{n+1} = S_n x_n$ and $\bigcap_{n=1}^\infty F(T_n) \subset \bigcap_{n=1}^\infty F(S_n)$. Set

$$U_{n,m} := S_{n+m-1} S_{n+m-2} \cdots S_n, \quad n, m \in \mathbb{N}.$$

Then $U_{n,m} x_n = x_{n+m}$ and $\bigcap_{n=1}^\infty F(T_n) \subset \bigcap_{n,m=1}^\infty F(U_{n,m})$. Moreover,

$$\|U_{n,m} x - U_{n,m} y\| \leq L_{n,m} \|x - y\|, \quad x, y \in K,$$

where $L_{n,m} = \prod_{j=n}^{n+m-1} L_j$. We note that $\lim_{n,m \rightarrow \infty} L_{n,m} = 1$. Set

$$b_{n,m}(t) := \|tU_{n,m}x_n + (1-t)p - U_{n,m}(tx_n + (1-t)p)\|.$$

By Lemma 2.3, we have

$$\gamma\left(\frac{1}{L_{n,m}}b_{n,m}(t)\right) \leq \|x_n - p\| - \frac{1}{L_{n,m}}\|U_{n,m}x_n - U_{n,m}p\| = \|x_n - p\| - \frac{1}{L_{n,m}}\|x_{n+m} - p\|,$$

where $\gamma : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing continuous convex and $\gamma(0) = 0$. By using Lemma 3.1 and $\lim_{n,m \rightarrow \infty} L_{n,m} = 1$, we have $\lim_{n,m \rightarrow \infty} \gamma\left(\frac{1}{L_{n,m}}b_{n,m}(t)\right) = 0$, and hence $\lim_{n,m \rightarrow \infty} b_{n,m} = 0$. Observe that

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &= \|tU_{n,m}x_n + (1-t)p - q\| \\ &\leq b_{n,m}(t) + \|U_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,m}(t) + \|tx_n + (1-t)p - q\| = b_{n,m}(t) + a_n(t). \end{aligned}$$

Consequently,

$$\limsup_{n \rightarrow \infty} a_n(t) \leq \lim_{n,m \rightarrow \infty} b_{n,m}(t) + \liminf_{n \rightarrow \infty} a_n(t) = \liminf_{n \rightarrow \infty} a_n(t).$$

This implies that $\lim_{n \rightarrow \infty} a_n(t)$ exists for any $t \in (0, 1)$ and the proof is finished. \square

Theorem 3.6. Let C be a nonempty closed convex subset of a uniformly convex Banach space E such that its dual E^* has the Kadec–Klee property. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings from C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $\{T_n\}$ satisfies the NST-condition, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. As in the proof of Theorem 3.2, we have $\omega_w\{x_n\} \subset \bigcap_{n=1}^{\infty} F(T_n)$. Then, by Lemma 3.5, the limit

$$\lim_{n \rightarrow \infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$. Hence $x_n \rightarrow z \in \bigcap_{n=1}^{\infty} F(T_n)$ by Lemma 3.4. This completes the proof. \square

Similarly, using the same arguments as in Theorems 3.3 and 3.6 yields the following.

Theorem 3.7. Let C be a nonempty closed convex subset of a uniformly convex Banach space E such that its dual E^* has the Kadec–Klee property. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $(\{T_n\}, T)$ satisfies AKTT-condition (I) and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, converges weakly to a common fixed point of $\{T_n\}$.

Setting $L \equiv 1$, we have the following.

Theorem 3.8. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and satisfy one of the following control conditions:

- (i) $\{T_n\}$ satisfies the NST-condition and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$;
- (ii) $(\{T_n\}, T)$ satisfies AKTT-condition (I), $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

If, in addition, E satisfies Opial's condition or E^* has the Kadec–Klee property, then the sequence $\{x_n\}$ in C defined by (1.2), converges weakly to a common fixed point of $\{T_n\}$.

Remark 3.9. Theorem 3.8(i) extends and improves Lemma 3.2 of [3] in the following ways:

- (i) The geometric property imposed on a Banach space is weakened.
- (ii) The AKTT-condition (II) is weakened and replaced by the more general NST-condition (see Lemma 2.9).

4. Common solutions of a fixed point problem and an equilibrium problem

In this section, we present several related results which can be deduced by corresponding convergence theorems obtained in Section 3. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Then

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle \tag{4.1}$$

for all $x, y \in H$. It is also known that H satisfies Opial's condition and has the Kadec–Klee property.

Let C be a nonempty closed convex subset of H . Then, for any $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

Such a mapping P_C is called the *metric projection* of H onto C . We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \text{if and only if} \quad \langle x - z, z - y \rangle \geq 0 \quad \text{for all } y \in C.$$

Lemma 4.1 ([4], Lemma 4). *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{x_n\}$ be a sequence in H and let $\{\delta_n\}$ be a sequence in $[0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and*

$$\|x_{n+1} - y\| \leq (1 + \delta_n)\|x_n - y\| \quad \text{for all } y \in C \text{ and } n \in \mathbb{N}.$$

Then the sequence $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and satisfy one of the following control conditions:*

- (i) $\{T_n\}$ satisfies the NST-condition and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$;
- (ii) [4, Theorem 5] $(\{T_n\}, T)$ satisfies AKTT-condition (I), $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then $\{x_n\}$ in C defined by (1.2), converges weakly to $z \in \bigcap_{n=1}^{\infty} F(T_n)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(T_n)}(x_n)$.

Proof. By Theorems 3.2, 3.3, 3.6 and 3.7, we have $x_n \rightarrow z$ for some $z \in \bigcap_{n=1}^{\infty} F(T_n)$. We prove that $\lim_{n \rightarrow \infty} z_n = z$, where $z_n = P_{\bigcap_{n=1}^{\infty} F(T_n)}(x_n)$. By (3.1) and Lemma 4.1, there is $z' \in \bigcap_{n=1}^{\infty} F(T_n)$ such that $z_n \rightarrow z'$. From $z_n = P_{\bigcap_{n=1}^{\infty} F(T_n)}(x_n)$ and $z \in \bigcap_{n=1}^{\infty} F(T_n)$, we have

$$\langle x_n - z_n, z_n - z \rangle \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

It follows from $z_n \rightarrow z'$ and $x_n \rightarrow z$ that

$$\langle z - z', z' - z \rangle \geq 0$$

and then $z' = z$. This completes the proof. \square

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction and let $A : C \rightarrow H$ be a nonlinear mapping. Then, we consider the following equilibrium problem:

$$\text{Find } z \in C \text{ such that } f(z, y) + \langle Az, y - z \rangle \geq 0 \quad \text{for all } y \in C. \tag{4.2}$$

The set of such z is denoted by $EP(f, A)$, i.e.,

$$EP(f, A) = \{z \in C : f(z, y) + \langle Az, y - z \rangle \geq 0 \text{ for all } y \in C\}.$$

In the case of $A \equiv 0$, $EP(f, A)$ is denoted by $EP(f)$. In the case of $f \equiv 0$, $EP(f, A)$ is denoted by $VI(C, A)$. The problem (4.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games (see [18–20]).

A mapping $A : C \rightarrow H$ is said to be

(1) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \text{for all } x, y \in C;$$

(2) α -*inverse-strongly-monotone*, where $\alpha > 0$, if

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \quad \text{for all } x, y \in C.$$

Note that every α -inverse-strongly-monotone mapping is monotone and $(1/\alpha)$ -Lipschitzian.

Recently Tada and Takahashi [6] considered iterative methods for finding an element of $EP(f) \cap F(S)$, where $S : C \rightarrow H$ is a nonexpansive mapping. On the other hand, Takahashi and Toyoda [21] introduced an iterative methods for finding an element of $VI(C, A) \cap F(S)$, where $A : C \rightarrow H$ is an α -inverse-strongly-monotone mapping. Very recently Moudafi [5] introduced an iterative methods for finding an element of $EP(f, A) \cap F(S)$ and then proved a weak convergence theorem.

Motivated by Tada–Takahashi [6] and Moudafi [5], we prove a weak convergence theorem for finding a common element of a common element of the common fixed point set for a countable family of mappings and the set of solutions of an equilibrium problem in a Hilbert space.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (see [18]):

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for any $x, y \in C$;

(A3) f is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [18, Corollary 1] and [22, Lemma 2.12], we have the following lemma.

Lemma 4.3. Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and let $r > 0$ and $x \in H$. Then there exists unique $x^* \in C$ such that

$$f(x^*, y) + \frac{1}{r} \langle y - x^*, x^* - x \rangle \geq 0 \quad \text{for all } y \in C.$$

Let T_r be a mapping of H into C defined by

$$T_r(x) = x^*$$

for all $x \in H$. Then, the following hold:

(i) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|(I - T_r)x - (I - T_r)y\|^2;$$

(ii) $F(T_r) = EP(f)$;

(iii) $EP(f)$ is closed and convex.

We need the following lemmas.

Lemma 4.4. Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and let $r > 0$. Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H with $EP(f, A) \neq \emptyset$, then

$$\|T_r(I - rA)x - u\|^2 \leq \|x - u\|^2 - \frac{2\alpha - r}{2\alpha} \|T_r(I - rA)x - x\|^2 \tag{4.3}$$

for all $x \in C$ and $u \in F(T_r(I - rA)) = EP(f, A)$. Furthermore, if $0 \leq r \leq 2\alpha$, then $T_r(I - rA)$ is a nonexpansive mapping of C into itself.

Proof. For $r > 0$, we note that $u \in EP(f, A)$ if and only if $u = T_r(I - rA)u$. That is, $F(T_r(I - rA)) = EP(f, A)$. Let $x \in C$ and $u \in EP(f, A)$. Since T_r is firmly nonexpansive, A is α -inverse-strongly-monotone and (4.1), we have

$$\begin{aligned} \|T_r(I - rA)x - u\|^2 &\leq \|(I - rA)x - (I - rA)u\|^2 - \|(I - T_r)(I - rA)x - (I - T_r)(I - rA)u\|^2 \\ &= \|(x - u) - r(Ax - Au)\|^2 - \|(x - T_r(I - rA)x) - r(Ax - Au)\|^2 \\ &= (\|x - u\|^2 - 2r \langle x - u, Ax - Au \rangle + r^2 \|Ax - Au\|^2) \\ &\quad - (\|x - T_r(I - rA)x\|^2 - 2r \langle x - T_r(I - rA)x, Ax - Au \rangle + r^2 \|Ax - Au\|^2) \\ &= \|x - u\|^2 - 2r \langle x - u, Ax - Au \rangle + 2r \langle x - T_r(I - rA)x, Ax - Au \rangle - \|x - T_r(I - rA)x\|^2 \\ &\leq \|x - u\|^2 - 2\alpha r \|Ax - Au\|^2 + 2r \|x - T_r(I - rA)x\| \|Ax - Au\| - \|x - T_r(I - rA)x\|^2 \\ &= \|x - u\|^2 - 2\alpha r \left(\|Ax - Au\| - \frac{1}{2\alpha} \|x - T_r(I - rA)x\| \right)^2 - \frac{2\alpha - r}{2\alpha} \|x - T_r(I - rA)x\|^2 \\ &\leq \|x - u\|^2 - \frac{2\alpha - r}{2\alpha} \|x - T_r(I - rA)x\|^2. \end{aligned}$$

Hence (4.3) holds. Finally, let $0 \leq r \leq 2\alpha$. Since T_r is nonexpansive and (4.1), we have

$$\begin{aligned} \|T_r(I - rA)x - T_r(I - rA)y\|^2 &\leq \|(I - rA)x - (I - rA)y\|^2 \\ &= \|x - y\|^2 - 2r \langle Ax - Ay, x - y \rangle + r^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2r\alpha \|Ax - Ay\|^2 + r^2 \|Ax - Ay\|^2 \\ &= \|x - y\|^2 - r(2\alpha - r) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 \end{aligned}$$

for all $x, y \in C$ and hence $T_r(I - rA)$ is nonexpansive. \square

Lemma 4.5. Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfies (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H with $EP(f, A) \neq \emptyset$. Let $\{W_n\}$ be a sequence of mappings of C into itself defined by

$$W_n = T_{r_n}(I - r_n A),$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$. Then $\{W_n\}$ satisfies NST-condition.

Proof. We note that $\bigcap_{n=1}^{\infty} F(W_n) = EP(f, A) \neq \emptyset$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - W_n z_n\| = 0$ and $z \in \omega_w\{z_n\}$. Then $z_{n_i} \rightarrow z$ for some subsequence $\{n_i\}$ of $\{n\}$. For each $n \in \mathbb{N}$, let $y_n = W_n z_n$. Then $y_{n_i} \rightarrow z$. Since $\liminf_{n \rightarrow \infty} r_n > 0$ and A is $(1/\alpha)$ -Lipschitzian, we have

$$\lim_{n \rightarrow \infty} \left\| \frac{z_n - y_n}{r_n} \right\| = \lim_{n \rightarrow \infty} \frac{1}{r_n} \|z_n - W_n z_n\| = 0 \quad (4.4)$$

and

$$\lim_{n \rightarrow \infty} \|Az_n - Ay_n\| \leq \lim_{n \rightarrow \infty} \frac{1}{\alpha} \|z_n - y_n\| = 0, \quad (4.5)$$

respectively. Notice that

$$f(y_n, y) + \frac{1}{r_n} \langle y - y_n, y_n - (z_n - r_n Az_n) \rangle \geq 0 \quad \text{for all } y \in C.$$

So, from (A2) and A is monotone,

$$\begin{aligned} f(y, y_n) &\leq \left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle + \langle y - y_n, Az_n \rangle \\ &= \left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle + \langle y - y_n, Az_n - Ay_n \rangle + \langle y - y_n, Ay_n - Ay \rangle + \langle y - y_n, Ay \rangle \\ &\leq \left\langle y - y_n, \frac{y_n - z_n}{r_n} \right\rangle + \langle y - y_n, Az_n - Ay_n \rangle + \langle y - y_n, Ay \rangle. \end{aligned}$$

In particular,

$$f(y, y_{n_i}) \leq \left\langle y - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{r_{n_i}} \right\rangle + \langle y - y_{n_i}, Az_{n_i} - Ay_{n_i} \rangle + \langle y - y_{n_i}, Ay \rangle.$$

This together with (4.4), (4.5), (A4) and $y_{n_i} \rightarrow z$ gives

$$f(y, z) \leq \langle y - z, Ay \rangle. \quad (4.6)$$

Put $z_t = ty + (1-t)z$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (4.6) we have

$$f(z_t, z) \leq \langle z_t - z, Az_t \rangle. \quad (4.7)$$

It follows from (A1), (A4) and (4.7) that

$$\begin{aligned} 0 &= f(z_t, z_t) \\ &\leq tf(z_t, y) + (1-t)f(z_t, z) \\ &\leq tf(z_t, y) + (1-t)\langle z_t - z, Az_t \rangle \\ &= tf(z_t, y) + (1-t)t\langle y - z, Az_t \rangle \end{aligned}$$

and hence

$$0 \leq f(z_t, y) + (1-t)\langle y - z, Az_t \rangle$$

Letting $t \rightarrow 0$ and using (A3), we get

$$0 \leq f(z, y) + \langle y - z, Az \rangle \quad \text{for all } y \in C$$

and hence $z \in EP(f, A) = \bigcap_{n=1}^{\infty} F(W_n)$. This implies that $\{W_n\}$ satisfies NST-condition. This completes the proof. \square

Theorem 4.6. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of

L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ such that $\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} f(y_n, y) + \langle Ax_n, y - y_n \rangle + \frac{1}{r_n}(y - y_n, y_n - x_n) \geq 0 & \text{for all } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n y_n, \end{cases} \tag{4.8}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $\{S_n\}$ satisfies NST-condition, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)}(x_n)$.

Proof. For each $n \in \mathbb{N}$, we have $y_n = T_{r_n}(I - r_n A)x_n$. By Lemmas 2.10 and 4.5, $\{S_n T_{r_n}(I - r_n A)\}$ is a family of L_n -Lipschitzian mappings of C into itself satisfying NST-condition. Applying Theorem 4.2, $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)}(x_n)$. \square

From Theorem 4.6, letting $L_n \equiv 1$ gives

Corollary 4.7. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (4.8), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $\{S_n\}$ satisfies NST-condition, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap EP(f, A)}(x_n)$.

From Corollary 4.7, letting $S_n \equiv S$ gives

Corollary 4.8 ([5], Theorem 3.1). Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let S be a nonexpansive mapping of C into itself such that $F(S) \cap EP(f, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} f(y_n, y) + \langle Ax_n, y - y_n \rangle + \frac{1}{r_n}(y - y_n, y_n - x_n) \geq 0 & \text{for all } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. Then the sequence $\{x_n\}$ converges weakly to $z \in F(S) \cap EP(f, A)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap EP(f, A)}(x_n)$.

We obtain [6, Theorem 4.1] from Corollary 4.8 by letting $A \equiv 0$.

Corollary 4.9. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and

$$\begin{cases} f(y_n, y) + \frac{1}{r_n}(y - y_n, y_n - x_n) \geq 0 & \text{for all } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S y_n, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap EP(f)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap EP(f)}(x_n)$.

Theorem 4.10. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n P_C(x_n - r_n A x_n), \tag{4.9}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $\{S_n\}$ satisfies NST-condition, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)}(x_n)$.

Proof. Setting $f \equiv 0$ in Theorem 4.6, we have $EP(f, A) = VI(C, A)$ and $T_r = P_C$ for all $r > 0$. \square

We immediately obtain [7, Theorem 4] from our Theorem 4.10 by letting $L_n \equiv 1$ and applying Lemmas 2.8 and 2.9.

Corollary 4.11. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H . Let $\{S_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by (4.9), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$. If $(\{S_n\}, S)$ satisfies AKTT-condition and $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$, then $\{x_n\}$ converges weakly to $z \in \bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$, where $z = \lim_{n \rightarrow \infty} P_{\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)}(x_n)$.

Remark 4.12 ([7, Theorem 4]). (See Corollary 4.11) is also a direct consequence of [3, Lemma 3.2] since $\{S_n P_C(I - r_n A)\}$ is a family of nonexpansive mappings satisfying AKTT-condition (II). In fact, by Lemma 2.10,

$$\bigcap_{n=1}^{\infty} F(S_n P_C(I - r_n A)) = \bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) = F(S) \cap VI(C, A) \neq \emptyset.$$

Let $\{n_i\}$ be a subsequence of $\{n\}$. Since $\{r_{n_i}\} \subset [c, d]$, there exists a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that $r_{n_{i_j}} \rightarrow r \in [c, d]$. Then $SP_C(I - rA)$ is a nonexpansive mapping of C into itself and

$$F(SP_C(I - rA)) = F(S) \cap VI(C, A) = \bigcap_{n=1}^{\infty} F(S_n P_C(I - r_n A)).$$

Let B be a bounded subset of C . Since A is $(1/\alpha)$ -Lipschitzian and $P_C(I - rA)$ is nonexpansive of C into itself, $\{Az : z \in B\}$ and $\tilde{B} = \{P_C(I - rA)z : z \in B\} \subset C$ are bounded. By Lemma 2.6, we have

$$\lim_{n \rightarrow \infty} \sup\{\|Sy - S_n y\| : y \in \tilde{B}\} = 0.$$

Let $M = \sup\{\|Az\| : z \in B\}$. Then

$$\begin{aligned} & \|SP_C(I - rA)z - S_{n_{i_j}} P_C(I - r_{n_{i_j}} A)z\| \\ & \leq \|SP_C(I - rA)z - S_{n_{i_j}} P_C(I - rA)z\| + \|S_{n_{i_j}} P_C(I - rA)z - S_{n_{i_j}} P_C(I - r_{n_{i_j}} A)z\| \\ & \leq \sup\{\|Sy + S_{n_{i_j}} y\| : y \in \tilde{B}\} + \|P_C(I - rA)z - P_C(I - r_{n_{i_j}} A)z\| \\ & \leq \sup\{\|Sy - S_{n_{i_j}} y\| : y \in \tilde{B}\} + \|(I - rA)z - (I - r_{n_{i_j}} A)z\| \\ & \leq \sup\{\|Sy - S_{n_{i_j}} y\| : y \in \tilde{B}\} + |r - r_{n_{i_j}}| M \end{aligned}$$

for all $z \in B$ and $j \in \mathbb{N}$. So, we get

$$\lim_{j \rightarrow \infty} \sup\{\|SP_C(I - rA)z - S_{n_{i_j}} P_C(I - r_{n_{i_j}} A)z\| : z \in B\} = 0.$$

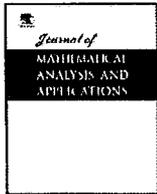
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Strong convergence theorems for a countable family of quasi-Lipschitzian mappings and its applications

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ABSTRACT

We use the hybrid method in mathematical programming to obtain strong convergence to common fixed points of a countable family of quasi-Lipschitzian mappings. As a consequence, several convergence theorems for quasi-nonexpansive mappings and asymptotically κ -strict pseudo-contractions are deduced. We also establish strong convergence of iterative sequences for finding a common element of the set of fixed point, the set of solutions of an equilibrium problem, and the set of solutions of a variational inequality. With an appropriate setting, we obtain the corresponding results due to Tada-Takahashi and Nakajo-Shimoji-Takahashi.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H . A mapping $T : C \rightarrow H$ is said to be *Lipschitzian* if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be L -Lipschitzian. We denote by $F(T)$ the set of fixed points of T . A mapping T is said to be *quasi-Lipschitzian* if $F(T) \neq \emptyset$ and there exists a positive constant L such that

$$\|Tx - y\| \leq L\|x - y\| \quad \text{for all } x \in C \text{ and } y \in F(T).$$

In this case we say that T is quasi- L -Lipschitzian.

Remark 1.1. It follows directly from the definitions above that:

- (1) If T is L -Lipschitzian with $F(T) \neq \emptyset$, then T is quasi- L -Lipschitzian.
- (2) If T is quasi- L_1 -Lipschitzian and $L_1 < L_2$, then T is quasi- L_2 -Lipschitzian.
- (3) T is (quasi-) 1-Lipschitzian if and only if T is (quasi-) nonexpansive.

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Throughout the paper, we deal with quasi- L -Lipschitzian where $L \geq 1$. There are many methods for approximating fixed points of a nonexpansive mapping. In 1953, Mann [10] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [16]. In an infinite-dimensional Hilbert space, strong convergence of Mann iteration is not guaranteed [5]. To obtain strong convergence results, Nakajo and Takahashi [13] proposed the following modification of Mann iteration method (1.1):

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C: \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C: \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_1, \quad n = 1, 2, 3, \dots, \end{cases} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H . Recently, the present authors [14] extended iteration (1.2) to obtain strong convergence to a common fixed point of a countable family of Lipschitzian mappings $\{T_n\}_{n=1}^\infty$ by the following iteration:

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ C_n = \{z \in C: \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C: \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_1, \quad n = 1, 2, 3, \dots, \end{cases} \tag{1.3}$$

where

$$\theta_n = (1 - \alpha_n)(L_n^2 - 1)(\text{diam } C)^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The main result of [14] is based on the condition introduced by Aoyama et al. [1].

In this paper we introduce a new condition, so-called the NST*-condition, for a class of mappings and use this condition to obtain strong convergence theorem for finding common fixed points of a countable family of quasi- L_n -Lipschitzian mappings in a real Hilbert space. As a consequence, several convergence theorems for quasi-nonexpansive mappings and asymptotically κ -strict pseudo-contractions are deduced. We also establish strong convergence of iterative sequences for finding a common element of the set of fixed point, the set of solutions of an equilibrium problem, and the set of solutions of a variational inequality. With an appropriate setting, we obtain the corresponding results due to Tada and Takahashi [18] and Nakajo, Shimoji, and Takahashi [11].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle \tag{2.1}$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. We write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x . It is also known that H satisfies:

(1) The Opial's condition [15], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

(2) The Kadec-Klee property [7,19], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ together imply $x_n \rightarrow x$.

Let C be a nonempty closed convex subset of H . Then, for any $x \in H$, there exists a unique nearest point in C , denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C.$$

Such a mapping P_C is called the *metric projection* of H onto C . We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C x \text{ if and only if } \langle x - z, z - y \rangle \geq 0 \text{ for all } y \in C.$$

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a Banach space, let $\{T_n\}$ be a family of mappings of C into H with $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\omega_w\{z_n\}$ denotes the set of all weak subsequential limits of a bounded sequence $\{z_n\}$ in C . $\{T_n\}$ is said to satisfy:

(a) The *AKTT-condition* [1] if for each bounded subset B of C ,

$$\sum_{n=1}^\infty \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

(b) The *NST-condition* [11] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \Rightarrow \omega_w\{z_n\} \subset \bigcap_{n=1}^\infty F(T_n).$$

Inspired by conditions above, we introduce the following one:

(c) $\{T_n\}$ is said to satisfy the *NST*-condition* [12] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = \lim_{n \rightarrow \infty} \|z_n - z_{n+1}\| = 0 \Rightarrow \omega_w\{z_n\} \subset \bigcap_{n=1}^\infty F(T_n).$$

Remark 2.1. It follows directly from the definitions above that:

- (i) If $\{T_n\}$ satisfies the NST-condition, then $\{T_n\}$ satisfies the NST*-condition.
- (ii) If $\{T_n\}$ satisfies the NST*-condition and $T_n z \rightarrow z \in C$, then $z \in \bigcap_{n=1}^\infty F(T_n)$.

Lemma 2.2. Let C be a nonempty closed convex subset of a real Banach space and let $\{T_n\}$ be a family of mappings of C into itself with $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Let $\{S_n\}$ be a family of mappings of C into itself defined by

$$S_n = \beta_n I + (1 - \beta_n)T_n,$$

for all $n \in \mathbb{N}$, where $\{\beta_n\}$ is a sequence in $[0, b] \subset [0, 1)$. Then $\bigcap_{n=1}^\infty F(T_n) = \bigcap_{n=1}^\infty F(S_n)$ and the following statements hold:

- (i) If $\{T_n\}$ satisfies the NST-condition, then $\{S_n\}$ satisfies the NST-condition.
- (ii) If $\{T_n\}$ satisfies the NST*-condition, then $\{S_n\}$ satisfies the NST*-condition.

Proof. It is easy to see that $\bigcap_{n=1}^\infty F(T_n) = \bigcap_{n=1}^\infty F(S_n)$. To see (i), let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - S_n z_n\| = 0$. It follows from $\beta_n \leq b < 1$ that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = \lim_{n \rightarrow \infty} \frac{1}{1 - \beta_n} \|z_n - S_n z_n\| = 0.$$

Since $\{T_n\}$ satisfies the NST-condition, we have $\omega_w\{z_n\} \subset \bigcap_{n=1}^\infty F(T_n) = \bigcap_{n=1}^\infty F(S_n)$. This implies that $\{S_n\}$ satisfies the NST-condition.

(ii) Since the proof is almost the same as (i), it is omitted. \square

Lemma 2.3. Let C and K be nonempty closed convex subsets of a real Banach space. Let $\{S_n\}$ be a family of L_n -Lipschitzian mappings of C into K with $L_n \geq 1$, $L_n \rightarrow 1$ and $F(S) := \bigcap_{n=1}^\infty F(S_n) \neq \emptyset$ and let $\{W_n\}$ be a family of mappings of K into C with $F(W) := \bigcap_{n=1}^\infty F(W_n) \neq \emptyset$ and

$$\|W_n x - u\|^2 \leq \|x - u\|^2 - a_n \|W_n x - x\|^2 \text{ for all } x \in K, u \in F(W) \text{ and } n \in \mathbb{N},$$

where $\{a_n\}$ is a sequence in $[a, \infty) \subset (0, \infty)$. Let $\{T_n\}$ be a family of mappings defined by

$$T_n = S_n W_n \text{ for all } n \in \mathbb{N}.$$

Then the following statements hold:

- (i) If $F := \bigcap_{n=1}^\infty F(T_n) = F(S) \cap F(W) \neq \emptyset$, then $\{T_n\}$ is a family of quasi- L_n -Lipschitzian mappings of K into itself, that is, for every $n \in \mathbb{N}$, $x \in K$ and $u \in F$, $\|T_n x - u\| \leq L_n \|x - u\|$ holds.

- (ii) If $\{S_n\}$ and $\{W_n\}$ satisfy NST-condition and $F(S) \cap F(W) \neq \emptyset$, then $\{T_n\}$ satisfies NST-condition and $\bigcap_{n=1}^{\infty} F(T_n) = F(S) \cap F(W)$.
- (iii) If $\{S_n\}$ and $\{W_n\}$ satisfy NST*-condition and $F(S) \cap F(W) \neq \emptyset$, then $\{T_n\}$ satisfies NST*-condition and $\bigcap_{n=1}^{\infty} F(T_n) = F(S) \cap F(W)$.

Proof. (i) is obvious.

(ii) Let $\{z_n\}$ be a bounded sequence in K such that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0.$$

Let $u \in F(S) \cap F(W)$ be given. Then

$$\begin{aligned} \|T_n z_n - u\|^2 &= \|S_n W_n z_n - u\|^2 \leq L_n^2 \|W_n z_n - u\|^2 \\ &\leq L_n^2 \|z_n - u\|^2 - a_n L_n^2 \|W_n z_n - z_n\|^2 \leq L_n^2 \|z_n - u\|^2 - a \|W_n z_n - z_n\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} a \|W_n z_n - z_n\|^2 &\leq (L_n^2 - 1) \|z_n - u\|^2 + (\|z_n - u\|^2 - \|T_n z_n - u\|^2) \\ &= (L_n^2 - 1) \|z_n - u\|^2 + (\|z_n - u\| - \|T_n z_n - u\|)(\|z_n - u\| + \|T_n z_n - u\|) \\ &\leq (L_n^2 - 1) \|z_n - u\|^2 + \|z_n - T_n z_n\| (\|z_n - u\| + \|T_n z_n - u\|). \end{aligned}$$

Since $\{z_n - u\}$ and $\{T_n z_n - u\}$ are bounded,

$$\lim_{n \rightarrow \infty} \|W_n z_n - z_n\| = 0.$$

So, we get

$$\|S_n W_n z_n - W_n z_n\| \leq \|S_n W_n z_n - z_n\| + \|W_n z_n - z_n\| = \|T_n z_n - z_n\| + \|W_n z_n - z_n\| \rightarrow 0.$$

Since $\{S_n\}$ and $\{W_n\}$ satisfy the NST-condition, we have $\omega_w \{z_n\} \subset F(S) \cap F(W)$. It easy to see that $F(S) \cap F(W) \subset \bigcap_{n=1}^{\infty} F(T_n)$. To see the reverse inclusion, let $z \in \bigcap_{n=1}^{\infty} F(T_n)$. Follow the first part of the proof above but now let $z_n \equiv z$. Then $z \in F(S) \cap F(W)$. This implies that $\{T_n\}$ satisfies the NST-condition.

The proof above does not invoke condition $\lim_{n \rightarrow \infty} \|z_{n+1} - z_n\|$ at all. So assertion (iii) holds for the NST*-condition as well. This completes the proof. \square

Lemma 2.4. (See [1, Lemma 3.2].) Let C be a nonempty closed subset of a Banach space and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition, then the mapping $T : C \rightarrow C$ defined by

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in C, \tag{2.3}$$

satisfies

$$\lim_{n \rightarrow \infty} \sup \{ \|Tz - T_n z\| : z \in B \} = 0$$

for each bounded subset B of C .

From now on, we will write $(\{T_n\}, T)$ satisfies AKTT-condition if $\{T_n\}$ satisfies AKTT-condition and T is defined by (2.3). Recall that a mapping T is *demi-closed* at y , if $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$.

Lemma 2.5. Let C be a nonempty closed convex subset of a Banach space and let $\{T_n\}$ be a family of quasi- L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition, $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ and $I - T$ is demi-closed at 0. Then $\{T_n\}$ satisfies the NST-condition. Consequently, $\{T_n\}$ satisfies the NST*-condition.

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$. We apply Lemma 2.4 to get

$$\|z_n - Tz_n\| \leq \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \leq \|z_n - T_n z_n\| + \sup \{ \|T_n z - Tz\| : z \in \{z_n\} \} \rightarrow 0.$$

Since $I - T$ is demi-closed at 0, $\omega_w \{z_n\} \subset F(T)$. This implies that $\{T_n\}$ satisfies the NST-condition. \square

Lemma 2.6. (See [7, Theorem 10.3].) Let C be a nonempty closed convex subset of a reflexive Banach space which satisfies Opial's condition and let T be a nonexpansive mapping of C into itself. Then $I - T$ is demi-closed at 0.

Lemma 2.7. Let C be a nonempty closed convex subset of a reflexive Banach space which satisfies Opial's condition and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition and $F(T) = \bigcap_{n=1}^\infty F(T_n)$. Then $\{T_n\}$ satisfies the NST-condition.

Proof. From the definition of T , we have T is nonexpansive. By Lemma 2.6, we have $I - T$ is demi-closed at 0. Applying Lemma 2.5, $\{T_n\}$ satisfies the NST-condition. \square

Lemma 2.8. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi- L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. If $\{T_n\}$ satisfies the NST*-condition, then $\bigcap_{n=1}^\infty F(T_n)$ is closed and convex.

Proof. To see that $\bigcap_{n=1}^\infty F(T_n)$ is closed, let $\{z_n\}$ be a sequence in $\bigcap_{n=1}^\infty F(T_n)$ with $z_n \rightarrow z$. Since

$$\|T_n z - z\| \leq \|T_n z - z_n\| + \|z_n - z\| \leq (L_n + 1)\|z_n - z\|,$$

we get

$$\lim_{n \rightarrow \infty} \|T_n z - z\| = 0.$$

By Remark 2.1(ii), we have $z \in \bigcap_{n=1}^\infty F(T_n)$ and hence $\bigcap_{n=1}^\infty F(T_n)$ is closed. Now we prove that $\bigcap_{n=1}^\infty F(T_n)$ is convex. To this end, let $x, y \in \bigcap_{n=1}^\infty F(T_n)$. Put $z = tx + (1 - t)y$, where $t \in (0, 1)$. From (2.2), we have

$$\begin{aligned} \|T_n z - z\|^2 &= \|t(T_n z - x) + (1 - t)(T_n z - y)\|^2 \\ &= t\|T_n z - x\|^2 + (1 - t)\|T_n z - y\|^2 - t(1 - t)\|x - y\|^2 \\ &\leq tL_n^2\|z - x\|^2 + (1 - t)L_n^2\|z - y\|^2 - t(1 - t)\|x - y\|^2 \\ &= t(1 - t)^2L_n^2\|x - y\|^2 + t^2(1 - t)L_n^2\|x - y\|^2 - t(1 - t)\|x - y\|^2 \\ &= t(1 - t)(L_n^2 - 1)\|x - y\|^2. \end{aligned}$$

This together with $L_n \rightarrow 1$ gives

$$\lim_{n \rightarrow \infty} \|T_n z - z\| = 0.$$

Then $z \in \bigcap_{n=1}^\infty F(T_n)$ and hence $\bigcap_{n=1}^\infty F(T_n)$ is convex. This completes the proof. \square

We also need the following lemma:

Lemma 2.9. (See [9, Lemma 2.2].) Let C be a nonempty closed convex subset of a real Hilbert space H . Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set

$$\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$$

is convex (and closed).

3. Strong convergence theorems

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi- L_n -Lipschitzian mappings of C into itself with $L_n \geq 1$ and $F := \bigcap_{n=1}^\infty F(T_n)$ is nonempty and bounded (that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in F$, $\|T_n x - u\| \leq L_n \|x - u\|$ holds). Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$. Let $\{x_n\}$ be a sequence in C defined as follows:

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2 + (1 - \alpha_n)\theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases} \tag{3.1}$$

where

$$\theta_n = \Delta_n^2 (L_n^2 - 1) \rightarrow 0, \quad \Delta_n = \sup\{\|x_n - z\| : z \in F\} < \infty.$$

If $\{T_n\}$ satisfies the NST*-condition, then $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. We first prove that C_n and Q_n are closed and convex for each $n \in \mathbb{N}$. From the definition of C_n and Q_n , it is obvious that C_n is closed and Q_n is closed and convex for each $n \in \mathbb{N}$. It follows from Lemma 2.9 that C_n is convex. Next, we prove that

$$F \subset C_n \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Let $p \in F$. Then

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)T_n x_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|T_n x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)L_n^2 \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2 \\ &= \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2 + (1 - \alpha_n)(L_n^2 - 1)\|x_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2 + (1 - \alpha_n)\theta_n, \end{aligned}$$

from which $p \in C_n$ and hence (3.2) holds. Next, we show that $F \subset Q_n$ for all $n \in \mathbb{N}$. For $n = 1$, we have $F \subset C = Q_1$. Suppose that $F \subset Q_k$ for some $k \in \mathbb{N}$. Then $\emptyset \neq F \subset C_k \cap Q_k$ and there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k} x_1$. Then

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0$$

for each $z \in C_k \cap Q_k$. In particular,

$$\langle x_{k+1} - p, x_1 - x_{k+1} \rangle \geq 0$$

for each $p \in F$. It follows that $F \subset Q_{k+1}$ and hence, by induction,

$$F \subset Q_n \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Now, from (3.2) and (3.3), we have

$$F \subset C_n \cap Q_n \quad \text{for all } n \in \mathbb{N}.$$

This implies that the sequence $\{x_n\}$ is well defined. It follows from $x_n = P_{Q_n} x_1$ that

$$\|x_n - x_1\| \leq \|z - x_1\| \quad \text{for all } z \in Q_n \text{ and all } n \in \mathbb{N}.$$

Let $z \in F \subset Q_n$ for all $n \in \mathbb{N}$. Then

$$\|x_n - x_1\| \leq \|z - x_1\| \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, from $x_{n+1} = P_{C_n \cap Q_n} x_1 \in Q_n$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\| \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $\{\|x_n - x_1\|\}$ is nondecreasing and bounded. So $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists and $\{x_n\}$ is bounded. Since $x_{n+1} = P_{C_n \cap Q_n} x_1 \in Q_n$, we have

$$\langle x_n - x_{n+1}, x_1 - x_n \rangle \geq 0.$$

It follows from (2.1) that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_1) - (x_n - x_1)\|^2 \\ &= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_{n+1} - x_n, x_n - x_1 \rangle \\ &\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Since $x_{n+1} \in C_n$, it follows that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2 + (1 - \alpha_n)\theta_n. \tag{3.5}$$

Moreover, since $y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n$, we have

$$\|y_n - x_{n+1}\|^2 = \alpha_n \|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\|T_n x_n - x_{n+1}\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2. \tag{3.6}$$

Substitute (3.6) into (3.5) to get

$$\|T_n x_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n \rightarrow 0.$$

This together with (3.4) gives

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

Finally, we show that $x_n \rightarrow w$, where $w = P_F x_1$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightarrow w'$. It follows from the NST*-condition of $\{T_n\}$ that $w' \in F$. Since $x_n = P_{Q_n} x_1$ and $w \in F \subset Q_n$, we have

$$\|x_n - x_1\| \leq \|w - x_1\|.$$

It follows from $w = P_F x_1$ and the lower semi-continuity of the norm that

$$\|w - x_1\| \leq \|w' - x_1\| \leq \liminf_{k \rightarrow \infty} \|x_{n_k} - x_1\| \leq \limsup_{k \rightarrow \infty} \|x_{n_k} - x_1\| \leq \|w - x_1\|.$$

That is,

$$\lim_{k \rightarrow \infty} \|x_{n_k} - x_1\| = \|w' - x_1\| = \|w - x_1\|.$$

Using the Kadec-Klee property of H , we obtain that

$$\lim_{k \rightarrow \infty} x_{n_k} = w' = w.$$

Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_F x_1$. \square

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi- L_n -Lipschitzian mappings of C into itself with $L_n \geq 1$ and $F := \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and bounded (that is, for every $n \in \mathbb{N}$, $x \in C$ and $u \in F$, $\|T_n x - u\| \leq L_n \|x - u\|$ holds). Assume that $\{\beta_n\}$ is a sequence in $[0, b] \subset [0, 1)$. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \beta_n x_n + (1 - \beta_n) T_n x_n, \\ C_n = \{z \in C: \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C: \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases} \tag{3.7}$$

where

$$\theta_n = \Delta_n^2 (L_n^2 - 1) \rightarrow 0, \quad \Delta_n = \sup\{\|x_n - z\|: z \in F\} < \infty.$$

If $\{T_n\}$ satisfies the NST*-condition, then $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. Let S_n be a mapping defined by

$$S_n = \beta_n I + (1 - \beta_n) T_n \quad \text{for all } n \in \mathbb{N}.$$

It is clear that S_n is quasi- L_n -Lipschitzian and $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(S_n)$. By Lemma 2.2(ii), we have $\{S_n\}$ satisfies the NST*-condition. Applying Theorem 3.1 for the case $\alpha_n \equiv 0$, we have $\{x_n\}$ converges strongly to $P_F x_1$. \square

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \geq 1$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty and bounded. Suppose that $\{T_n\}$ satisfies the NST*-condition. Then the following statements hold:*

- (i) If $\{\alpha_n\}$ is a sequence in $[0, 1)$, then $\{x_n\}$ defined by (3.1) converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)} x_1$.
- (ii) If $\{\beta_n\}$ is a sequence in $[0, b] \subset [0, 1)$, then $\{x_n\}$ defined by (3.7) converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)} x_1$.

Remark 3.4. Corollary 3.3(ii) includes [14, Theorem 10] as a special case since the AKTT-condition implies the NST*-condition (see Lemma 2.7).

4. Deduced theorems

4.1. Convergence theorems for a countable family of quasi-nonexpansive mappings

It is not difficult to see from the proof of the preceding theorems in Section 3 that the boundedness of a common fixed point set can be discarded if $\{T_n\}$ is a sequence of quasi-nonexpansive mappings. So, from Theorem 3.1, we immediately obtain the following theorem.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into itself with a nonempty common fixed point set. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\ C_n = \{z \in C: \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T_n x_n\|^2\}, \\ Q_n = \{z \in C: \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1)$. If $\{T_n\}$ satisfies the NST*-condition, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^\infty F(T_n)} x_1$.

Corollary 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\{T_n\}$ be a family of quasi-nonexpansive mappings of C into itself with a nonempty common fixed point set. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \beta_n x_n + (1 - \beta_n) T_n x_n, \\ C_n = \{z \in C: \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C: \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases}$$

where $\{\beta_n\}$ is a sequence in $[0, b] \subset [0, 1)$. If $\{T_n\}$ satisfies the NST*-condition, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^\infty F(T_n)} x_1$.

4.2. Strong convergence theorems for an asymptotically κ -strict pseudo-contraction

Let C be a subset of a real Hilbert space H and $\kappa \in [0, 1)$. A mapping $T : C \rightarrow C$ is said to be an asymptotically κ -strict pseudo-contraction if there exists a sequence $\{k_n\}$ of real numbers such that $k_n \geq 1$, $k_n \rightarrow 1$, and

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \kappa \|(I - T^n)x - (I - T^n)y\|^2$$

for all $x, y \in C$ and $n \in \mathbb{N}$. Note that the class of asymptotically κ -strict pseudo-contractions strictly includes the class of asymptotically nonexpansive mappings, a concept introduced by Goebel and Kirk [6] in 1972. We note that T is asymptotically nonexpansive if and only if T is asymptotically 0-strict pseudo-contraction. In 1991, Schu [17] introduced the modified Mann iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n \quad \text{for all } n \in \mathbb{N},$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. In 2008, Kim and Xu [9] introduced the hybrid method as follows:

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ C_n = \{z \in C: \|y_n - z\|^2 \leq \|x_n - z\|^2 + (\kappa - \alpha_n(1 - \alpha_n))\|x_n - T^n x_n\|^2 + \bar{\theta}_n\}, \\ Q_n = \{z \in C: \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases} \tag{4.1}$$

where

$$\bar{\theta}_n = \Delta_n^2 (1 - \alpha_n)(k_n - 1) \rightarrow 0 \quad (n \rightarrow \infty), \quad \Delta_n = \sup\{\|x_n - z\|: z \in F(T)\} < \infty$$

and proved that if $\{\alpha_n\}$ is a sequence in $[0, b] \subset [0, 1 - \kappa)$ and $F(T)$ is nonempty and bounded, then $\{x_n\}$ defined by (4.1) converges strongly to $P_{F(T)} x_1$.

We present a convergence theorem for an asymptotically κ -strict pseudo-contraction with a new control parameter which is complementary to Kim and Xu's result [9, Theorem 4.1]. Let us start with some auxiliary lemmas.

Lemma 4.3. (See [9, Proposition 2.6].) Let C be a nonempty closed subset of a real Hilbert space H and let T be an asymptotically κ -strict pseudo-contraction of C into itself with $0 \leq \kappa < 1$ and a sequence $\{k_n\} \subset [1, \infty)$ satisfying $k_n \rightarrow 1$.

(i) For each $n \in \mathbb{N}$, T^n satisfies the Lipschitz condition:

$$\|T^n x - T^n y\| \leq \ell_n \|x - y\| \quad \text{for all } x, y \in C,$$

$$\text{where } \ell_n := \frac{\kappa + \sqrt{1 + (1 - \kappa)(k_n - 1)}}{1 - \kappa}.$$

(ii) $I - T$ is demi-closed at 0.

Lemma 4.4. Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be an asymptotically κ -strict pseudo-contraction of C into itself for some $0 \leq \kappa < 1$ and a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \rightarrow 1$. Assume that $F(T) \neq \emptyset$. Let S_n be a mapping defined by

$$S_n = \lambda_n I + (1 - \lambda_n)T^n \quad \text{for all } n \in \mathbb{N},$$

where $\kappa \leq \lambda_n \leq b < 1$. Then $\{S_n\}$ is a family of $\sqrt{k_n}$ -Lipschitzian mappings of C into itself satisfies the NST*-condition and $\bigcap_{n=1}^\infty F(S_n) = F(T)$.

Proof. We first observe that, since $\kappa \leq \lambda_n$,

$$\begin{aligned} \|S_n x - S_n y\|^2 &= \|\lambda_n(x - y) + (1 - \lambda_n)(T^n x - T^n y)\|^2 \\ &= \lambda_n \|x - y\|^2 + (1 - \lambda_n) \|T^n x - T^n y\|^2 - \lambda_n(1 - \lambda_n) \|(I - T^n)x - (I - T^n)y\|^2 \\ &\leq \lambda_n \|x - y\|^2 + (1 - \lambda_n)(k_n \|x - y\|^2 + \kappa \|(I - T^n)x - (I - T^n)y\|^2) \\ &\quad - \lambda_n(1 - \lambda_n) \|(I - T^n)x - (I - T^n)y\|^2 \\ &= (\lambda_n + (1 - \lambda_n)k_n) \|x - y\|^2 - (1 - \lambda_n)(\lambda_n - \kappa) \|(I - T^n)x - (I - T^n)y\|^2 \\ &\leq k_n \|x - y\|^2, \end{aligned}$$

for all $x, y \in C$. That is, S_n is $\sqrt{k_n}$ -Lipschitzian. We now prove that $\{S_n\}$ satisfies the NST*-condition, let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n \rightarrow \infty} \|z_n - S_n z_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z_n - z_{n+1}\| = 0.$$

Since $z_n - S_n z_n = (1 - \lambda_n)(z_n - T^n z_n)$ and $\lambda_n \leq b < 1$,

$$\lim_{n \rightarrow \infty} \|z_n - T^n z_n\| = 0.$$

Note that $\{\ell_n\}$ defined in Lemma 4.3(i) is bounded and

$$\begin{aligned} \|z_{n+1} - T z_{n+1}\| &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + \|T^{n+1} z_{n+1} - T z_{n+1}\| \\ &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + \ell_1 \|T^n z_{n+1} - z_{n+1}\| \\ &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + \ell_1 (\|T^n z_{n+1} - T^n z_n\| + \|T^n z_n - z_n\| + \|z_n - z_{n+1}\|) \\ &\leq \|z_{n+1} - T^{n+1} z_{n+1}\| + \ell_1 (\ell_n + 1) \|z_{n+1} - z_n\| + \ell_1 \|T^n z_n - z_n\|. \end{aligned}$$

Then

$$\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0.$$

By Lemma 4.3(ii), we have $\omega_w \{z_n\} \subset F(T)$. It is easy to see that $F(T) = \bigcap_{n=1}^\infty F(S_n)$. It follows that $\{S_n\}$ satisfies the NST*-condition. \square

Theorem 4.5. Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be an asymptotically κ -strict pseudo-contraction of C into itself for some $0 \leq \kappa < 1$ and a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$. Assume that $\{\alpha_n\}$ is a sequence in $[\kappa, 1)$ and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (\kappa - \alpha_n)(1 - \alpha_n) \|x_n - T^n x_n\|^2 + \bar{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases} \tag{4.2}$$

where

$$\tilde{\theta}_n = \Delta_n^2 \left(\frac{1 - \alpha_n}{1 - \kappa} \right) (k_n - 1) \rightarrow 0 \quad (n \rightarrow \infty), \quad \Delta_n = \sup \{ \|x_n - z\| : z \in F(T) \} < \infty.$$

Then the sequence $\{x_n\}$ defined by (4.2) converges strongly to $P_{F(T)}x_1$.

Proof. Let S_n be a mapping defined by

$$S_n = \kappa I + (1 - \kappa)T^n \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 4.4, we have $\{S_n\}$ is a family of $\sqrt{k_n}$ -Lipschitzian mappings of C into itself satisfies the NST*-condition and $F(T) = \bigcap_{n=1}^{\infty} F(S_n)$. For a given sequence $\{\alpha_n\}$ in $[\kappa, 1)$, we define $\beta_n \equiv \frac{\alpha_n - \kappa}{1 - \kappa} \in [0, 1)$. Then

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)T^n x_n = (\beta_n + (1 - \beta_n)\kappa)x_n + (1 - \beta_n)(1 - \kappa)T^n x_n \\ &= \beta_n x_n + (1 - \beta_n)(\kappa x_n + (1 - \kappa)T^n x_n) = \beta_n x_n + (1 - \beta_n)S_n x_n, \end{aligned}$$

and

$$(\kappa - \alpha_n)(1 - \alpha_n) \|x_n - T^n x_n\|^2 = -\beta_n(1 - \beta_n) \|x_n - S_n x_n\|^2.$$

Applying Theorem 3.1, we have $\{x_n\}$ converges strongly to $P_{F(T)}x_1$ as desired. \square

Setting $\kappa = 0$, we have the following.

Corollary 4.6. Let C be a nonempty closed convex subset of a real Hilbert space H . Let T be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 \in C \text{ is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1 - \alpha_n) \|x_n - T^n x_n\|^2 + \tilde{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases} \tag{4.3}$$

where

$$\tilde{\theta}_n = \Delta_n^2 (1 - \alpha_n)(k_n - 1) \rightarrow 0 \quad (n \rightarrow \infty), \quad \Delta_n = \sup \{ \|x_n - z\| : z \in F(T) \} < \infty.$$

Then the sequence $\{x_n\}$ defined by (4.3) converges strongly to $P_{F(T)}x_1$.

Remark 4.7. Corollary 4.6 is an improvement of [9, Corollary 4.3]. More precisely, the condition $\limsup_{n \rightarrow \infty} \alpha_n < 1$ is dropped away.

4.3. Common solutions of a fixed point problem and an equilibrium problem

Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction of $C \times C$ into \mathbb{R} , where \mathbb{R} is the set of real numbers. The equilibrium problem for $f : C \times C \rightarrow \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \quad \text{for all } y \in C. \tag{4.4}$$

Numerous problems in physics, optimization, and economics reduce to find a solution of (4.4). The set of solutions of (4.4) is denoted by $EP(f)$. Some methods have been proposed to solve the equilibrium problem (see [2–4,14,18]). In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $EP(f)$ is nonempty and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions (see [2]):

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for any $x, y \in C$;
- (A3) f is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} f(tz + (1 - t)x, y) \leq f(x, y);$$

- (A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [2, Corollary 1] and [3, Lemma 2.12], we have the following lemma.

Lemma 4.8. Let C be a nonempty closed convex subset of a real Hilbert space H , let f be a bifunction from $C \times C$ into \mathbb{R} satisfies (A1)–(A4) and let $r > 0$ and $x \in H$. Then there exists unique $x^* \in C$ such that

$$f(x^*, y) + \frac{1}{r}(y - x^*, x^* - x) \geq 0 \quad \text{for all } y \in C.$$

Let T_r be a mapping of H into C defined by

$$T_r(x) = x^*$$

for all $x \in H$. Then, the following hold:

(i) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \|x - y\|^2 - \|(T_r x - x) - (T_r y - y)\|^2;$$

(ii) $F(T_r) = EP(f)$;

(iii) $EP(f)$ is closed and convex.

In order to apply Theorem 3.1, we first prove the following lemmas.

Lemma 4.9. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and $EP(f) \neq \emptyset$. If $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$, then $\{T_{r_n}\}$ is a family of firmly nonexpansive mappings of H into C and satisfies NST-condition.

Proof. It easy to see that $\{T_{r_n}\}$ is a family of firmly nonexpansive mappings of H into C and $\bigcap_{n=1}^{\infty} F(T_{r_n}) = EP(f)$. Let $\{z_n\}$ be a bounded sequence in H such that $\lim_{n \rightarrow \infty} \|z_n - T_{r_n} z_n\| = 0$ and $w \in \omega_w\{z_n\}$. Then $z_{n_i} \rightarrow w$ for some subsequence $\{n_i\}$ of $\{n\}$. For each $n \in \mathbb{N}$, let $w_n = T_{r_n} z_n$. Since $\liminf_{n \rightarrow \infty} r_n > 0$, it follows that

$$\lim_{n \rightarrow \infty} \left\| \frac{z_n - w_n}{r_n} \right\| = \lim_{n \rightarrow \infty} \frac{1}{r_n} \|z_n - T_{r_n} z_n\| = 0. \tag{4.5}$$

Notice that

$$f(w_n, y) + \frac{1}{r_n}(y - w_n, w_n - z_n) \geq 0 \quad \text{for all } y \in C.$$

So, from (A2) we have

$$\left\langle y - w_n, \frac{w_n - z_n}{r_n} \right\rangle \geq f(y, w_n) \quad \text{for all } y \in C.$$

In particular,

$$\left\langle y - w_{n_i}, \frac{w_{n_i} - z_{n_i}}{r_{n_i}} \right\rangle \geq f(y, w_{n_i}) \quad \text{for all } y \in C.$$

This together with (4.5), (A4) and $z_{n_i} \rightarrow w$ gives

$$0 \geq f(y, w) \quad \text{for all } y \in C.$$

Then, for $t \in (0, 1]$ and $y \in C$,

$$\begin{aligned} \text{?} \quad 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\leq tf(ty + (1-t)w, y) \end{aligned}$$

hence

$$f(ty + (1-t)w, y) \geq 0.$$

Letting $t \rightarrow 0^+$ and using (A3), we get

$$f(w, y) \geq 0 \quad \text{for all } y \in C,$$

and hence $w \in EP(f) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. This implies that $\{T_{r_n}\}$ satisfies the NST-condition. This completes the proof. \square

Using Theorem 3.1, we have the following theorem.

Theorem 4.10. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and let $\{S_n\}$ be a family of L_n -Lipschitzian mappings of C into H with $L_n \geq 1$ such that $F := \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f)$ is nonempty and bounded. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \text{ for all } y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n u_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1 - \alpha_n) \|x_n - S_n u_n\|^2 + (1 - \alpha_n) \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases}$$

where

$$\theta_n = \Delta_n^2 (L_n^2 - 1) \rightarrow 0, \quad \Delta_n = \sup\{\|x_n - z\| : z \in F\} < \infty.$$

If $\{S_n\}$ satisfies NST^* -condition, then $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. For each $n \in \mathbb{N}$, we have $u_n = T_{r_n} x_n$. By Lemmas 4.9 and 2.3, we have $\{S_n T_{r_n}\}$ is a family of quasi- L_n -Lipschitzian mappings of H into itself satisfies NST^* -condition. Applying Theorem 3.1, we have $\{x_n\}$ converges strongly to $P_F(x_1)$. \square

Using Corollary 4.2, we have the following theorem.

Theorem 4.11. Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and let $\{S_n\}$ be a family of nonexpansive mappings of C into H such that $F := \bigcap_{n=1}^{\infty} F(S_n) \cap EP(f) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in $[0, b] \subset [0, 1)$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \text{ for all } y \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) S_n u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots \end{cases}$$

If $\{S_n\}$ satisfies NST^* -condition, then $\{x_n\}$ converges strongly to $P_F x_1$.

Corollary 4.12. (See [18, Theorem 3.1].) Let C be a nonempty closed convex subset of a real Hilbert space H . Let f be a bifunction from $C \times C$ into \mathbb{R} satisfying (A1)–(A4) and let S be a nonexpansive mapping from C into H with $F(S) \cap EP(f) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in $[0, b] \subset [0, 1)$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfying $\liminf_{n \rightarrow \infty} r_n > 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \text{ for all } y \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) S u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots \end{cases}$$

Then $\{x_n\}$ converges strongly to $P_{F(S) \cap EP(f)}(x_1)$.

Proof. Setting $S_n \equiv S$ in Theorem 4.11. \square

4.4. Common solutions of a fixed point problem and a variational inequality problem

In this subsection, we apply Corollary 3.2 to the problem of approximating a common element of the common fixed point set for a countable family of mappings and the set of solutions of the variational inequality for an inverse strongly monotone mapping.

Let C be a nonempty closed convex subset of a real Hilbert space H and A a mapping of C into H . The classical variational inequality is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \text{ for all } y \in C.$$

The set of solutions of classical variational inequality is denoted by $VI(C, A)$. We know that

$$u \in VI(C, A) \iff u = P_C(u - \lambda Au)$$

for all $\lambda > 0$, where I is the identity mapping (see [8,20]). A mapping A of C into H is said to be

(1) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \text{for all } x, y \in C;$$

(2) α -*inverse-strongly-monotone*, where $\alpha > 0$, if

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \quad \text{for all } x, y \in C.$$

Note that every α -inverse-strongly-monotone mapping is monotone and $(1/\alpha)$ -Lipschitzian.

We need the following lemma.

Lemma 4.13. (See [11, Lemma 2.3].) *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\alpha > 0$ and let A be an α -inverse-strongly-monotone mapping of C into H with $VI(C, A) \neq \emptyset$, then*

$$\|P_C(I - \lambda A)x - z\|^2 \leq \|x - z\|^2 - \frac{2\alpha - \lambda}{2\alpha} \|P_C(I - \lambda A)x - x\|^2$$

for all $\lambda > 0, x \in C$ and $z \in VI(C, A)$. Furthermore, if $0 < \lambda < 2\alpha$, then $P_C(I - \lambda A)$ is a nonexpansive mapping of C into itself.

Lemma 4.14. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\alpha > 0$. Let A be an α -inverse-strongly-monotone mapping of C into H such that $VI(C, A) \neq \emptyset$. Let $\{W_n\}$ be a sequence of nonexpansive mappings of C into itself defined by*

$$W_n = P_C(I - \lambda_n A),$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[c, d] \subset (0, 2\alpha)$. Then $\{W_n\}$ satisfies NST-condition.

Proof. We note that $\bigcap_{n=1}^\infty F(W_n) = VI(C, A) \neq \emptyset$. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - W_n z_n\| = 0$ and $w \in \omega_w\{z_n\}$. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that $z_{n_i} \rightarrow w$ and $\lambda_{n_i} \rightarrow \lambda \in [c, d]$. Then $P_C(I - \lambda A)$ is a nonexpansive mapping of C into itself and

$$F(P_C(I - \lambda A)) = VI(C, A) = \bigcap_{n=1}^\infty F(W_n).$$

Since A is $(1/\alpha)$ -Lipschitzian and $\{z_{n_i}\}$ is bounded, let $M = \sup\{\|Az_{n_i}\| : i \in \mathbb{N}\}$. Then

$$\|P_C(I - \lambda A)z_{n_i} - P_C(I - \lambda_{n_i} A)z_{n_i}\| \leq \|(I - \lambda A)z_{n_i} - (I - \lambda_{n_i} A)z_{n_i}\| = |\lambda - \lambda_{n_i}| \|Az_{n_i}\| \leq |\lambda - \lambda_{n_i}| M \rightarrow 0.$$

So, we get

$$\|P_C(I - \lambda A)z_{n_i} - z_{n_i}\| \leq \|P_C(I - \lambda A)z_{n_i} - P_C(I - \lambda_{n_i} A)z_{n_i}\| + \|W_{n_i} z_{n_i} - z_{n_i}\| \rightarrow 0.$$

By Lemma 2.6, we have $w \in F(P_C(I - \lambda A))$ and hence $\omega_w\{z_n\} \subset \bigcap_{n=1}^\infty F(W_n)$. This implies that $\{W_n\}$ satisfies the NST-condition. \square

Using Corollary 3.2, we have the following theorem.

Theorem 4.15. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\alpha > 0$. Let A be an α -inverse-strongly-monotone mapping of C into H and let $\{S_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \geq 1$ such that $F := \bigcap_{n=1}^\infty F(S_n) \cap VI(C, A)$ is nonempty and bounded. Assume that $\{\beta_n\}$ is a sequence in $[0, b] \subset [0, 1)$ and $\{\lambda_n\}$ is a sequence in $[c, d] \subset (0, 2\alpha)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_n P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad n = 1, 2, 3, \dots, \end{cases}$$

where

$$\theta_n = \Delta_n^2 (L_n^2 - 1) \rightarrow 0, \quad \Delta_n = \sup\{\|x_n - z\| : z \in F\} < \infty.$$

If $\{S_n\}$ satisfies NST*-condition, then $\{x_n\}$ converges strongly to $P_F x_1$.

Proof. For each $n \in \mathbb{N}$, let $T_n = S_n P_C(I - \lambda_n A)$. By Lemmas 4.13, 4.14 and 2.3, we have $\{T_n\}$ is a family of quasi- L_n -Lipschitzian mappings of C into itself satisfying NST*-condition. Applying Corollary 3.2, $\{x_n\}$ converges strongly to $P_F(x_1)$. \square

Using Corollary 4.2 and Lemmas 4.13 and 4.14, we have the following theorem.

Theorem 4.16. Let C be a nonempty closed convex subset of a real Hilbert space H and let $\alpha > 0$. Let A be an α -inverse-strongly-monotone mapping of C into H and $\{S_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = S_n P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[c, d] \subset (0, 2\alpha)$. If $\{S_n\}$ satisfies NST*-condition, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)} x_1$.

Setting $S_n \equiv S$ in Theorem 4.16, we have the following result.

Corollary 4.17. (See [11, Theorem 3.9].) Let C be a nonempty closed convex subset of a real Hilbert space H and let $\alpha > 0$. Let A be an α -inverse-strongly-monotone mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = S P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[c, d] \subset (0, 2\alpha)$. Then $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(C, A)}(x_1)$.

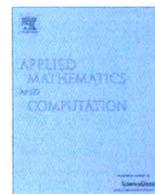
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Convergence theorems for a countable family of Lipschitzian mappings

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ABSTRACT

This paper is concerned with convergence of an approximating common fixed point sequence of countable Lipschitzian mappings in a uniformly convex Banach space. We introduce a new condition for a class of mappings to obtain several weak and strong convergence theorems. This new condition is implied by many previous known conditions introduced by many authors. We also apply our results for a class of nonexpansive mappings and asymptotically nonexpansive mappings and we immediately obtain convergence theorems proved by Song–Chen, Kimura–Takahashi, Tan–Xu, and many others.

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1. Introduction

A real Banach space E is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{and} \quad \|x - y\| \geq \varepsilon \quad \text{imply} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Let C be a subset of a real Banach space E . A mapping $T : C \rightarrow E$ is said to be *Lipschitzian* if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

In this case, T is also said to be L -Lipschitzian. Clearly, if T is L_1 -Lipschitzian and $L_1 < L_2$, then T is L_2 -Lipschitzian. Throughout the paper, we assume that every Lipschitzian mapping is L -Lipschitzian with $L \geq 1$. If $L = 1$, then T is known as a nonexpansive mapping. We denote by $F(T)$ the set of fixed points of T . If C is a nonempty bounded closed convex subset of a uniformly convex Banach space and T is a nonexpansive self-mapping of C , then $F(T) \neq \emptyset$ (see [10]). There are many methods for approximating fixed points of a nonexpansive mapping. In 1953, Mann [13] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Song and Chen [22] extended iteration (1.1) to obtain weak and strong convergence theorems for finding common fixed points of a countable family of uniformly asymptotically regular nonexpansive mappings $\{T_n\}$ by the following iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \tag{1.2}$$

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where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Recently, the present authors [17] proved that the iteration (1.2) converges weakly to a common fixed point of a countable family of L_n -Lipschitzian mappings in a Hilbert space.

In this paper we introduce a new condition, so-called the NST*-condition, for a class of mappings and use this condition to obtain weak and strong convergence theorems for finding common fixed points of a countable family of L_n -Lipschitzian mappings in a uniformly convex Banach space. Consequently, we immediately obtain the corresponding results recently proved by Song–Chen [22], and Kimura–Takahashi [12]. It is noted that the NST*-condition is weaker than the additional condition on the mappings studied in [22]. Moreover, we also apply our results for asymptotically nonexpansive mappings.

2. Preliminaries

We need some facts and tools in a uniformly convex Banach space which are listed as lemmas below.

Lemma 2.1 [20]. *Let E be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq d, \limsup_{n \rightarrow \infty} \|y_n\| \leq d$ and $\lim_{n \rightarrow \infty} \|\lambda_n x_n + (1 - \lambda_n)y_n\| = d$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

By Xu's inequality [27, Theorem 2], we have the following lemma.

Lemma 2.2. *Let E be a uniformly convex Banach space and $B_r := \{x \in E : \|x\| \leq r\}, r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$.

Lemma 2.3 [2, Lemma 1.1]. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E . Then there is a strictly increasing and continuous convex function $\gamma : [0, \infty) \rightarrow [0, \infty)$ with $\gamma(0) = 0$ such that, for every L -Lipschitzian mapping $T : C \rightarrow E$ and, the following inequality holds, for all $x, y \in C$ and $t \in [0, 1]$,*

$$\gamma\left(\frac{1}{L}\|tTx + (1 - t)Ty - T(tx + (1 - t)y)\|\right) \leq \|x - y\| - \frac{1}{L}\|Tx - Ty\|.$$

We write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x .

Lemma 2.4 [11, Lemma 2.2]. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E . Then there is a strictly increasing and continuous convex function $h : [0, \infty) \rightarrow [0, \infty)$ with $h(0) = 0$ such that, for every nonexpansive mapping $T : C \rightarrow E$ and $x_n \rightarrow x$ with $x_n \in C$,*

$$h(\|x - Tx\|) \leq \liminf_{n \rightarrow \infty} \|x_n - Tx_n\|.$$

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a real Banach space E , let $\{T_n\}$ and \mathcal{T} be families of mappings of C with $F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(\mathcal{T})$ is the set of all common fixed points of all mappings in \mathcal{T} . $\{T_n\}$ is said to satisfy

(a) the AKTT-condition [1] if for each bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

(b) the NST-condition (I) with \mathcal{T} [14] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \text{ implies } \lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0 \text{ for all } T \in \mathcal{T}.$$

In particular, if $\mathcal{T} = \{T\}$, i.e., \mathcal{T} consists of one mapping T , then $\{T_n\}$ is simply said to satisfy the NST-condition (I) with T .

(c) the NST-condition (II) [14] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0 \text{ implies } \lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0 \text{ for all } m \in \mathbb{N}.$$

(d) the uniformly asymptotic regularity (in short, u.a.r.) [22] if

$$\limsup_{n \rightarrow \infty} \sup_{z \in C} \|T_n z - T_m(T_n z)\| = 0 \text{ for all } m \in \mathbb{N}.$$

Inspired by conditions above, we introduce the following one:

(e) $\{\mathcal{T}_n\}$ is said to satisfy the NST*-condition with \mathcal{T} if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|z_n - z_{n+1}\| = 0$$

imply that $\lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0$ for all $T \in \mathcal{T}$. In particular, if $\mathcal{T} = \{T\}$, then, we simply say that $\{T_n\}$ satisfies the NST*-condition with T .

Remark 2.5.

- (1) If $\{T_n\}$ satisfies the NST-condition (I) with \mathcal{T} , then $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} .
- (2) If $\{T_n\}$ satisfies the NST-condition (II), then $\{T_n\}$ satisfies the NST*-condition with $\{T_n\}$.

Lemma 2.6. [1, Lemma 3.2]. *Let C be a nonempty closed subset of a Banach space E and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition, then the mapping $T : C \rightarrow C$ defined by*

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in C$$

satisfies

$$\lim_{n \rightarrow \infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of C .

From now on, we will write $(\{T_n\}, T)$ satisfies AKTT-condition if $\{T_n\}$ satisfies AKTT-condition and T is defined by Lemma 2.6.

Lemma 2.7. *Let C be a nonempty closed subset of a Banach space E . Suppose that $(\{T_n\}, T)$ satisfies AKTT-condition and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ satisfies the NST-condition (I) with T . Consequently, $\{T_n\}$ satisfies the NST*-condition with T .*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$. We apply Lemma 2.6 to get

$$\|z_n - Tz_n\| \leq \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \leq \|z_n - T_n z_n\| + \sup\{\|T_n z - Tz\| : z \in \{z_n\}\} \rightarrow 0.$$

Hence, we obtain that $\{T_n\}$ satisfies the NST-condition (I) with T . This completes the proof. \square

Lemma 2.8. *Let C be a nonempty closed subset of a Banach space E and let $\{T_n\}$ be a family of uniformly continuous mappings of C into itself. If $\{T_n\}$ satisfies the condition u.a.r., then $\{T_n\}$ satisfy the NST-condition (I) with $\{T_n\}$ and the NST-condition (II).*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$. It follows from the uniform continuity of each mapping T_m that

$$\lim_{n \rightarrow \infty} \|T_m z_n - T_m(T_n z_n)\| = 0.$$

Since $\{T_n\}$ satisfies the condition u.a.r., we get

$$\begin{aligned} \|z_n - T_m z_n\| &\leq \|z_n - T_n z_n\| + \|T_n z_n - T_m(T_n z_n)\| + \|T_m z_n - T_m(T_n z_n)\| \\ &\leq \|z_n - T_n z_n\| + \sup_{z \in C} \|T_n z - T_m(T_n z)\| + \|T_m z_n - T_m(T_n z_n)\| \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Hence, we obtain that $\{T_n\}$ satisfies the NST-condition (I) with $\{T_n\}$. The rest of the proof is similar to the one above, so it is left for the reader to verify. \square

The following lemma presents a demiclosedness-like principle of a countable family of Lipschitzian mappings.

Lemma 2.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $L_n \rightarrow 1$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C such that $x_n \rightarrow x \in C$ and $\lim_{n \rightarrow \infty} \|x_n - T_m x_n\| = 0$ for all $m \in \mathbb{N}$. If $\{T_n\}$ satisfies the NST*-condition with $\{T_n\}$, then $x \in \bigcap_{n=1}^{\infty} F(T_n)$.*

Proof. Since $\{x_n\}$ is bounded, let $R > 0$ be such that $\{x_n\} \subset K := B_R \cap C$. Then K is a nonempty closed convex bounded subset of E . For each $m \in \mathbb{N}$, $\hat{T}_m := \frac{1}{L_m} T_m$ is a nonexpansive mapping of K into E . This implies that

$$h\left(\|x - \hat{T}_m x\|\right) \leq \liminf_{n \rightarrow \infty} \|x_n - \hat{T}_m x_n\|,$$

where h is the function given in Lemma 2.4. Since $L_n \rightarrow 1$, $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and K is bounded, there exists $M > 0$ such that

$$\|T_m z\| \leq M, \quad \text{for all } m \in \mathbb{N} \text{ and } z \in K.$$

In particular,

$$\|x_n - \hat{T}_m x_n\| = \left\| x_n - \frac{1}{L_m} T_m x_n \right\| \leq \|x_n - T_m x_n\| + \left(1 - \frac{1}{L_m}\right) \|T_m x_n\| \leq \|x_n - T_m x_n\| + \left(1 - \frac{1}{L_m}\right) M.$$

This implies that

$$\limsup_{n \rightarrow \infty} \|x_n - \widehat{T}_m x_n\| \leq \left(1 - \frac{1}{L_m}\right)M$$

and hence

$$h\left(\|x - \widehat{T}_m x\|\right) \leq \left(1 - \frac{1}{L_m}\right)M.$$

So, we get

$$\lim_{m \rightarrow \infty} h\left(\|x - \widehat{T}_m x\|\right) = 0.$$

From the definition of h , we have

$$\lim_{m \rightarrow \infty} \|x - \widehat{T}_m x\| = 0.$$

Since

$$\|x - T_m x\| \leq \|x - \widehat{T}_m x\| + \|\widehat{T}_m x - T_m x\| \leq \|x - \widehat{T}_m x\| + \left(1 - \frac{1}{L_m}\right)M,$$

we get

$$\lim_{m \rightarrow \infty} \|x - T_m x\| = 0.$$

Since $\{T_n\}$ satisfies the NST^{*}-condition with $\{T_n\}$, we have $\|x - T_k x\| = 0$ for all $k \in \mathbb{N}$. That is, $x \in \bigcap_{n=1}^{\infty} F(T_n)$ and the proof is finished. \square

Lemma 2.10 [24, Lemma 1]. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of nonnegative real numbers such that

$$a_{n+1} \leq (1 + b_n)a_n, \quad \text{for all } n \in \mathbb{N}$$

and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.11. Let C be a nonempty closed convex subset of a Banach space E and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by (1.2), then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for each $p \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Then

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|T_n x_n - p\| \leq \alpha_n \|x_n - p\| + (1 - \alpha_n) L_n \|x_n - p\| \\ &= (1 + (1 - \alpha_n)(L_n - 1)) \|x_n - p\| \leq (1 + (L_n - 1)) \|x_n - p\| \end{aligned} \tag{2.1}$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} (L_n - 1) < \infty$, the conclusion follows from Lemma 2.10. This completes the proof. \square

Lemma 2.12. Let C be a nonempty closed convex subset of a Banach space E and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings from C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by (1.2), where $\lim_{n \rightarrow \infty} \alpha_n = 0$. Suppose that $\{T_n\}$ satisfies the NST-condition (II). Then

$$\lim_{n \rightarrow \infty} \|x_n - T_m x_n\| = 0 \quad \text{for all } m \in \mathbb{N}.$$

Proof. It follows from Lemma 2.11 that $\{x_n\}$ and $\{x_n - T_n x_n\}$ are bounded. Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T_n x_n\| = \alpha_n \|x_n - T_n x_n\| = 0.$$

The conclusion follows since $\{T_n\}$ satisfies the NST-condition (II). \square

Lemma 2.13. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by (1.2), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Suppose that \mathcal{T} is a family of mappings of C into itself such that $F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$. If $\{T_n\}$ satisfies the NST^{*}-condition with \mathcal{T} , then

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0 \quad \text{for all } T \in \mathcal{T}.$$

Proof. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. By Lemma 2.11, let $\lim_n \|x_n - p\| = d$ for some $d \geq 0$. It follows from (1.2) that

$$d = \lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \lim_{n \rightarrow \infty} \|\alpha_n(x_n - p) + (1 - \alpha_n)(T_n x_n - p)\|. \tag{2.2}$$

Consequently,

$$\limsup_{n \rightarrow \infty} a_n(t) \leq \lim_{n, m \rightarrow \infty} b_{n,m}(t) + \liminf_{n \rightarrow \infty} a_n(t) = \liminf_{n \rightarrow \infty} a_n(t).$$

This implies that $\lim_{n \rightarrow \infty} a_n(t)$ exists for any $t \in (0, 1)$ and the proof is finished. \square

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E such that its dual E^* has the Kadec–Klee property. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings from C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that \mathcal{F} is a family of mappings from C into itself such that $F(\mathcal{F}) = \bigcap_{n=1}^{\infty} F(T_n)$ and $I - T$ is demi-closed at 0 for all $T \in \mathcal{F}$. If $\{T_n\}$ satisfies the NST^{*}-condition with \mathcal{F} , then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, converges weakly to $z \in F(\mathcal{F})$.*

Proof. It follows from Lemma 2.13 that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \quad \text{for all } T \in \mathcal{F}.$$

Since $\{x_n\}$ is bounded, let $z \in \omega_w\{x_n\}$. There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. So, we get $z \in F(\mathcal{F})$. This implies that $\omega_w\{x_n\} \subset F(\mathcal{F})$. Then, by Lemma 3.5, the limit

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$. Hence $x_n \rightharpoonup z \in F(\mathcal{F})$ by Lemma 3.4. This completes the proof. \square

Similarly, using the same arguments as in Theorems 3.3 and 3.6 yields the following.

Theorem 3.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E such that its dual E^* has the Kadec–Klee property. Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined by (1.2) and one of the following control conditions:*

- (i) $\{T_n\}$ satisfies the NST-condition (II) and $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\{T_n\}$ satisfies the NST^{*}-condition with $\{T_n\}$ and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$.

Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}$.

4. Strong convergence theorems

We prove strong convergence of the iteration (1.2) in a uniformly convex Banach space. Recall that a mapping $T : C \rightarrow C$ is

- (a) *semi-compact* if every sequence $\{x_n\}$ in C satisfying $x_n - Tx_n \rightarrow 0$ has a convergent subsequence;
- (b) *closed* if whenever $\{x_n\}$ is a sequence in C satisfying $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $x \in C$ and $Tx = y$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that $\{T_n\}$ satisfies the NST^{*}-condition with \mathcal{F} , where \mathcal{F} is a family of closed mappings of C into itself such that $F(\mathcal{F}) = \bigcap_{n=1}^{\infty} F(T_n)$. If there exists some $\hat{T} \in \mathcal{F}$ which is semi-compact, then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, converges strongly to $z \in F(\mathcal{F})$.*

Proof. Suppose that $\hat{T} \in \mathcal{F}$ is semi-compact. It follows from Lemma 2.13 that

$$\lim_{n \rightarrow \infty} \|x_n - \hat{T}x_n\| = 0.$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow z \in C$ as $j \rightarrow \infty$. As

$$\lim_{j \rightarrow \infty} \|x_{n_j} - Tx_{n_j}\| = 0 \quad \text{for all } T \in \mathcal{F}.$$

Since T is closed, we have $Tz = z$ for all $T \in \mathcal{F}$ and hence $z \in F(\mathcal{F})$. By $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{F})$ and Lemma 2.11, we have $x_n \rightarrow z \in F(\mathcal{F})$. \square

Using Lemma 2.12 and following the proof of Theorem 4.1, we also have the following.

Theorem 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ and $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. If $\{T_n\}$ satisfies the NST-condition (II) and one of the mappings $\{T_n\}$ is semi-compact, then the sequence $\{x_n\}$ in C defined by (1.2), where $\lim_{n \rightarrow \infty} \alpha_n = 0$, converges strongly to a common fixed point of $\{T_n\}$.*

We next prove a strong convergence theorem of our iteration in the presence of another condition. We recall that a mapping T of C with $F(T) \neq \emptyset$ satisfies condition (A) (see [21]) if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$,

$$f(d(x, F(T))) \leq \|x - Tx\|.$$

Analogous to condition (A), a finite family \mathcal{T} of mappings of C with $F(\mathcal{T}) \neq \emptyset$ satisfies condition (B) if there exists a function f as above such that for all $x \in C$,

$$f(d(x, F(\mathcal{T}))) \leq \max\{\|x - Tx\| : T \in \mathcal{T}\}.$$

Theorem 4.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of L_n -Lipschitzian mappings of C into itself with $\sum_{n=1}^\infty (L_n - 1) < \infty$ and $\bigcap_{n=1}^\infty F(T_n)$ is nonempty. Suppose that $\{T_n\}$ satisfies the NST^{*}-condition with \mathcal{T} , where \mathcal{T} is a family of closed mappings of C into itself such that $F(\mathcal{T}) = \bigcap_{n=1}^\infty F(T_n)$. If \mathcal{T} is finite and satisfies condition (B), then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, converges strongly to $z \in F(\mathcal{T})$.*

Proof. By Lemma 2.13, we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \quad \text{for all } T \in \mathcal{T}. \tag{4.1}$$

Let f be the function corresponding to the condition (B) of the finite family \mathcal{T} . Then

$$f(d(x_n, F(\mathcal{T}))) \leq \max\{\|x_n - Tx_n\| : T \in \mathcal{T}\} \rightarrow 0$$

and so $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$. By Lemma 2.11, we know that $\{x_n - p\}$ is bounded. Then there is a constant M such that

$$M = \sup\{\|x_n - p\| : n \in \mathbb{N}\}.$$

This together with (2.1), shows that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + M(L_n - 1).$$

We also have

$$\|x_{n+m} - p\| \leq \|x_n - p\| + M \sum_{j=n}^{n+m-1} (L_j - 1) \tag{4.2}$$

for all $n, m \in \mathbb{N}$ and $p \in F(\mathcal{T})$.

We now prove that $\{x_n\}$ is a Cauchy sequence in C . Let $\varepsilon > 0$. Since

$$\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0 \quad \text{and} \quad \sum_{n=1}^\infty (L_n - 1) < \infty,$$

there exists a positive integer N such that

$$d(x_N, F(\mathcal{T})) < \frac{\varepsilon}{4} \quad \text{and} \quad \sum_{j=N}^\infty (L_j - 1) \leq \frac{\varepsilon}{4M}.$$

There must exist $q \in F(\mathcal{T})$ such that

$$\|x_N - q\| < \frac{\varepsilon}{4}.$$

From (4.2) it follows that, for all $n \geq N$ and $m \geq 1$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \leq 2\|x_n - q\| + M \sum_{j=n}^{n+m-1} (L_j - 1) \leq 2\|x_N - q\| + 2M \sum_{j=N}^{n-1} (L_j - 1) + M \sum_{j=n}^{n+m-1} (L_j - 1) \\ &\leq 2\|x_N - q\| + 2M \sum_{j=N}^\infty (L_j - 1) < 2 \frac{\varepsilon}{4} + 2M \frac{\varepsilon}{4M} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in C . In virtue of the completeness of C , we may assume that $x_n \rightarrow z \in C$. Since T is closed and (4.1), we have $z = Tz$ for all $T \in \mathcal{T}$, so $z \in F(\mathcal{T})$. This completes the proof. \square

5. Deduced theorems

In this section, we present several related results which can be deduced by corresponding convergence theorems obtained in Sections 3 and 4.

5.1. Convergence theorems for a countable family of nonexpansive mappings

Lemma 5.1 [9, Theorem 10.4]. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E , and $T : C \rightarrow C$ be a nonexpansive mapping. Then $I - T$ is demi-closed at 0.*

Theorem 5.2. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and satisfy one of the following control conditions:

- (i) $\{T_n\}$ satisfies the NST^* -condition with \mathcal{F} , where \mathcal{F} is a family of nonexpansive mappings of C into itself such that $F(\mathcal{F}) = \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$;
- (ii) $\{T_n\}$ satisfies the NST -condition (II) and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

If, in addition, E satisfies Opial's condition or E^* has the Kadec–Klee property, then the sequence $\{x_n\}$ in C defined by (1.2), converges weakly to a common fixed point of $\{T_n\}$.

Since the condition u.a.r. studied in [22] implies both NST^* -condition and NST -condition (II) (see Lemma 2.8), we immediately, obtain the following corollary.

Corollary 5.3 [22, Theorems 3.3–3.5]. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of u.a.r. nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. If, in addition, E satisfies Opial's condition or E^* has the Kadec–Klee property, the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ or $\lim_{n \rightarrow \infty} \alpha_n = 0$, converges weakly to a common fixed point of $\{T_n\}$.

Theorem 5.4. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and satisfy one of the following control conditions:

- (i) $\{T_n\}$ satisfies the NST^* -condition with \mathcal{F} , where \mathcal{F} is a family of nonexpansive mappings of C into itself such that $F(\mathcal{F}) = \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and one of the following holds:
 - (a) there exists some $\hat{T} \in \mathcal{F}$ which is semi-compact;
 - (b) \mathcal{F} is finite and satisfies condition (B);
- (ii) $\{T_n\}$ satisfies the NST -condition (II) and one of the mappings $\{T_n\}$ is semi-compact, and $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Then the sequence $\{x_n\}$ in C defined by (1.2), converges strongly to a common fixed point of $\{T_n\}$.

Corollary 5.5 [22, Theorem 3.9]. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{T_n\}$ be a sequence of u.a.r. nonexpansive mappings of C into itself such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Suppose that one of the mappings $\{T_n\}$ is semi-compact. Then the sequence $\{x_n\}$ in C defined by (1.2), where $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ or $\lim_{n \rightarrow \infty} \alpha_n = 0$, converges strongly to a common fixed point of $\{T_n\}$.

Let C be a nonempty closed convex subset of a Banach space E . Let $\{\alpha_{n,k}\}$ be a family of real numbers in $[0, 1]$ with indices $n, k \in \mathbb{N}$ with $1 \leq k \leq n$ and let $\{S_k\}$ be a family of nonexpansive mappings with a common fixed point, we define a family $\{W_n\}$ of nonexpansive mappings from C into itself by

$$\begin{aligned} U_{n,n} &= \alpha_{n,n}S_n + (1 - \alpha_{n,n})I, \\ U_{n,n-1} &= \alpha_{n,n-1}S_{n-1}U_{n,n} + (1 - \alpha_{n,n-1})I, \\ &\vdots \\ U_{n,2} &= \alpha_{n,2}S_2U_{n,3} + (1 - \alpha_{n,2})I, \\ W_n &= U_{n,1} = \alpha_{n,1}S_1U_{n,2} + (1 - \alpha_{n,1})I. \end{aligned}$$

Such a mapping W_n is call a W -mapping (see [12]) generated by S_n, S_{n-1}, \dots, S_1 and $\alpha_{n,n}, \alpha_{n,n-1}, \dots, \alpha_{n,1}$.

Lemma 5.6. Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{\alpha_{n,k}\}$ be a family of real numbers in $[a, b]$ with indices $n, k \in \mathbb{N}$ with $1 \leq k \leq n$ for some $a, b \in (0, 1)$ and let $\{S_k\}$ be a family of nonexpansive mappings with a common fixed point. Then $\{S_1U_{n,2}\}$ satisfies the NST -condition (I) with $\{S_k\}$.

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - S_1U_{n,2}z_n\| = 0$. Let $p \in \bigcap_{k=0}^\infty F(S_k)$. It follows from Lemma 2.2 that

$$\begin{aligned} \|z_n - p\|^2 &\leq (\|z_n - S_1U_{n,2}z_n\| + \|S_1U_{n,2}z_n - p\|)^2 = \|z_n - S_1U_{n,2}z_n\|^2 + 2\|z_n - S_1U_{n,2}z_n\|\|S_1U_{n,2}z_n - p\| + \|S_1U_{n,2}z_n - p\|^2 \\ &\leq \|z_n - S_1U_{n,2}z_n\|M + \|U_{n,2}z_n - p\|^2 = \|z_n - S_1U_{n,2}z_n\|M + \|\alpha_{n,2}(S_2U_{n,3}z_n - p) + (1 - \alpha_{n,2})(z_n - p)\|^2 \\ &\leq \|z_n - S_1U_{n,2}z_n\|M + \alpha_{n,2}\|S_2U_{n,3}z_n - p\|^2 + (1 - \alpha_{n,2})\|z_n - p\|^2 - \alpha_{n,2}(1 - \alpha_{n,2})g(\|S_2U_{n,3}z_n - z_n\|) \\ &\leq \|z_n - S_1U_{n,2}z_n\|M + \|z_n - p\|^2 - a(1 - b)g(\|z_n - S_2U_{n,3}z_n\|) \end{aligned}$$

for all $n \in \mathbb{N}$, where $M = \sup\{\|z_n - S_1U_{n,2}z_n\| + 2\|S_1U_{n,2}z_n - p\| : n \in \mathbb{N}\}$. So, we get

$$\lim_{n \rightarrow \infty} \|z_n - S_2U_{n,3}z_n\| = 0.$$

By induction, we have

$$\lim_{n \rightarrow \infty} \|z_n - S_k U_{n,k+1} z_n\| = 0 \quad \text{for all } k \in \mathbb{N}.$$

Since

$$\begin{aligned} \|z_n - S_k z_n\| &\leq \|z_n - S_k U_{n,k+1} z_n\| + \|S_k z_n - S_k U_{n,k+1} z_n\| \leq \|z_n - S_k U_{n,k+1} z_n\| + \|z_n - U_{n,k+1} z_n\| \\ &= \|z_n - S_k U_{n,k+1} z_n\| + \alpha_{n,k+1} \|z_n - S_{k+1} U_{n,k+2} z_n\| \end{aligned}$$

for all $n \in \mathbb{N}$, we get

$$\lim_{n \rightarrow \infty} \|z_n - S_k z_n\| = 0 \quad \text{for all } k \in \mathbb{N}.$$

It is easy to see that $\bigcap_{n=1}^{\infty} F(S_k) \subset \bigcap_{n=1}^{\infty} F(S_1 U_{n,2})$. To see the reverse inclusion, let $z \in \bigcap_{n=1}^{\infty} F(S_1 U_{n,2})$. Follow from the first part of the proof above but now let $z_n \equiv z$. Then $z \in \bigcap_{n=1}^{\infty} F(S_k)$. This implies that $\{S_1 U_{n,2}\}$ satisfies the NST-condition with $\{S_k\}$. \square

Theorem 5.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{\beta_{n,k}\}$ be a family of real numbers in $[a, b]$ with indices $n, k \in \mathbb{N}$ with $1 \leq k \leq n$ for some $a, b \in (0, 1)$ and let $\{S_k\}$ be a family of nonexpansive mappings with a common fixed point. If, in addition, E satisfies Opial's condition or E^* has the Kadec–Klee property, then the sequence $\{x_n\}$ in C defined by $x_1 \in C$ and $x_{n+1} = W_n x_n$, converges weakly to a common fixed point of $\{S_k\}$.*

Proof. Let $\alpha_n = 1 - \alpha_{n,1}$ for all $n \in \mathbb{N}$. Then $\alpha_n \in [a, b]$ and

$$x_{n+1} = W_n x_n = \alpha_n x_n + (1 - \alpha_n) S_1 U_{n,2} x_n.$$

By Lemma 5.6 and Theorem 5.2, it follows that $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}$. \square

Remark 5.8. Theorem 5.7 improves [12, Theorem 3.1] in the sense that the geometric property imposed on a Banach space is weakened.

Using Lemma 5.6 and Theorem 5.4, we also have the following result.

Theorem 5.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E . Let $\{\beta_{n,k}\}$ be a family of real numbers in $[a, b]$ with indices $n, k \in \mathbb{N}$ with $1 \leq k \leq n$ for some $a, b \in (0, 1)$ and let $\{S_k\}$ be a family of nonexpansive mappings with a common fixed point. Suppose that one of the mappings $\{S_k\}$ is semi-compact. Then the sequence $\{x_n\}$ in C defined by $x_1 \in C$ and $x_{n+1} = W_n x_n$, converges strongly to a common fixed point of $\{S_k\}$.*

5.2. Convergence theorems for an asymptotically nonexpansive mapping

Let C be a subset of a Banach space E . A mapping $T : C \rightarrow C$ is said to be an *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers such that $k_n \in [1, \infty)$, $\lim_{n \rightarrow \infty} k_n = 1$, and

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and $n \in \mathbb{N}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [8] as an important generalization of the class of nonexpansive mappings. It is also proved that if C is a nonempty bounded closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of C , then T has a fixed point. There are many methods for approximating fixed points of an asymptotically nonexpansive mapping. In 1991, Schu [19,20] introduced the following:

(a) the modified Mann iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n x_n \tag{5.1}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$.

(b) the modified Ishikawa iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^n (\beta_n x_n + (1 - \beta_n) T^n x_n), \tag{5.2}$$

where $x_1 \in C$ and $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0, 1]$. If $\beta_n \equiv 1$, then (5.2) becomes (5.1).

Tan and Xu [25] proved the weak convergence theorem of the iterations (5.1) and (5.2) for an asymptotically nonexpansive mapping in a uniformly convex Banach space which satisfies Opial's condition or has a Fréchet differentiable norm.

In this subsection, we use the NST*-condition to obtain results proved by Tan and Xu. We start with the following auxiliary result.

Lemma 5.10. Let C be a nonempty closed subset of a Banach space E and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ satisfying $\lim_{n \rightarrow \infty} k_n = 1$ and $F(T) \neq \emptyset$. Define a mapping

$$U_n := T^n(\beta_n I + (1 - \beta_n)T^n)$$

for all $n \in \mathbb{N}$, where $\{\beta_n\} \subset [a, 1]$ for some $a \in (0, 1)$. Then $\{U_n\}$ is a family of k_n^2 -Lipschitzian mappings of C into itself and satisfies the NST^{*}-condition with T .

Proof. We first observe that

$$\begin{aligned} \|U_n x - U_n y\| &\leq k_n \|\beta_n(x - y) + (1 - \beta_n)(T^n x - T^n y)\| \leq k_n(\beta_n \|x - y\| + (1 - \beta_n)\|T^n x - T^n y\|) \\ &\leq k_n(\beta_n + (1 - \beta_n)k_n)\|x - y\| \leq k_n^2 \|x - y\| \end{aligned}$$

for all $x, y \in C$. Then $\{U_n\}$ is a family of k_n^2 -Lipschitzian mappings of C into itself. Next, we prove that $\{U_n\}$ satisfies the NST^{*}-condition with T . Let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n \rightarrow \infty} \|z_n - U_n z_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0.$$

Without loss of generality, we may assume that $(1 - \beta_n)k_n < 1$ for all $n \in \mathbb{N}$. We consider

$$\|z_n - T^n z_n\| \leq \|z_n - U_n z_n\| + \|T^n z_n - T^n(\beta_n z_n + (1 - \beta_n)T^n z_n)\| \leq \|z_n - U_n z_n\| + (1 - \beta_n)k_n \|z_n - T^n z_n\|,$$

we get

$$\|z_n - T^n z_n\| \leq \frac{1}{1 - (1 - \beta_n)k_n} \|z_n - U_n z_n\| \rightarrow 0.$$

Since

$$\begin{aligned} \|z_{n+1} - Tz_{n+1}\| &\leq \|z_{n+1} - T^{n+1}z_{n+1}\| + \|T^{n+1}z_{n+1} - Tz_{n+1}\| \leq \|z_{n+1} - T^{n+1}z_{n+1}\| + k_1 \|T^n z_{n+1} - z_{n+1}\| \\ &\leq \|z_{n+1} - T^{n+1}z_{n+1}\| + k_1(\|T^n z_{n+1} - T^n z_n\| + \|T^n z_n - z_n\| + \|z_{n+1} - z_n\|) \\ &\leq \|z_{n+1} - T^{n+1}z_{n+1}\| + k_1(k_n + 1)\|z_{n+1} - z_n\| + k_1 \|T^n z_n - z_n\|, \end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0.$$

It is easy to see that $F(T) \subset \bigcap_{n=1}^{\infty} F(U_n)$. To see the reverse inclusion, let $z \in \bigcap_{n=1}^{\infty} F(U_n)$. Follow from the first part of the proof above but now let $z_n \equiv z$. Then $z \in F(T)$. This implies that $\{U_n\}$ satisfies the NST^{*}-condition with T . \square

Lemma 5.11 [4, Theorem 1]. Let C be a nonempty closed convex subset of a uniformly convex Banach space E , and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a fixed point set nonempty. Then $I - T$ is demi-closed at 0.

The following theorem improves [25, Theorem 3.2] in the sense that the Fréchet differentiability norm is replaced by the Kadec–Klee property of the dual.

Theorem 5.12. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by (5.2), where $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, 1]$ for some $a, b \in (0, 1)$. If, in addition, E satisfies Opial's condition or E^* has the Kadec–Klee property, then $\{x_n\}$ converges weakly to a fixed point of T .

Theorem 5.13. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined by (5.2), where $\{\alpha_n\} \subset [a, b]$ and $\{\beta_n\} \subset [a, 1]$ for some $a, b \in (0, 1)$. If T is semi-compact or T satisfies condition (A), then $\{x_n\}$ converges strongly to a fixed point of T .

Remark 5.14. Theorem 5.13 includes Theorem 2.2 of [26] as a special case.

Remark 5.15. By using the same ideas and techniques, we can also obtain the weak and strong convergence theorems for finitely many asymptotically nonexpansive mappings and thereby unify the results obtained by many authors [3–5,15,16,19,20,23,26].

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AN IMPLICIT ITERATION FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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□ *We introduce an implicit sequence for an infinite family of nonexpansive mappings in a uniformly convex Banach space. We prove weak and strong convergence theorems for finding a common fixed point of the mappings. Our results not only include Plubtieng et al. (Numer. Funct. Anal. Optim. 2007; 28:737-749), Kikkawa and Takahashi (Ann. Univ. Mariae Curie-Sklodowska Sect. A 2004; 58:69-78), Kimura and Takahashi (Set-Valued Anal. 2008; 16:597-619) as special cases but also are established under the weaker assumptions.*

Keywords Implicit iteration; Nonexpansive mapping; Uniformly convex Banach space; Weak and strong convergence.

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1. INTRODUCTION

Let C be a nonempty subset of a real Banach space X . A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T . If C is a bounded closed convex subset of a uniformly convex Banach space X and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [6, 9, 18]). We write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges strongly (weakly, resp.) to x . There are many methods for approximating fixed points of a nonexpansive mapping. In 2001, Xu and Ori [25] introduced

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the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, with $\{\alpha_n\}$ is a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:

$$\begin{aligned}x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 \\x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2 \\&\vdots \\x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N \\x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1} \\&\vdots\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1, \quad (1.1)$$

where $T_n \equiv T_{n \bmod N}$, (here the $\bmod N$ function takes values in $\{1, 2, \dots, N\}$) and proved that this process converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$ in a real Hilbert space. In [26], Zhou and Chang studied the weak and strong convergence of this implicit process to a common fixed point for a finite family of nonexpansive mappings in a uniformly convex Banach space. Liu [10] and Chidume and Shahzad [3] independently proved that the iteration (1.1) converges strongly to a common fixed point of a finite family of nonexpansive mappings if one of the mappings is semicompact. Plubtieng et al. [12] introduced the following implicit iteration process for two finite families of nonexpansive mappings $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$, with $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$, and an initial point $x_0 \in C$:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n (\beta_n x_n + (1 - \beta_n) S_n x_n), \quad n \geq 1, \quad (1.2)$$

where $T_n \equiv T_{n \bmod N}$ and $S_n \equiv S_{n \bmod N}$. They proved that the weak and strong convergence of the iteration (1.2) to a common fixed point of $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$. Their results extend and improve the corresponding results of Xu and Ori [25], Zhou and Chang [26], Liu [10], and Chidume and Shahzad [3].

Recently, Kikkawa and Takahashi [7] proposed an implicit sequence for finding a common fixed point of an infinite family of nonexpansive mappings. Their method involves the W-mapping introduced by Takahashi [17].

In this paper, we introduce a simpler implicit sequence for an infinite family of nonexpansive mappings in a uniformly convex Banach space and

prove weak and strong convergence theorems for finding a common fixed point of the mappings. Several convergence theorems proved by Plubtieng et al. [12], Kikkawa and Takahashi [7], and Kimura and Takahashi [8] are not only essentially special cases of our corresponding theorems but also are established under the weaker assumptions.

2. PRELIMINARIES

Let X be a real Banach space. We say that X is *strictly convex* if the following implication holds for $x, y \in X$:

$$\|x\| = \|y\| = 1 \quad \text{and} \quad x \neq y \quad \text{imply} \quad \left\| \frac{x+y}{2} \right\| < 1.$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{and} \quad \|x - y\| \geq \varepsilon \quad \text{imply} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is known that every uniformly convex Banach space is strictly convex.

By Xu's inequality [24, Theorem 2], we have the following lemma.

Lemma 2.1. *Let X be a uniformly convex Banach space and $B_r := \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$.

Lemma 2.2 ([6, Theorem 10.4]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space X , and $T : C \rightarrow C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \rightarrow x$ and $x_n - Tx_n \rightarrow 0$, then $x \in F(T)$.*

Lemma 2.3 ([2, Lemma 1.1]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X . Then there is a strictly increasing and continuous convex function $\gamma : [0, \infty) \rightarrow [0, \infty)$ with $\gamma(0) = 0$ such that, for every nonexpansive mapping $T : C \rightarrow X$ and, for all $x, y \in C$ and $t \in [0, 1]$, the following inequality holds:*

$$\gamma(\|tTx + (1 - t)Ty - T(tx + (1 - t)y)\|) \leq \|x - y\| - \|Tx - Ty\|.$$

The following lemma was proved by Tan and Xu [23, Lemma 1].

Lemma 2.4. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of non-negative real numbers such that

$$a_{n+1} \leq a_n + b_n \quad \text{for all } n \geq 1,$$

and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists. In particular, if $\liminf_{n \rightarrow \infty} a_n = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

The following lemma is also well known (see [16, Lemma 1] or [19, Problem 6.2.4]).

Lemma 2.5. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n b_n < \infty$. Then $\liminf_{n \rightarrow \infty} b_n = 0$.

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a real Banach space X , let $\{T_n\}$ and \mathcal{T} be families of mappings of C with $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$, where $F(\mathcal{T})$ is the set of all common fixed points of all mappings in \mathcal{T} . $\{T_n\}$ is said to satisfy

(a) the *AKTT-condition* [1, 11] if for each bounded subset B of C ,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

(b) the *NST-condition (I)* with \mathcal{T} [11] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \quad \text{implies} \quad \lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0 \quad \text{for all } T \in \mathcal{T}.$$

In particular, if $\mathcal{T} = \{T\}$, i.e., \mathcal{T} consists of one mapping T , then $\{T_n\}$ is simply said to satisfy the *NST-condition (I)* with T .

(c) the *NST-condition (II)* [11] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0 \quad \text{implies} \quad \lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0 \quad \text{for all } m \in \mathbb{N}.$$

These three conditions were first named in [20, 21]. Inspired by conditions above, we introduce the following one:

(d) $\{T_n\}$ is said to satisfy the *NST*-condition with \mathcal{T}* if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z_n - z_{n+1}\| = 0$$

imply that $\lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0$ for all $T \in \mathcal{T}$. In particular, if $\mathcal{T} = \{T\}$, then, we simply say that $\{T_n\}$ satisfies the *NST*-condition with T* .

Remark 2.6.

- (1) If $\{T_n\}$ satisfies the NST-condition (I) with \mathcal{T} , then $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} .
- (2) If $\{T_n\}$ satisfies the NST-condition (II), then $\{T_n\}$ satisfies the NST*-condition with $\{T_n\}$.

Lemma 2.7 ([1, Lemma 3.2]). *Let C be a nonempty closed subset of a Banach space X and let $\{T_n\}$ be a family of mappings of C into itself that satisfies the AKTT-condition, then the mapping $T : C \rightarrow C$ defined by*

$$Tx = \lim_{n \rightarrow \infty} T_n x \quad \text{for all } x \in C$$

satisfies

$$\limsup_{n \rightarrow \infty} \{\|Tz - T_n z\| : z \in B\} = 0$$

for each bounded subset B of C .

Lemma 2.8. *Let C be a nonempty closed subset of a Banach space X and let $\{T_n\}$ be a family of mappings of C into itself that satisfies the AKTT-condition and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let T be the mapping from C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{T_n\}$ satisfies the NST-condition (I) with T . Consequently, $\{T_n\}$ satisfies the NST*-condition with T .*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$. We apply Lemma 2.7 to get

$$\begin{aligned} \|z_n - Tz_n\| &\leq \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \\ &\leq \|z_n - T_n z_n\| + \sup\{\|T_n z - Tz\| : z \in \{z_n\}\} \rightarrow 0. \end{aligned}$$

Hence, we obtain that $\{T_n\}$ satisfies the NST-condition (I) with T . This completes the proof. □

Lemma 2.9. *Let C be a nonempty closed subset of a Banach space X and let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Then $\{T_n\}$ satisfies the NST*-condition with $\mathcal{T} = \{T_1, T_2, \dots, T_N\}$, where $T_n \equiv T_{n \bmod N}$.*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0.$$

Obviously, it is easy to see that $\lim_{n \rightarrow \infty} \|z_{n+i} - z_n\| = 0$ for each $i = 1, 2, \dots, N$. Consequently,

$$\begin{aligned} \|z_n - T_{n+i}z_n\| &\leq \|z_n - z_{n+i}\| + \|z_{n+i} - T_{n+i}z_{n+i}\| + \|T_{n+i}z_{n+i} - T_{n+i}z_n\| \\ &\leq 2\|z_n - z_{n+i}\| + \|z_{n+i} - T_{n+i}z_{n+i}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0$$

for each $m = 1, 2, \dots, N$. □

Remark 2.10. There are families of mappings $\{T_n\}$ and \mathcal{T} such that

- (1) $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} and
- (2) $\{T_n\}$ fails the NST-condition (I) with \mathcal{T} and the NST-condition (II).

The following example shows that the NST*-condition with \mathcal{T} is strictly weaker than the NST-condition (I) with \mathcal{T} and the NST-condition (II).

Example 2.11. Let $x := \mathbb{R}^2$ and $C := [0, 1] \times [0, 1]$. Define $T_1, T_2 : C \rightarrow C$ as follows:

$$T_1(x, y) = (x, 1 - y) \quad \text{and} \quad T_2(x, y) = (1 - x, y),$$

for all $(x, y) \in C$. Hence, T_1 and T_2 are nonexpansive mappings with

$$F(T_1) \cap F(T_2) = ([0, 1] \times \{1/2\}) \cap (\{1/2\} \times [0, 1]) = \{(1/2, 1/2)\} \neq \emptyset.$$

Let $T_n = T_{n(\text{mod}2)}$. By Lemma 2.9, we have $\{T_n\}$ satisfies NST*-condition with $\{T_1, T_2\}$.

- (a) $\{T_n\}$ fails the NST-condition (I) with $\mathcal{T} = \{T_1, T_2\}$. In fact, let $z_{2n-1} = (1, 1/2)$ and $z_{2n} = (1/2, 1)$ for all $n \in \mathbb{N}$. Then $z_{2n-1} \in F(T_{2n-1}) = F(T_1)$ and $z_{2n} \in F(T_{2n}) = F(T_2)$. In particular, $\|z_n - T_n z_n\| \equiv 0$. Clearly,

$$\|z_n - T_1 z_n\| \not\equiv 0 \quad \text{and} \quad \|z_n - T_2 z_n\| \not\equiv 0.$$

Hence $\{T_n\}$ fails the NST-condition (I) with $\{T_1, T_2\}$.

- (b) $\{T_n\}$ fails the NST-condition (II). To this end, let $z_{4n-3} = (1/4, 1/4)$, $z_{4n-2} = (1/4, 3/4)$, $z_{4n-1} = (3/4, 3/4)$ and $z_{4n} = (3/4, 1/4)$ for all $n \in \mathbb{N}$. Then $\|z_{n+1} - T_n z_n\| \equiv 0$. But

$$\|z_n - T_1 z_n\| \not\equiv 0 \quad \text{and} \quad \|z_n - T_2 z_n\| \not\equiv 0.$$

Hence $\{T_n\}$ fails the NST-condition (II).

Lemma 2.12 ([22]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X , S and T be two nonexpansive mappings of C into itself with a common fixed point, and $0 < \beta < 1$. Let U be a mapping defined by*

$$U = T(\beta I + (1 - \beta)S),$$

where I is the identity mapping. Then U is nonexpansive and $F(U) = F(T) \cap F(S)$.

Lemma 2.13. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}$, $\{S_n\}$, \mathcal{T}_1 , and \mathcal{T}_2 be families of nonexpansive mappings of C into itself with $F(\mathcal{T}_1) = \bigcap_{n=1}^{\infty} F(T_n)$, $F(\mathcal{T}_2) = \bigcap_{n=1}^{\infty} F(S_n)$ and $F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. Let $\{U_n\}$ be a family of nonexpansive mappings of C into itself defined by*

$$U_n = T_n(\beta_n I + (1 - \beta_n)S_n) \quad \text{for all } n \geq 1.$$

If $\{T_n\}$ and $\{S_n\}$ satisfy the NST*-condition with \mathcal{T}_1 and \mathcal{T}_2 , respectively, then $\{U_n\}$ satisfies the NST*-condition with $\mathcal{T}_1 \cup \mathcal{T}_2$.

Proof. By Lemma 2.12, we have U_n is a nonexpansive mapping of C into itself and $F(U_n) = F(T_n) \cap F(S_n)$. Then

$$\bigcap_{n=1}^{\infty} F(U_n) = F(\mathcal{T}_1) \cap F(\mathcal{T}_2) = F(\mathcal{T}_1 \cup \mathcal{T}_2).$$

Let $\{z_n\}$ be a bounded sequence in C such that

$$\lim_{n \rightarrow \infty} \|z_n - U_n z_n\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0.$$

Let $p \in F(\mathcal{T}_1) \cap F(\mathcal{T}_2)$. Then $\{z_n - U_n z_n\}$ and $\{z_n - p\}$ are bounded. Let $r = \sup\{\|z_n - U_n z_n\|, \|z_n - p\| : n \in \mathbb{N}\}$. It follows from Lemma 2.1 that

$$\begin{aligned} \|z_n - p\|^2 &\leq (\|z_n - U_n z_n\| + \|U_n z_n - p\|)^2 \\ &= \|z_n - U_n z_n\|^2 + 2\|z_n - U_n z_n\|\|U_n z_n - p\| + \|U_n z_n - p\|^2 \\ &\leq 3r\|z_n - U_n z_n\| + \|T_n(\beta_n(z_n(1 - \beta_n)S_n z_n) - T_n p)\|^2 \\ &\leq 3r\|z_n - U_n z_n\| + \|\beta_n(z_n - p) + (1 - \beta_n)(S_n z_n - p)\|^2 \\ &\leq 3r\|z_n - U_n z_n\| + \beta_n\|z_n - p\|^2 + (1 - \beta_n)\|S_n z_n - p\|^2 \\ &\quad - \beta_n(1 - \beta_n)g(\|z_n - S_n z_n\|) \\ &\leq 3r\|z_n - U_n z_n\| + \|z_n - p\|^2 - \beta_n(1 - \beta_n)g(\|z_n - S_n z_n\|) \end{aligned}$$

for all $n \in \mathbb{N}$. So, we get

$$\lim_{n \rightarrow \infty} \|z_n - S_n z_n\| = 0.$$

Because $\{S_n\}$ satisfies the NST*-condition with \mathcal{T}_2 , we have

$$\lim_{n \rightarrow \infty} \|z_n - S z_n\| = 0 \quad \text{for all } S \in \mathcal{T}_2.$$

Because

$$\begin{aligned} \|z_n - T_n z_n\| &\leq \|z_n - U_n z_n\| + \|U_n z_n - T_n z_n\| \\ &\leq \|z_n - U_n z_n\| + (1 - \beta_n) \|z_n - S_n z_n\|, \end{aligned}$$

it follows that

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0.$$

Because $\{T_n\}$ satisfies the NST*-condition with \mathcal{T}_1 , we have

$$\lim_{n \rightarrow \infty} \|z_n - T z_n\| = 0 \quad \text{for all } T \in \mathcal{T}_1.$$

Hence, $\{U_n\}$ satisfies the NST*-condition with $\mathcal{T}_1 \cup \mathcal{T}_2$. □

Lemma 2.14. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself with a common fixed point. Let $\{x_n\}$ be a sequence in C defined by $x_0 \in C$ and*

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n \tag{2.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$. Then we have the following conclusions.

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for each $p \in \bigcap_{n=1}^{\infty} F(T_n)$.
- (ii) $\lim_{n \rightarrow \infty} d(x_n, \bigcap_{n=1}^{\infty} F(T_n))$ exists, where $d(x, K)$ denotes the distance from x to the set K .
- (iii) There exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that $\sum_{n=1}^{\infty} (1 - \alpha_n) g(\|x_{n-1} - T_n x_n\|) < \infty$.

Proof. Observe that if C is a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \rightarrow C$ is a nonexpansive mapping, then for every $u \in C, \alpha \in (0, 1)$, the mapping $S = S_{(\alpha, u, T)} : C \rightarrow C$ defined by

$$Sx = \alpha u + (1 - \alpha)Tx \quad \text{for all } x \in C$$

is a $(1 - \alpha)$ -contraction. In fact, for all $x, y \in C$,

$$\|Sx - Sy\| = (1 - \alpha)\|Tx - Ty\| \leq (1 - \alpha)\|x - y\|.$$

Consequently, S has a unique fixed point $x^* \in C$. Thus there exists a unique $x^* \in C$, that is,

$$x^* = \alpha u + (1 - \alpha)Tx^*.$$

This implies that the implicit iteration scheme (2.1) is well defined. To see (i) and (ii), let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Then

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T_n x_n - p)\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|x_n - p\|. \end{aligned}$$

It follows from $\alpha_n > 0$ that

$$\|x_n - p\| \leq \|x_{n-1} - p\|. \tag{2.2}$$

So, $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, that is, (i) holds. Moreover, from (2.2), we immediately have (ii). Finally, as the sequence $\{x_n - p\}$ is bounded, let $r = \sup\{\|x_n - p\| : n \in \mathbb{N}\}$. It follows from Lemma 2.1 that $x_n - p$ and $T_n x_n - p$ belong to B_r , and

$$\begin{aligned} \|x_n - p\|^2 &\leq \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) \|T_n x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_{n-1} - T_n x_n\|) \\ &\leq \alpha_n \|x_{n-1} - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) g(\|x_{n-1} - T_n x_n\|). \end{aligned}$$

Because $\alpha_n > 0$, we have

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\|^2 - (1 - \alpha_n) g(\|x_{n-1} - T_n x_n\|),$$

that is,

$$(1 - \alpha_n) g(\|x_{n-1} - T_n x_n\|) \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2.$$

Summing from 1 to m and tending to infinity for m , we have (iii). This completes the proof. \square

3. WEAK CONVERGENCE THEOREMS

In this section, we prove several weak convergence theorems.

Lemma 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings of*

C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . Let $\{\alpha_n\} \subset (0, b)$ for some $b \in (0, 1)$ and $\{x_n\}$ be the sequence in C defined by (2.1), then

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \quad \text{for all } T \in \mathcal{T}.$$

Proof. It follows from Lemma 2.14(i) that $\{x_n\}$ is bounded. By Lemma 2.14(iii) and $\alpha_n \leq b < 1$, we have

$$\sum_{n=1}^{\infty} g(\|x_{n-1} - T_n x_n\|) < \infty.$$

In particular, $\lim_{n \rightarrow \infty} g(\|x_{n-1} - T_n x_n\|) = 0$ and so $\lim_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0$. Thus

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = \lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0.$$

The conclusion follows $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . \square

We first prove a weak convergence theorem of the sequence $\{x_n\}$ that defined by (2.1) in a uniformly convex Banach space satisfying Opial's condition. Recall that X satisfies *Opial's condition* [13] if $x_n \rightharpoonup x$ and $x \neq y$ imply that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

It is well known [13] that all Hilbert spaces and ℓ^p spaces, $1 \leq p < \infty$, have this property, whereas all L^p spaces do not have this property unless $p = 2$.

Lemma 3.2 ([14, Lemma 2.7]). *Let X be a Banach space that satisfies Opial's condition and let $\{x_n\}$ be a sequence in X . Let $u, v \in X$ be elements such that $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ that converge weakly to u and v , respectively, then $u = v$.*

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X that satisfies Opial's condition. Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . Then the sequence $\{x_n\}$ in C defined by (2.1), where $\{\alpha_n\} \subset (0, b)$ for some $b \in (0, 1)$, converges weakly to $z \in F(\mathcal{T})$.*

Proof. It follows from Lemma 3.1 that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \quad \text{for all } T \in \mathcal{T}.$$

Because $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ for some $z \in C$. By Lemma 2.2, $z \in F(\mathcal{T})$. To prove that $x_n \rightharpoonup z$, suppose that there exists another subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $x_{m_j} \rightharpoonup z'$. So, we have $z' \in F(\mathcal{T})$. It follows from Lemmas 2.14(i) and 3.2 that $z = z'$. Hence the whole sequence $\{x_n\}$ converges weakly to z , as required. \square

Using Lemmas 2.9 and 2.13 [12, Theorem 3.8] is a special case of Theorem 3.3.

Corollary 3.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X that satisfies Opial's condition. Let $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$ be two finite families of nonexpansive mappings of C into itself with $F(T) \cap F(S) \neq \emptyset$, where $F(T) := \bigcap_{n=1}^N F(T_n)$ and $F(S) := \bigcap_{n=1}^N F(S_n)$. Assume that $\{\alpha_n\} \subset (0, b]$ and $\{\beta_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then the sequence $\{x_n\}$ in C defined by (1.2), where $T_n \equiv T_{n \bmod N}$ and $S_n \equiv S_{n \bmod N}$, converges weakly to a common fixed point of $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$.*

Remark 3.5. The restriction $\alpha_n \in [a, b]$ for some $a, b \in (0, 1)$ assumed in [12, Theorem 3.8] is relaxed.

In the presence of the stronger condition than NST*-condition with \mathcal{T} , a variable control sequence $\{\alpha_n\}$ is taken into consideration.

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X that satisfies Opial's condition. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself that satisfies the AKTT-condition and $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Let T be the mapping from C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^\infty F(T_n)$. Then the sequence in C defined by (2.1), where $\{\alpha_n\} \subset (0, 1)$ satisfies $\sum_{n=1}^\infty (1 - \alpha_n) = \infty$, converges weakly to $z \in F(T)$.*

Proof. By Lemmas 2.5 and 2.14(iii), and $\sum_{n=1}^\infty (1 - \alpha_n) = \infty$, we have

$$\liminf_{n \rightarrow \infty} g(\|x_{n-1} - T_n x_n\|) = 0.$$

Because g is continuous and strictly increasing, we have $\liminf_{n \rightarrow \infty} \|x_{n-1} - T_n x_n\| = 0$ and so

$$\liminf_{n \rightarrow \infty} \|x_n - T_n x_n\| = \liminf_{n \rightarrow \infty} \alpha_n \|x_{n-1} - T_n x_n\| = 0.$$

We next prove that the limit $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ actually exists. Because $\{x_n\}$ is bounded, it follows that

$$\sum_{n=1}^{\infty} \sup\{\|T_n z - T_{n-1} z\| : z \in \{x_n\}\} < \infty. \quad (3.1)$$

Notice that

$$\begin{aligned} & \|x_n - x_{n-1}\| \\ &= (1 - \alpha_n) \|x_{n-1} - T_n x_n\| \\ &\leq (1 - \alpha_n) (\|x_{n-1} - T_{n-1} x_{n-1}\| + \|T_{n-1} x_{n-1} - T_{n-1} x_n\| + \|T_{n-1} x_n - T_n x_n\|) \\ &\leq (1 - \alpha_n) \|x_{n-1} - T_{n-1} x_{n-1}\| + (1 - \alpha_n) \|x_{n-1} - x_n\| \\ &\quad + (1 - \alpha_n) \sup\{\|T_n z - T_{n-1} z\| : z \in \{x_n\}\}, \end{aligned}$$

so we have

$$\begin{aligned} \alpha_n \|x_n - x_{n-1}\| &\leq (1 - \alpha_n) \|x_{n-1} - T_{n-1} x_{n-1}\| \\ &\quad + (1 - \alpha_n) \sup\{\|T_n z - T_{n-1} z\| : z \in \{x_n\}\}. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - T_n x_n\| &= \frac{\alpha_n}{1 - \alpha_n} \|x_n - x_{n-1}\| \\ &\leq \|x_{n-1} - T_{n-1} x_{n-1}\| + \sup\{\|T_n z - T_{n-1} z\| : z \in \{x_n\}\}. \end{aligned}$$

By Lemma 2.4 and (3.1), we have $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\|$ exists. Thus

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

From the definition of T , we have T is nonexpansive. By Lemma 2.8, we have $\{T_n\}$ satisfies the NST*-condition with T . As in the proof of Theorem 3.3, $\{x_n\}$ converges weakly to $z \in F(T)$. \square

Next, we discuss the weak convergence of the iteration (2.1) in a uniformly convex Banach space X whose dual X^* has the Kadec–Klee property. Before going on, let us recall some geometric properties.

- (1) A Banach space X has the *Kadec–Klee property* [6] if for every $\{x_n\}$ in X , $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$ together imply $\|x_n - x\| \rightarrow 0$.
- (2) A Banach space X has a *Fréchet differentiable norm* if, for any $x \in S_X$, the limit

$$\lim_{t \rightarrow 0} \frac{\|x + tx\| - \|x\|}{t}$$

exists and is approached uniformly as y varies over S_X , where S_X denotes the unit sphere of X .

It is known that if X is reflexive and has a Fréchet differentiable norm, then X^* has the Kadec–Klee property (see also [4, Lemma 1]). Most weak convergence theorems are proved in a uniformly convex Banach space and the presence of Opial’s condition or the Fréchet differentiability of the norm (see, e.g., [7, 8, 14, 23, 26]). However, there exist uniformly convex Banach spaces that have neither the Opial’s condition nor a Fréchet differentiable norm but their duals do have the Kadec–Klee property.

As an example, let us take $X_1 = \mathbb{R}^2$ with the norm defined by

$$\|x\| = \sqrt{\|x\|_2^2 + \|x\|_1^2}$$

and $X_2 = L^p[0, 1]$ with $1 < p < \infty$ and $p \neq 2$. The Cartesian product of X_1 and X_2 furnished with the l^2 -norm is uniformly convex. It does not have the Opial’s condition and its norm is not Fréchet differentiable, but its dual does have the Kadec–Klee property (see also [5, 6]).

We will denote by $\omega_w\{x_n\}$ the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in X .

The following lemma is our main tool for proving the weak convergence theorem.

Lemma 3.7 ([5, Lemma 3.2]). *Let X be a uniformly convex Banach space such that its dual X^* has the Kadec–Klee property. Suppose $\{x_n\}$ is a bounded sequence in X such that*

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$. Then $\omega_w\{x_n\}$ is a singleton.

Lemma 3.8. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself with a common fixed point, and let $\{x_n\}$ be a sequence in C defined by (2.1). Then the limit*

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \bigcap_{n=1}^{\infty} F(T_n)$.

Proof. By Lemma 2.14(i), we have $\{x_n\}$ is bounded. Take $R > 0$ such that $\{x_n\} \subset K := B_R \cap C$. Then K is a nonempty closed convex bounded subset of X . Let $p, q \in \bigcap_{n=1}^{\infty} F(T_n)$ and set

$$\delta_n(t) = \|tx_n + (1 - t)p - q\|.$$

Then $\lim_{n \rightarrow \infty} \delta_n(0) = \|p - q\|$, and from Lemma 2.14(i), $\lim_{n \rightarrow \infty} \delta_n(1) = \lim_n \|x_n - q\|$ exists. Now, we consider the case $t \in (0, 1)$. Let $u \in K$ and $\alpha \in (0, 1)$, the mapping $S_{(\alpha, u, T)} : C \rightarrow C$ defined by

$$S_{(\alpha, u, T)}x = \alpha u + (1 - \alpha)Tx \quad \text{for all } x \in C$$

is a $(1 - \alpha)$ -contraction. Consequently, $S_{(\alpha, u, T)}$ has a unique fixed point $x^* \in C$. Thus there exists a unique $x^* \in C$, that is,

$$x^* = \alpha u + (1 - \alpha)Tx^*.$$

Hence, we can define $U_n : C \rightarrow C$ by letting $U_n x$ be the unique fixed point of the contraction $S_{(\alpha_{n+1}, x, T_{n+1})}$, i.e.,

$$U_n x = \alpha_{n+1} x + (1 - \alpha_{n+1})T_{n+1} U_n x.$$

In particular, U_n is nonexpansive. From (2.1), we also have $x_{n+1} = U_n x_n$. Let $z \in \bigcap_{n=1}^{\infty} F(T_n)$. Then

$$\|U_n z - z\| = (1 - \alpha_{n+1})\|T_{n+1} U_n z - z\| \leq (1 - \alpha_{n+1})\|U_n z - z\|$$

and so $U_n z = z$. It follows that $\bigcap_{n=1}^{\infty} F(T_n) \subset \bigcap_{n=1}^{\infty} F(U_n)$. Because $K \subset C$, we define a nonexpansive mapping $V_{n,m}$ from K into C by

$$V_{n,m} = U_{n+m-1} U_{n+m-2} \cdots U_n, \quad n, m \geq 1.$$

Then $V_{n,m} x_n = x_{n+m}$ and $\bigcap_{n=1}^{\infty} F(T_n) \subset \bigcap_{n,m=1}^{\infty} F(V_{n,m})$. Set

$$\lambda_{n,m}(t) = \|tV_{n,m}x_n + (1-t)p - V_{n,m}(tx_n + (1-t)p)\|.$$

By Lemma 2.3, we have

$$\gamma(\lambda_{n,m}(t)) \leq \|x_n - p\| - \|V_{n,m}x_n - V_{n,m}p\| = \|x_n - p\| - \|x_{n+m} - p\|,$$

where $\gamma : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing continuous convex function with $\gamma(0) = 0$. Using Lemma 2.14(i), we have $\lim_{n,m \rightarrow \infty} \gamma(\lambda_{n,m}(t)) = 0$, and hence $\lim_{n,m \rightarrow \infty} \lambda_{n,m} = 0$. Observe that

$$\begin{aligned} \delta_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &= \|tV_{n,m}x_n + (1-t)p - q\| \\ &\leq \lambda_{n,m}(t) + \|V_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq \lambda_{n,m}(t) + \|tx_n + (1-t)p - q\| = \lambda_{n,m}(t) + \delta_n(t). \end{aligned}$$

Consequently,

$$\limsup_{n \rightarrow \infty} \delta_n(t) \leq \lim_{n,m \rightarrow \infty} \lambda_{n,m}(t) + \liminf_{n \rightarrow \infty} \delta_n(t) = \liminf_{n \rightarrow \infty} \delta_n(t).$$

This implies that $\lim_{n \rightarrow \infty} \delta_n(t)$ exists for any $t \in (0, 1)$ and the proof is finished. \square

Theorem 3.9. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X such that its dual X^* has the Kadec-Klee property. Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . Then the sequence $\{x_n\}$ in C defined by (2.1), where $\{\alpha_n\} \subset (0, b)$ for some $b \in (0, 1)$, converges weakly to $z \in F(\mathcal{T})$.*

Proof. It follows from Lemma 3.1 that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \quad \text{for all } T \in \mathcal{T}.$$

Because $\{x_n\}$ is bounded, let $z \in \omega_w\{x_n\}$. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$. By Lemma 2.2, we have $z \in F(\mathcal{T})$. This implies that $\omega_w\{x_n\} \subset F(\mathcal{T})$. Then, by Lemma 3.8, the limit

$$\lim_{n \rightarrow \infty} \|tx_n + (1 - t)p - q\|$$

exists for all $t \in [0, 1]$ and $p, q \in \omega_w\{x_n\}$. Hence $x_n \rightharpoonup z \in F(\mathcal{T})$ by Lemma 3.7. This completes the proof. \square

The following corollary is proved in [12, Theorem 3.10] with the restriction $\alpha_n \in [a, b]$ for some $a, b \in (0, 1)$.

Corollary 3.10. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X such that its dual X^* has the Kadec-Klee property. Let $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$ be two finite families of nonexpansive mappings of C into itself with $F(T) \cap F(S) \neq \emptyset$, where $F(T) := \bigcap_{n=1}^N F(T_n)$ and $F(S) := \bigcap_{n=1}^N F(S_n)$. Assume that $\{\alpha_n\} \subset (0, b)$ and $\{\beta_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then the sequence $\{x_n\}$ in C defined by (1.2), where $T_n \equiv T_{n \bmod N}$ and $S_n \equiv S_{n \bmod N}$, converges weakly to a common fixed point of $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$.*

As in the proofs of Theorems 3.6 and 3.9, we also have the following.

Theorem 3.11. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X such that its dual X^* has the Kadec-Klee property. Let $\{T_n\}$ be a family of nonexpansive mappings of C into itself that satisfies the AKTT-condition*

and $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let T be the mapping from C into itself defined by $Tz = \lim_{n \rightarrow \infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then the sequence $\{x_n\}$ in C defined by (2.1), where $\{\alpha_n\} \subset (0, 1)$ satisfying $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$, converges weakly to a common fixed point of $\{T_n\}$.

4. STRONG CONVERGENCE THEOREMS

In this section, we discuss the strong convergence of the iteration (2.1) in a uniformly convex Banach space.

We recall that a mapping $T : C \rightarrow C$ is *semicompact* if every sequence $\{u_n\}$ in C satisfying $\|u_n - Tu_n\| \rightarrow 0$ has a convergent subsequence.

Theorem 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . If there exists some $\widehat{T} \in \mathcal{T}$ that is semicompact, then the sequence $\{x_n\}$ in C defined by (2.1), where $\{\alpha_n\} \subset (0, b)$ for some $b \in (0, 1)$, converges strongly to $z \in F(\mathcal{T})$.*

Proof. Suppose that $\widehat{T} \in \mathcal{T}$ is semicompact. It follows from Lemma 3.1 that

$$\lim_{n \rightarrow \infty} \|x_n - \widehat{T}x_n\| = 0.$$

So there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $x_{n_j} \rightarrow z \in C$ as $j \rightarrow \infty$. As

$$\lim_{j \rightarrow \infty} \|x_{n_j} - Tx_{n_j}\| = 0 \quad \text{for all } T \in \mathcal{T},$$

we have $\|z - Tz\| = 0$ for all $T \in \mathcal{T}$. This implies that $z \in F(\mathcal{T})$. By $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T})$ and Lemma 2.14(i), we have $x_n \rightarrow z \in F(\mathcal{T})$. \square

Using Theorem 4.1 and Lemmas 2.9 and 2.13, we have the following results, which is an improvement of [12, Theorem 3.4].

Corollary 4.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$ be two finite families of nonexpansive mappings of C into itself with $F(T) \cap F(S) \neq \emptyset$, where $F(T) := \bigcap_{n=1}^N F(T_n)$ and $F(S) := \bigcap_{n=1}^N F(S_n)$. Assume that $\{\alpha_n\} \subset (0, b)$ and $\{\beta_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Suppose that one of the mapping in $\{T_n, S_n\}_{n=1}^N$ is semicompact. Then the sequence $\{x_n\}$ in C defined by (1.2), where $T_n \equiv T_{n \bmod N}$ and $S_n \equiv S_{n \bmod N}$, converges strongly to a common fixed point of $\{T_n, S_n\}_{n=1}^N$.*

Remark 4.3.

(1) Corollary 4.2 improves [12, Theorem 3.4] in the following ways:

- (a) The restriction on $\{\alpha_n\}$ is weakened from $\alpha_n \in [a, b] \subset (0, 1)$ to $\alpha_n \in (0, b] \subset (0, 1)$.
- (b) The strong convergence is proved in [12, Theorem 3.4] under the assumption that there exist at least two mappings from the families $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$ that are semicompact, while our Corollary 4.2 needs only one mapping that is semicompact.

(2) In [12], the authors claimed that by letting $S_1 = S_2 = \dots = S_N = I$, [3, Theorem 3.3] is a consequence of [12, Theorem 3.4]. However, the identity I is not semicompact.

Now we present another generalization of [3, Theorem 3.3] with the weaker assumption on $\{\alpha_n\}$. It should be noted here that [3, Theorem 3.3] is also independently obtained in [10, Theorem 2].

Corollary 4.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mappings of C into itself with a common fixed point. Suppose that one of the mappings in $\{T_n\}_{n=1}^N$ is semicompact. Then the sequence $\{x_n\}$ in C defined by (2.1), where $T_n \equiv T_{n \bmod N}$ and $\{\alpha_n\} \subset (0, b]$ for some $b \in (0, 1)$, converges strongly to a common fixed point of $\{T_n\}_{n=1}^N$.*

We next prove a strong convergence theorem of our iteration in the presence of another condition. We recall that a finite family \mathcal{T} of mappings of C with $F(\mathcal{T}) \neq \emptyset$ satisfies *condition (B)* (see [3]) if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in C$,

$$f(d(x, F(\mathcal{T}))) \leq \max\{\|x - Tx\| : T \in \mathcal{T}\}.$$

Theorem 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}$ and \mathcal{T} be two families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = F(\mathcal{T}) \neq \emptyset$ and suppose that $\{T_n\}$ satisfies the NST*-condition with \mathcal{T} . If \mathcal{T} is finite and satisfies condition (B), then the sequence $\{x_n\}$ in C defined by (2.1), where $\{\alpha_n\} \subset (0, b]$ for some $b \in (0, 1)$, converges strongly to $z \in F(\mathcal{T})$.*

Proof. By Lemma 3.1, we have

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0 \quad \text{for all } T \in \mathcal{T}. \tag{4.1}$$

Let f be the function corresponding to the condition (B) of the finite family \mathcal{T} . Then

$$f(d(x_n, F(\mathcal{T}))) \leq \max\{\|x_n - Tx_n\| : T \in \mathcal{T}\} \rightarrow 0.$$

and so $\lim_{n \rightarrow \infty} d(x_n, F(\mathcal{T})) = 0$. We now prove that $\{x_n\}$ is a Cauchy sequence in C . Let $\varepsilon > 0$. There exists a positive integer N such that

$$d(x_N, F(\mathcal{T})) < \frac{\varepsilon}{2}.$$

There must exist $q \in F(\mathcal{T})$ such that

$$\|x_N - q\| < \frac{\varepsilon}{2}.$$

From (2.2) it follows that, for all $m, n \geq N$,

$$\|x_m - x_n\| \leq \|x_m - q\| + \|x_n - q\| \leq \|x_N - q\| + \|x_N - q\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C . In virtue of the completeness of C , we may assume that $x_n \rightarrow z \in C$. By the continuity of T and (4.1), we have $z = Tz$ for all $T \in \mathcal{T}$, so $z \in F(\mathcal{T})$. This completes the proof. \square

Using Theorem 4.5 and Lemmas 2.9 and 2.13, we have the following result, which is an improvement of [12, Theorem 3.3].

Corollary 4.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$ be two finite families of nonexpansive mappings of C into itself with $F(T) \cap F(S) \neq \emptyset$, where $F(T) := \bigcap_{n=1}^N F(T_n)$ and $F(S) := \bigcap_{n=1}^N F(S_n)$. Assume that $\{\alpha_n\} \subset (0, b]$ and $\{\beta_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. If $\{T_n, S_n\}_{n=1}^N$ satisfies condition (B), then the sequence $\{x_n\}$ in C defined by (1.2), where $T_n \equiv T_{n \bmod N}$ and $S_n \equiv S_{n \bmod N}$, converges strongly to a common fixed point of $\{T_n\}_{n=1}^N$ and $\{S_n\}_{n=1}^N$.*

5. DEDUCED THEOREMS

In the last section, we show that our results not only include ones in [7, 8] as special cases but also give a new weak convergence theorem in the presence of Kadec–Klee property of the dual space.

Let C be a nonempty closed convex subset of a Banach space X . Let $\{\beta_n\}$ be a sequence in $[0, 1]$ and let $\{S_k\}$ be a family of nonexpansive

mappings with a common fixed point, we define a family $\{W_n\}$ of nonexpansive mappings from C into itself by

$$\begin{aligned}
 U_{n,n+1} &= I \\
 U_{n,n} &= \beta_n S_n U_{n,n+1} + (1 - \beta_n)I \\
 U_{n,n-1} &= \beta_{n-1} S_{n-1} U_{n,n} + (1 - \beta_{n-1})I \\
 &\vdots \\
 U_{n,k} &= \beta_k S_k U_{n,k+1} + (1 - \beta_k)I \\
 U_{n,k-1} &= \beta_{k-1} S_{k-1} U_{n,k} + (1 - \beta_{k-1})I \\
 &\vdots \\
 U_{n,2} &= \beta_2 S_2 U_{n,3} + (1 - \beta_2)I \\
 W_n = U_{n,1} &= \beta_1 S_1 U_{n,2} + (1 - \beta_1)I.
 \end{aligned}$$

Such a mapping W_n is call a *W-mapping* [17] generated by S_n, S_{n-1}, \dots, S_1 and $\beta_n, \beta_{n-1}, \dots, \beta_1$ (see also [11, 15]).

Lemma 5.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X . Let $\{S_k\}$ be a family of nonexpansive mappings of C into itself with a common fixed point and let $\{\beta_n\}$ be a sequence in $(0, 1)$. Then $\{W_n\}$ satisfies the NST-condition (I) with $\{S_k\}$. Consequently, $\{W_n\}$ satisfies the NST*-condition with $\{S_k\}$.*

Proof. Let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - W_n z_n\| = 0$. Because $z_n - W_n z_n = z_n - U_{n,1} z_n = \beta_1(z_n - S_1 U_{n,2} z_n)$, we have

$$\lim_{n \rightarrow \infty} \|z_n - S_1 U_{n,2} z_n\| = 0.$$

Let $p \in \bigcap_{k=0}^{\infty} F(S_k)$. It follows from Lemma 2.1 that

$$\begin{aligned}
 \|z_n - p\|^2 &\leq (\|z_n - S_1 U_{n,2} z_n\| + \|S_1 U_{n,2} z_n - p\|)^2 \\
 &= \|z_n - S_1 U_{n,2} z_n\|^2 + 2\|z_n - S_1 U_{n,2} z_n\| \|S_1 U_{n,2} z_n - p\| + \|S_1 U_{n,2} z_n - p\|^2 \\
 &\leq \|z_n - S_1 U_{n,2} z_n\| M + \|U_{n,2} z_n - p\|^2 \\
 &= \|z_n - S_1 U_{n,2} z_n\| M + \|\beta_2(S_2 U_{n,3} z_n - p) + (1 - \beta_2)(z_n - p)\|^2 \\
 &\leq \|z_n - S_1 U_{n,2} z_n\| M + \beta_2 \|S_2 U_{n,3} z_n - p\|^2 + (1 - \beta_2) \|z_n - p\|^2 \\
 &\quad - \beta_2(1 - \beta_2)g(\|S_2 U_{n,3} z_n - z_n\|) \\
 &\leq \|z_n - S_1 U_{n,2} z_n\| M + \|z_n - p\|^2 - \beta_2(1 - \beta_2)g(\|z_n - S_2 U_{n,3} z_n\|)
 \end{aligned}$$

for all $n \in \mathbb{N}$, where $M = \sup\{\|z_n - S_1 U_{n,2} z_n\| + 2\|S_1 U_{n,2} z_n - p\| : n \in \mathbb{N}\}$. So, we get

$$\lim_{n \rightarrow \infty} \|z_n - S_2 U_{n,3} z_n\| = 0.$$

By induction, we have

$$\lim_{n \rightarrow \infty} \|z_n - S_k U_{n,k+1} z_n\| = 0 \quad \text{for all } k \in \mathbb{N}.$$

Because

$$\begin{aligned} \|z_n - S_k z_n\| &\leq \|z_n - S_k U_{n,k+1} z_n\| + \|S_k z_n - S_k U_{n,k+1} z_n\| \\ &\leq \|z_n - S_k U_{n,k+1} z_n\| + \|z_n - U_{n,k+1} z_n\| \\ &= \|z_n - S_k U_{n,k+1} z_n\| + \beta_{k+1} \|z_n - S_{k+1} U_{n,k+2} z_n\|, \end{aligned}$$

for all $n \in \mathbb{N}$, we get

$$\lim_{n \rightarrow \infty} \|z_n - S_k z_n\| = 0 \quad \text{for all } k \in \mathbb{N}.$$

Hence, $\{W_n\}$ satisfies the NST-condition (I) with $\{S_k\}$. \square

Lemma 5.2 ([11, Lemma 3.6(ii)]). *Let C be a nonempty closed convex subset of a strictly convex Banach space X . Let $\{S_k\}$ be a family of nonexpansive mappings of C into itself with a common fixed point and let $\{\beta_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, the family $\{W_n\}$ satisfies AKTT-condition and the mapping $W : C \rightarrow C$ defined by $Wz = \lim_{n \rightarrow \infty} W_n z$ ($z \in C$) satisfies $F(W) = \bigcap_{n=1}^{\infty} F(W_n) = \bigcap_{k=0}^{\infty} F(S_k)$.*

Theorem 5.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $\{S_k\}$ be a family of nonexpansive mappings of C into itself with a common fixed point. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ and let $b \in (0, 1)$. Assume that one the of following conditions holds:*

- (i) $\alpha_n \in (0, b]$;
- (ii) $\sum_{n=1}^{\infty} (1 - \alpha_n) = \infty$ and $\beta_n \in (0, b]$.

If, in addition, X satisfies Opial's condition or X^ has the Kadec–Klee property, then the sequence $\{x_n\}$ in C defined by (2.1), where $T_n \equiv W_n$, converges weakly to a common fixed point of $\{S_k\}$.*

Remark 5.4. Theorem 5.3 improves [8, Theorem 5.1] in the following ways:

- (1) The geometric property of a Banach space is weakened.
- (2) The restrictions on the parameters $\{\alpha_n\}$ and $\{\beta_n\}$ are relaxed.

The last theorem is an improvement of [7, Theorem 8] with the weaker restriction on the parameters $\{\beta_n\}$.

Theorem 5.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $\{S_k\}$ be a family of nonexpansive mappings of C into itself with a common fixed point. Let $\{\alpha_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$ and $\{\beta_n\}$ be a sequence in $(0, 1)$. Suppose that one of the mappings $\{S_k\}$ is semicompact. Then the sequence $\{x_n\}$ in C defined by (2.1), where $T_n \equiv W_n$, converges strongly to a common fixed point of $\{S_k\}$.*

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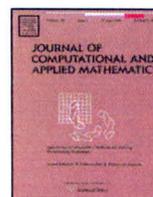
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Weak and strong convergence theorems of an implicit iteration for a countable family of continuous pseudocontractive mappings

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ABSTRACT

To approximate a common fixed point of a countable family of continuous pseudocontractive mappings, we introduce an implicit iteration sequence. A necessary and sufficient condition for the convergence of a sequence of such iterates for countably many continuous pseudocontractive mappings is given. We also prove the convergence theorems of an implicit iteration sequence for a countable family of strictly pseudocontractive mappings.

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1. Introduction

Let E be a real Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. We say that the norm of E is Gâteaux differentiable or E is smooth if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. The norm of E is said to be Fréchet differentiable if for each $x \in S(E)$, the limit above is attained uniformly for $y \in S(E)$. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\},$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. If E is smooth, then J is single-valued.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in a Banach space is called

- (i) pseudocontractive [1], if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2; \quad (1.1)$$

equivalently, for all $x, y \in D(T)$ and for all $s > 0$,

$$\|x - y\| \leq \|x - y + s[(I - T)x - (I - T)y]\|; \quad (1.2)$$

- (ii) λ -strictly pseudocontractive (in the terminology of Browder and Petryshyn) [1] for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(x - Tx) - (y - Ty)\|^2; \quad (1.3)$$

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(iii) strongly pseudocontractive if there exists $\lambda \in (0, 1)$ for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \lambda \|x - y\|^2;$$

(iv) L -Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$,

$$\|Tx - Ty\| \leq L\|x - y\|.$$

Remark 1.2. (1) The expression (1.3) is equivalent to there exists $\lambda \in (0, 1)$, for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle (x - Tx) - (y - Ty), j(x - y) \rangle \geq \lambda \|(x - Tx) - (y - Ty)\|^2.$$

(2) Every λ -strictly pseudocontractive mapping is $(\frac{1+\lambda}{\lambda})$ -Lipschitzian. (see e.g. [2, Remark 1.1]).

The concept of pseudocontractive mappings is closely related to accretive operators which were independently introduced in 1967 in [3,4]. It is known that T is pseudocontractive if and only if $I - T$ is accretive, where I is an identity mapping. The importance of accretive mappings is from their connection with theory of solutions for nonlinear evolution equations in Banach spaces [5]. Heat, wave, or Schrödinger equations can be modeled in terms of an initial value problem:

$$\frac{du}{dt} = Tu - u, \quad u(0) = u_0$$

where T is a pseudocontractive mapping in an appropriate Banach space.

There are many methods for approximating a fixed point of a given mapping (see e.g. [6,7]). In 1974, Ishikawa [8] introduced a new iteration in the approximation of fixed point of Lipschitzian pseudocontractive mappings. Since then, an question of whether or not the Ishikawa iteration can be replaced by the simpler Mann iteration has remained open. Recently, Chidume and Mutangadura [9] solved this question by constructing an example of a Lipschitzian pseudocontractive mapping with a unique fixed point for which every Mann-type iteration fails to converge.

Inspired by the implicit iteration introduced by Xu and Ori [10], Chen et al. [2] and Zhou [11] proposed and studied convergence theorems for an implicit iteration process for finite many continuous pseudocontractive mappings. They also presented convergence theorems for a finite family of strictly pseudocontractive mappings.

The purpose of this paper is to present simple and natural generalizations from the finite case [2,11] to the countable one. A careful discussion on [11, Theorem 3.1] is also given in Remark 2.4. We also present a countable version of [11, Theorem 3.1] under an appropriate assumption.

For a mapping T , we denoted by $F(T)$ the set of fixed points of T , that is, $F(T) = \{x \in D(T) : x = Tx\}$. To prove our main results, we need the following lemmas.

Lemma 1.3 ([2,12]). *If $J : E \rightarrow 2^{E^*}$ is a normalized duality mapping, then for all $x, y \in E$ and $j(x + y) \in J(x + y)$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$

Throughout the paper, we write $x_n \rightarrow x$ ($x_n \rightharpoonup x$, resp.) if the sequence $\{x_n\}$ converges in a Banach space strongly (weakly, resp.) to an element x . We also write $\omega_w(x_n) = \{x \in X : x_{n_k} \rightharpoonup x \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}$.

Lemma 1.4 ([11, Tool 3]). *Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E , and T a continuous pseudocontractive mapping of K , Then $I - T$ is demiclosed at zero, that is, for all sequences $\{x_n\} \subset K$ with $x_n \rightharpoonup p$ and $x_n - Tx_n \rightarrow 0$ it follows that $p = Tp$.*

Lemma 1.5 ([13, Lemma 2.3]). *Let K be a closed convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable. Let $\{T_n\}_{n=1}^\infty$ be a family of L_n -Lipschitzian self-mappings of K such that $\sum_{n=1}^\infty (L_n - 1) < \infty$. Suppose that $F = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$, $x_1 \in K$, and $x_{n+1} = T_n x_n$ for all $n \geq 1$. Then for every $p, q \in F$,*

$$\lim_{n \rightarrow \infty} \langle x_n, j(p - q) \rangle \text{ exists.}$$

In particular, for all $u, v \in \omega_w(x_n)$ and $p, q \in F$, we have

$$\langle u - v, j(p - q) \rangle = 0.$$

2. Continuous pseudocontractive mappings

In this section, we study an implicit iteration for a countable family of continuous pseudocontractive mappings. Deimling [14] proved the following fixed point theorem.

Lemma 2.1. Let K be a nonempty closed convex subset of a real Banach space E and T a continuous and strongly pseudocontractive mapping of K . Then T has a unique fixed point in K .

Let K be a nonempty closed convex subset of a real Banach space E and T a continuous pseudocontractive mapping of K . For every $u \in K$ and $t \in (0, 1)$, the mapping $S_t : K \rightarrow K$ defined by

$$S_t x = tu + (1 - t)Tx, \quad x \in K,$$

is a continuous and strongly pseudocontractive mapping; by utilizing Lemma 2.1, there exists a unique fixed point $x_t \in K$ of S_t which satisfies

$$x_t = tu + (1 - t)Tx_t, \quad t \in (0, 1). \tag{2.1}$$

Moreover, it is proved in [15] that there exists an iterative scheme approximating such a unique fixed point of the mapping S_t with desired convergence rates.

Let $\{T_n\}_{n=1}^\infty$ be a countable family of continuous pseudocontractive mappings on a closed convex subset K of a real Banach space E . From (2.1), we can consider the following iteration: $x_0 \in K$ and

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n \quad (n \geq 1) \tag{2.2}$$

where $\{\alpha_n\}_{n=1}^\infty$ is a real sequence in $(0, 1)$. It is easy to see that

$$\alpha_n(x_n - x_{n-1}) = (1 - \alpha_n)(T_n x_n - x_n).$$

Consequently,

- (1) if $x_n - x_{n-1} \rightarrow 0$ and $\alpha_n \leq b < 1$ for all $n \geq 1$, then $T_n x_n - x_n \rightarrow 0$;
- (2) if $T_n x_n - x_n \rightarrow 0$ and $\alpha_n \geq a > 0$ for all $n \geq 1$, then $x_n - x_{n-1} \rightarrow 0$.

Theorem 2.2. Let $\{T_n\}_{n=1}^\infty$ be defined as above and $F := \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. Let $\{x_n\}_{n=1}^\infty$ is defined by the iteration (2.2) where $\{\alpha_n\}_{n=1}^\infty$ is a real sequence in $(0, 1)$. Then

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$;
- (ii) $\lim_{n \rightarrow \infty} d(x_n, F)$ exists, where $d(x_n, F) = \inf\{\|x_n - p\| : p \in F\}$;
- (iii) $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}_{n=1}^\infty$ if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.

Proof. Let $p \in F$ and $n \geq 1$. By (1.1) we choose $j(x_n - p) \in J(x_n - p)$ such that

$$\langle Tx_n - Tp, j(x_n - p) \rangle \leq \|x_n - p\|^2.$$

Then

$$\begin{aligned} \|x_n - p\|^2 &= \langle x_n - p, j(x_n - p) \rangle \\ &= \langle \alpha_n(x_{n-1} - p) + (1 - \alpha_n)(T_n x_n - p), j(x_n - p) \rangle \\ &= \alpha_n \langle x_{n-1} - p, j(x_n - p) \rangle + (1 - \alpha_n) \langle T_n x_n - p, j(x_n - p) \rangle \\ &\leq \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1 - \alpha_n) \|x_n - p\|^2. \end{aligned}$$

So

$$\|x_n - p\|^2 \leq \|x_{n-1} - p\| \|x_n - p\|.$$

Consequently,

$$\|x_n - p\| \leq \|x_{n-1} - p\|. \tag{2.3}$$

Taking infimum over all $p \in F$, we have

$$d(x_n, F) = \inf_{p \in F} \|x_n - p\| \leq \inf_{p \in F} \|x_{n-1} - p\| = d(x_{n-1}, F),$$

hence the both limits $\lim_{n \rightarrow \infty} \|x_n - p\|$ and $\lim_{n \rightarrow \infty} d(x_n, F)$ exist. The proofs of (i) and (ii) are complete.

We finally prove (iii). Suppose that $x_n \rightarrow p \in F$. From (ii) and $d(x_n, F) \leq \|x_n - p\| \rightarrow 0$, we have $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. By (ii), $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. For $n, m \in \mathbb{N}$ and $p \in F$, it follows from (2.3) that

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_n - p\|.$$

Consequently,

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F) \rightarrow 0.$$

Therefore $\{x_n\}$ is a Cauchy sequence. Suppose $\lim_{n \rightarrow \infty} x_n = u$ for some $u \in E$. Then

$$d(u, F) = \lim_{n \rightarrow \infty} d(x_n, F) = 0.$$

Since F is a closed set, $u \in F$. The proof is complete. \square

Remark 2.3. Theorem 2.2 above is a countable version of [2, Theorem 2.3].

We next present a weak convergence theorem of the iteration (2.2) in the presence of the following condition (see [16]):

A countable family of mappings $\{T_n\}$ on a subset K of a Banach space is said to satisfy the NST-condition if for every bounded sequence $\{z_n\}$ in K , $z_{n+1} - T_n z_n \rightarrow 0$ implies

$$z_n - T_m z_n \rightarrow 0 \quad \text{for all } m \in \mathbb{N}.$$

Remark 2.4. The following result is very related to [11, Theorem 3.1] but the proof of the mentioned result is doubtful. More precisely, in Step 3, the author claims that $\lim_{n \rightarrow \infty} \langle x_n, j(p - q) \rangle$ exists for all $p, q \in F$ by applying Lemma 1.5. But we think that the condition $\sum_{n=1}^{\infty} (L_n - 1) < \infty$ of Lemma 1.5 seems to be overlooked in [11, Theorem 3.1].

Theorem 2.5. Let K be a closed convex subset of a real uniformly convex Banach space E with a Fréchet differentiable norm, $\{T_n\}_{n=1}^{\infty}$ a family of L_n -Lipschitzian pseudocontractive mappings of K such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\sum_{n=1}^{\infty} (L_n - 1) < \infty$. Let $\{x_n\}$ be defined by (2.2) where

$$0 < 1 - \frac{1 - \varepsilon}{L_n} \leq \alpha_n \leq b < 1 \quad \text{for all } n \geq 1.$$

Suppose that $\{T_n\}$ satisfies the NST-condition. Then $x_n \rightarrow p$ for some $p \in \bigcap_{n=1}^{\infty} F(T_n)$.

Before proving this result, we need the following lemma.

Lemma 2.6 ([17]). Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n . Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$ for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Proof of Theorem 2.5. Let $p \in F$. It follows from Theorem 2.2 that

$$\|x_n - p\| \leq \|x_{n-1} - p\| \quad \text{for all } n \in \mathbb{N},$$

and $d := \lim_{n \rightarrow \infty} \|x_n - p\|$ exists and hence the sequence $\{x_n\}$ is bounded. By (1.2), we have

$$\begin{aligned} \|x_n - p\| &\leq \left\| x_n - p + \frac{1 - \alpha_n}{2\alpha_n} (x_n - T_n x_n) \right\| \\ &= \left\| x_n - p + \frac{1 - \alpha_n}{2} (x_{n-1} - T_n x_n) \right\| \\ &= \left\| \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n - p + \frac{1 - \alpha_n}{2} (x_{n-1} - T_n x_n) \right\| \\ &= \left\| \frac{1}{2} (x_{n-1} + x_n) - p \right\| \\ &\leq \frac{1}{2} \|x_{n-1} - p\| + \frac{1}{2} \|x_n - p\| \leq \|x_{n-1} - p\|. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{2} (x_{n-1} - p) + \frac{1}{2} (x_n - p) \right\| = d.$$

From Lemma 2.6, we get that

$$x_n - x_{n-1} = (x_n - p) - (x_{n-1} - p) \rightarrow 0,$$

and hence

$$x_n - T_n x_n \rightarrow 0.$$

Since $\{T_n\}$ satisfies the NST-condition and $x_{n+1} - T_n x_n \rightarrow 0$,

$$x_n - T_m x_n \rightarrow 0 \quad \text{for all } m.$$

Moreover, since each mapping T_m is demiclosed at zero,

$$\omega_w(x_n) \subset F. \tag{2.4}$$

Next, we will apply Lemma 1.5. We observe that, for $u \in K$, the continuous and strongly pseudocontractive mapping $S_{(\alpha, u, T)} : K \rightarrow K$ defined by

$$S_{(\alpha, u, T)}x = \alpha u + (1 - \alpha)Tx \quad \text{for all } x \in K$$

has a unique fixed point $p \in K$, or

$$p = S_{(\alpha, u, T)}p = \alpha u + (1 - \alpha)Tp.$$

We now define

$$U_n : K \rightarrow K \quad (n \geq 0)$$

by letting $U_n x$ be the unique fixed point of the mapping $S_{(\alpha_{n+1}, x, T_n)}$, that is,

$$U_n x = \alpha_{n+1}x + (1 - \alpha_{n+1})T_{n+1}U_n x.$$

Then

$$x_{n+1} = U_n x_n.$$

We observe that

$$\begin{aligned} \|U_n x - U_n y\| &\leq \alpha_{n+1}\|x - y\| + (1 - \alpha_{n+1})\|T_{n+1}U_n x - T_{n+1}U_n y\| \\ &\leq \alpha_{n+1}\|x - y\| + (1 - \alpha_{n+1})L_{n+1}\|U_n x - U_n y\|, \end{aligned}$$

so

$$\begin{aligned} \|U_n x - U_n y\| &\leq \frac{\alpha_{n+1}}{1 - (1 - \alpha_{n+1})L_{n+1}}\|x - y\| \\ &= \left(\frac{(1 - \alpha_{n+1})(L_{n+1} - 1)}{1 - (1 - \alpha_{n+1})L_{n+1}} + 1 \right) \|x - y\| \\ &\leq \left(\frac{1}{\varepsilon}(L_{n+1} - 1) + 1 \right) \|x - y\|. \end{aligned}$$

By Lemma 1.5 and $\sum_{n=1}^{\infty} (L_n - 1) < \infty$, it follows that

$$\langle u - v, j(p - q) \rangle = 0, \tag{2.5}$$

for all $u, v \in \omega_w(x_n)$ and $p, q \in \bigcap_{n=1}^{\infty} F(U_n)$. We next show that

$$\bigcap_{n=0}^{\infty} F(U_n) = F. \tag{2.6}$$

For $p \in F$ and $n \geq 0$,

$$\begin{aligned} \|U_n p - p\|^2 &= \langle U_n p - p, j(U_n p - p) \rangle \\ &= (1 - \alpha_{n+1})\langle T_{n+1}U_n p - T_{n+1}p, j(U_n p - p) \rangle \\ &\leq (1 - \alpha_{n+1})\|U_n p - p\|^2 \end{aligned}$$

and so $U_n p = p$. It follows that $F \subset \bigcap_{n=0}^{\infty} F(U_n)$. On the other hand, let $q \in \bigcap_{n=0}^{\infty} F(U_n)$, that is, $q = U_n q$ for all $n \geq 0$. Then

$$q = U_n q = \alpha_{n+1}q + (1 - \alpha_{n+1})T_{n+1}U_n q = \alpha_{n+1}q + (1 - \alpha_{n+1})T_{n+1}q.$$

Thus $q = T_{n+1}q$ and hence $\bigcap_{n=0}^{\infty} F(U_n) \subset F$. So (2.6) is proved.

Finally, to show that $x_n \rightarrow p$ for some $p \in F$, it suffices to show that $\omega_w(x_n)$ is a singleton. Suppose that $u, v \in \omega_w(x_n)$. From (2.4) and Lemma 1.4, we get that $u, v \in \bigcap_{n=1}^{\infty} F(U_n)$. By Lemma 1.5, $\lim_{n \rightarrow \infty} \langle x_n, j(u - v) \rangle$ exists. Suppose that $\{n_k\}$ and $\{m_k\}$ are subsequences of $\{n\}$ so that

$$x_{n_k} \rightarrow u \quad \text{and} \quad x_{m_k} \rightarrow v.$$

Then $0 = \lim_{k \rightarrow \infty} \langle x_{n_k} - x_{m_k}, j(u - v) \rangle = \langle u - v, j(u - v) \rangle = \|u - v\|^2$, and hence $u = v$. \square

3. Strictly pseudocontractive mappings

We now restrict ourselves to a subclass of continuous pseudocontractive mappings. In this section, we present convergence theorems of an implicit iteration for a countable family of strictly pseudocontractive mappings. Our results generalizes the recent ones due to [2] from a finite family of mapping to a countable one. We first start with the following auxiliary results which is inspired by Bruck’s paper [18].

Lemma 3.1. *Let K be a closed convex subset of a smooth Banach space E . Suppose that $\{T_n\}_{n=1}^\infty$ is a family of λ -strictly pseudocontractive mappings from K into E with $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and $\{\beta_n\}_{n=1}^\infty$ is a real sequence in $(0, 1)$ such that $\sum_{n=1}^\infty \beta_n = 1$. Then the following conclusions hold:*

- (1) $G := \sum_{n=1}^\infty \beta_n T_n : K \rightarrow E$ is a λ -strictly pseudocontractive mapping;
- (2) $F(G) = \bigcap_{n=1}^\infty F(T_n)$.

Proof. (1) Fix $p \in \bigcap_{n=1}^\infty F(T_n)$ and let $x \in K$. Then

$$\|T_n x\| \leq \|T_n x - T_n p\| + \|T_n p\| \leq \frac{1 + \lambda}{\lambda} \|x - p\| + \|p\|.$$

Thus $\sum \beta_n T_n x$ converges absolutely for each $x \in K$, that is $G := \sum_{n=1}^\infty \beta_n T_n : K \rightarrow E$ is well defined. Note that J is single-valued since E is smooth. To show G is λ -strictly pseudocontractive, let $x, y \in K$. Then

$$\begin{aligned} \langle (x - Gx) - (y - Gy), j(x - y) \rangle &= \left\langle \left(x - \sum_{n=1}^\infty \beta_n T_n x \right) - \left(y - \sum_{n=1}^\infty \beta_n T_n y \right), j(x - y) \right\rangle \\ &= \sum_{n=1}^\infty \beta_n \langle (x - T_n x) - (y - T_n y), j(x - y) \rangle \\ &\geq \lambda \sum_{n=1}^\infty \beta_n \| (x - T_n x) - (y - T_n y) \|^2 \\ &\geq \lambda \left\| \sum_{n=1}^\infty \beta_n ((x - T_n x) - (y - T_n y)) \right\|^2 \\ &= \lambda \left\| \left(x - \sum_{n=1}^\infty \beta_n T_n x \right) - \left(y - \sum_{n=1}^\infty \beta_n T_n y \right) \right\|^2 \\ &= \lambda \| (x - Gx) - (y - Gy) \|^2. \end{aligned}$$

Hence G is λ -strictly pseudocontractive.

- (2) Clearly, $\bigcap_{n=1}^\infty F(T_n) \subseteq F(G)$. Conversely, let $x \in F(G)$ and fix $p \in \bigcap_{n=1}^\infty F(T_n)$. Then

$$\begin{aligned} \|x - p\|^2 &= \langle x - p, j(x - p) \rangle \\ &= \sum_{n=1}^\infty \beta_n \langle (T_n x - p), j(x - p) \rangle \\ &\leq \sum_{n=1}^\infty \beta_n (\|x - p\|^2 - \lambda \|x - T_n x\|^2) \\ &= \|x - p\|^2 - \lambda \sum_{n=1}^\infty \beta_n \|x - T_n x\|^2, \end{aligned}$$

that is,

$$\lambda \sum_{n=1}^\infty \beta_n \|x - T_n x\|^2 \leq 0.$$

This implies that $x = T_n x$ for all $n \in \mathbb{N}$. Hence $x \in \bigcap_{n=1}^\infty F(T_n)$. The proof is finished. \square

From the preceding lemma, we immediately obtain the following corollary.

Corollary 3.2. *Let K be a nonempty convex subset of a smooth Banach space E . Given an integer $r \geq 1$, assume that $\{T_i\}_{i=1}^r$ is a finite family of λ_i -strictly pseudocontractive mappings of K such that $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$ and $0 \leq \lambda_i < 1$ for all $i = 1, \dots, r$. Assume that μ_1, \dots, μ_r are positive real numbers such that $\sum_{i=1}^r \mu_i = 1$. Then the following statements hold:*

- (1) [11, Proposition 3.1] $\sum_{i=1}^r \mu_i T_i : K \rightarrow K$ is a λ -strictly pseudocontractive mapping where $\lambda = \min\{\lambda_i : 1 \leq i \leq r\}$.
- (2) [11, Proposition 3.2] $F(\sum_{i=1}^r \mu_i T_i) = F$.

Recall that a Banach space E satisfies Opial's condition if

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$$

for all sequences $\{x_n\}$ in E with $x_n \rightharpoonup x$ and for all elements $y \neq x$.

Theorem 3.3. For a given countable family of λ_k -strictly pseudocontractive mappings $\{S_k\}_{k=1}^\infty$ of a nonempty closed convex subset K of a smooth and uniformly convex Banach space E with $\lambda := \inf\{\lambda_k : k \in \mathbb{N}\} > 0$ and $F := \bigcap_{k=1}^\infty F(S_k)$, let $\{x_n\}$ be a sequence defined by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) \sum_{k=1}^n \beta_n^k S_k x_n, \quad n \geq 1, \tag{3.1}$$

where $0 < \alpha_n \leq b < 1$ and $\{\beta_n^k\}$ is a family of nonnegative numbers with indices $n, k \in \mathbb{N}$ with $k \leq n$ such that

- (i) $\sum_{k=1}^n \beta_n^k = 1$ for all $n \in \mathbb{N}$;
- (ii) $\lim_{n \rightarrow \infty} \beta_n^k > 0$ for all $k \in \mathbb{N}$;
- (iii) $\sum_{n=1}^\infty \sum_{k=1}^n |\beta_{n+1}^k - \beta_n^k| < \infty$.

Then the following statements hold.

- (1) If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of the mappings $\{S_k\}_{k=1}^\infty$.
- (2) If K is compact, then $\{x_n\}$ converges strongly to a common fixed point of the mapping $\{S_k\}_{k=1}^\infty$.

Proof. The iteration (3.1) is well defined since each mapping $T_n := \sum_{k=1}^n \beta_n^k S_k$ is λ -strictly pseudocontractive. For convenience, we rewrite the iteration (3.1) as

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1. \tag{3.2}$$

Then (the proof is the same as that of [19])

- (a) $\sum_{n=1}^\infty \sup\{\|T_{n+1}x - T_n x\| : x \in B\} < \infty$ for all bounded subset B of K ;
- (b) $\sum_{k=1}^\infty \beta^k = 1$ and the mapping $T : K \rightarrow K$ defined by

$$Tx = \sum_{k=1}^\infty \beta^k S_k x$$

satisfies

$$\sup\{\|T_n x - Tx\| : x \in B\} \rightarrow 0$$

for all bounded subset B of K .

Moreover, it follows from Lemma 3.1 that T above is a λ -strictly pseudocontractive mapping and

$$F(T) = \bigcap_{n=1}^\infty F(T_n) = F.$$

Let $p \in F$. Since each T_n is a λ_n -strictly pseudocontractive mapping, there exists $j(x_n - p) \in J(x_n - p)$ such that

$$\langle (I - T_n)x_n - (I - T_n)p, j(x_n - p) \rangle \geq \lambda_n \|(I - T_n)x_n - (I - T_n)p\|^2.$$

Clearly,

$$x_n - x_{n-1} = \frac{\alpha_n - 1}{\alpha_n} (x_n - T_n x_n),$$

and so

$$\langle x_n - x_{n-1}, j(x_n - p) \rangle = \frac{\alpha_n - 1}{\alpha_n} \langle x_n - T_n x_n, j(x_n - p) \rangle.$$

It follows then that

$$\begin{aligned} \|x_n - p\|^2 &= \|(x_{n-1} - p) + (x_n - x_{n-1})\|^2 \\ &\leq \|x_{n-1} - p\|^2 + 2\langle x_n - x_{n-1}, j(x_n - p) \rangle \\ &= \|x_{n-1} - p\|^2 + 2\frac{\alpha_n - 1}{\alpha_n} \langle x_n - T_n x_n, j(x_n - p) \rangle \\ &= \|x_{n-1} - p\|^2 - 2\frac{1 - \alpha_n}{\alpha_n} \langle (x_n - T_n x_n) - (p - T_n p), j(x_n - p) \rangle \\ &\leq \|x_{n-1} - p\|^2 - 2\lambda_n \frac{1 - \alpha_n}{\alpha_n} \|x_n - T_n x_n\|^2. \end{aligned}$$

From $0 < \alpha_n \leq b < 1$ and $\lambda_n \geq \lambda$, we have

$$2\lambda \frac{1 - b}{b} \|x_n - T_n x_n\|^2 \leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2.$$

It follows from the existence of the limit $\lim_{n \rightarrow \infty} \|x_n - p\|$ that

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0.$$

Consequently, since $\{x_n\}$ is bounded,

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|Tx_n - T_n x_n\| + \|T_n x_n - x_n\| \\ &\leq \sup\{\|Tz - T_n z\| : z \in \{x_n\}\} + \|T_n x_n - x_n\| \rightarrow 0. \end{aligned}$$

(1) Since T is a λ -strictly pseudocontractive mapping,

$$\omega_w(x_n) \subset F(T) = \bigcap_{n=1}^{\infty} F(T_n) = F.$$

If E satisfies Opial's condition, we get that $\omega_w(x_n)$ is a singleton. Otherwise, suppose that $p, q \in \omega_w(x_n)$ and $p \neq q$. Then $p, q \in F$. By Theorem 2.2(i) for the iteration (3.2), both limits $\lim_{n \rightarrow \infty} \|x_n - p\|$ and $\lim_{n \rightarrow \infty} \|x_n - q\|$ exist. Let $\{x_{n_k}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_k} \rightarrow p$ and $x_{m_j} \rightarrow q$. By Opial's condition, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - p\| &= \lim_{k \rightarrow \infty} \|x_{n_k} - p\| < \lim_{k \rightarrow \infty} \|x_{n_k} - q\| = \lim_{n \rightarrow \infty} \|x_n - q\| \\ &= \lim_{j \rightarrow \infty} \|x_{m_j} - q\| < \lim_{j \rightarrow \infty} \|x_{m_j} - p\| = \lim_{n \rightarrow \infty} \|x_n - p\|, \end{aligned}$$

which is a contradiction. Hence $x_n \rightarrow p$ for some $p \in F$.

(2) Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p$ for some $p \in K$. By the continuity of T , we get that $p \in F(T) = F$. Using Theorem 2.2(i) and (3.2), we have $\lim_{n \rightarrow \infty} \|x_n - p\| = 0$. Hence $x_n \rightarrow p$, as desired. \square

The following example [19] shows that there exists $\{\beta_n^k\}$ satisfying the condition of Theorem 3.3.

Example 3.4. Let $\{\beta_n^k\}$ be defined by

$$\beta_n^k = \begin{cases} 2^{-k} & (k < n) \\ 2^{1-k} & (k = n), \end{cases}$$

for all $n, k \in \mathbb{N}$ with $k \leq n$. In this case, the sequence $\{T_n\}$ of mappings generated by $\{S_k\}$ is defined as follows: For $x \in C$,

$$\begin{aligned} T_1 x &= S_1 x, \\ T_2 x &= \frac{1}{2} S_1 x + \frac{1}{2} S_2 x, \\ T_3 x &= \frac{1}{2} S_1 x + \frac{1}{4} S_2 x + \frac{1}{4} S_3 x, \\ T_4 x &= \frac{1}{2} S_1 x + \frac{1}{4} S_2 x + \frac{1}{8} S_3 x + \frac{1}{8} S_4 x, \\ &\vdots \\ T_n x &= \frac{1}{2} S_1 x + \frac{1}{4} S_2 x + \frac{1}{8} S_3 x + \frac{1}{16} S_4 x + \cdots + \frac{1}{2^{n-1}} S_{n-1} x + \frac{1}{2^{n-1}} S_n x. \end{aligned}$$

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Research Article

Approximation of Common Fixed Points of a Countable Family of Relatively Nonexpansive Mappings

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We introduce two general iterative schemes for finding a common fixed point of a countable family of relatively nonexpansive mappings in a Banach space. Under suitable setting, we not only obtain several convergence theorems announced by many authors but also prove them under weaker assumptions. Applications to the problem of finding a common element of the fixed point set of a relatively nonexpansive mapping and the solution set of an equilibrium problem are also discussed.

1. Introduction and Preliminaries

Let C be a nonempty subset of a Banach space E , and let T be a mapping from C into itself. When $\{x_n\}$ is a sequence in E , we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightarrow x$ and weak convergence by $x_n \rightharpoonup x$. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in the dual E^* by $x_n^* \xrightarrow{*} x^*$. A point $p \in C$ is an asymptotic fixed point of T if there exists $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Tx_n \rightarrow 0$. We denote $F(T)$ and $\widehat{F}(T)$ by the set of fixed points and of asymptotic fixed points of T , respectively. A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $x, y \in S(E) = \{z \in E : \|z\| = 1\}$ and $x \neq y$. It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. The space E is said to be smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + tx\| - \|x\|}{t} \quad (1.1)$$

exists for all $x, y \in S(E)$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in S(E)$.

Many problems in nonlinear analysis can be formulated as a problem of finding a fixed point of a certain mapping or a common fixed point of a family of mappings. This paper deals with a class of nonlinear mappings, so-called relatively nonexpansive mappings introduced by Matsushita and Takahashi [1]. This type of mappings is closely related to the resolvent of maximal monotone operators (see [2–4]).

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed convex subset of E . Throughout this paper, we denote by ϕ the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \forall x, y \in E, \quad (1.2)$$

where J is the normalized duality mapping from E to the dual space E^* given by the following relation:

$$\langle x, Jx \rangle = \|x\|^2 = \|Jx\|^2. \quad (1.3)$$

We know that if E is smooth, strictly convex, and reflexive, then the duality mapping J is single-valued, one-to-one, and onto. The duality mapping J is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $Jx_n \xrightarrow{*} Jx$ (see [5] for more details).

Following Matsushita and Takahashi [6], a mapping $T : C \rightarrow E$ is said to be relatively nonexpansive if the following conditions are satisfied:

- (R1) $F(T)$ is nonempty;
- (R2) $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$, $x \in C$;
- (R3) $\hat{F}(T) = F(T)$.

If T satisfies (R1) and (R2), then T is called relatively quasi-nonexpansive [7]. Obviously, relative nonexpansiveness implies relative quasi-nonexpansiveness but the converse is not true. Relatively quasi-nonexpansive mappings are sometimes called hemirelatively nonexpansive mappings. But we do prefer the former name because in a Hilbert space setting, relatively quasi-nonexpansive mappings are nothing but quasi-nonexpansive.

In [2], Alber introduced the generalized projection Π_C from E onto C as follows:

$$\Pi_C(x) = \arg \min_{y \in C} \phi(y, x) \quad \forall x \in E. \quad (1.4)$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C becomes the metric projection of E onto C . Alber's generalized projection is an example of relatively nonexpansive mappings. For more example, see [1, 8].

In 2004, Masushita and Takahashi [1, 6] also proved weak and strong convergence theorems for finding a fixed point of a single relatively nonexpansive mapping. Several iterative methods, as a generalization of [1, 6], for finding a common fixed point of the family of relatively nonexpansive mappings have been further studied in [7, 9–14].

Recently, a problem of finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping is studied by Takahashi and Zembayashi in [15, 16]. The purpose of this paper is to introduce a new iterative scheme which unifies several ones studied by many authors and to deduce the corresponding convergence theorems under the weaker assumptions. More precisely, many restrictions as were the case in other papers are dropped away.

First, we start with some preliminaries which will be used throughout the paper.

Lemma 1.1 (see [7, Lemma 2.5]). *Let C be a nonempty closed convex subset of a strictly convex and smooth Banach space E and let T be a relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 1.2 (see [17, Proposition 5]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E . Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad (1.5)$$

for all $x \in C$ and $y \in E$.

Lemma 1.3 (see [17]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $h : [0, 2r] \rightarrow \mathbb{R}$ such that $h(0) = 0$ and*

$$h(\|x - y\|) \leq \phi(x, y) \quad (1.6)$$

for all $x, y \in B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 1.4 (see [17, Proposition 2]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 1.5 (see [2]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , let $x \in E$, and let $z \in C$. Then*

$$z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C. \quad (1.7)$$

Lemma 1.6 (see [18]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \rightarrow \mathbb{R}$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|) \quad (1.8)$$

for all $x, y \in B_r$ and $t \in [0, 1]$.

We next prove the following three lemmas which are very useful for our main results.

Lemma 1.7. Let C be a closed convex subset of a smooth Banach space E . Let T be a relatively quasi-nonexpansive mapping from E into E and let $\{S_i\}_{i=1}^N$ be a family of relatively quasi-nonexpansive mappings from C into itself such that $F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. The mapping $U : C \rightarrow E$ is defined by

$$Ux = TJ^{-1} \sum_{i=1}^N \omega_i (\alpha_i Jx + (1 - \alpha_i) JS_i x) \quad (1.9)$$

for all $x \in C$ and $\{\omega_i\}, \{\alpha_i\} \subset [0, 1]$, $i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \omega_i = 1$. If $x \in C$ and $z \in F(T) \cap \bigcap_{i=1}^N F(S_i)$, then

$$\phi(z, Ux) \leq \phi(z, x). \quad (1.10)$$

Proof. The proof of this lemma can be extracted from that of Lemma 1.8; so it is omitted. \square

If E has a stronger assumption, we have the following lemma.

Lemma 1.8. Let C be a closed convex subset of a uniformly smooth Banach space E . Let $r > 0$. Then, there exists a strictly increasing, continuous, and convex function $g^* : [0, 6r] \rightarrow \mathbb{R}$ such that $g^*(0) = 0$ and for each relatively quasi-nonexpansive mapping $T : E \rightarrow E$ and each finite family of relatively quasi-nonexpansive mappings $\{S_i\}_{i=1}^N : C \rightarrow C$ such that $F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$,

$$\sum_{i=1}^N \omega_i \alpha_i (1 - \alpha_i) g^*(\|Jz - JS_i z\|) \leq \phi(u, z) - \phi(u, Uz) \quad (1.11)$$

for all $z \in C \cap B_r$ and $u \in F(T) \cap \bigcap_{i=1}^N F(S_i) \cap B_r$, where

$$Ux = TJ^{-1} \sum_{i=1}^N \omega_i (\alpha_i Jx + (1 - \alpha_i) JS_i x) \quad (1.12)$$

$x \in C$ and $\{\omega_i\}, \{\alpha_i\} \subset [0, 1]$, $i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \omega_i = 1$.

Proof. Let $r > 0$. From Lemma 1.6 and E^* is uniformly convex, then there exists a strictly increasing, continuous, and convex function $g^* : [0, 6r] \rightarrow \mathbb{R}$ such that $g^*(0) = 0$ and

$$\|tx^* + (1 - t)y^*\|^2 \leq t\|x^*\|^2 + (1 - t)\|y^*\|^2 - t(1 - t)g^*(\|x^* - y^*\|) \quad (1.13)$$

for all $x^*, y^* \in \{z^* \in E^* : \|z^*\| \leq 3r\}$ and $t \in [0, 1]$. Let $T : E \rightarrow E$ and $\{S_i\}_{i=1}^N : C \rightarrow C$ be relatively quasi-nonexpansive for all $i = 1, 2, \dots, N$ such that $F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. For $z \in C \cap B_r$ and $u \in F(T) \cap \bigcap_{i=1}^N F(S_i) \cap B_r$. It follows that

$$(\|u\| - \|S_i z\|)^2 \leq \phi(u, S_i z) \leq \phi(u, z) \leq (\|u\| + \|z\|)^2 \leq (2r)^2 \quad (1.14)$$

and hence $\|S_i z\| \leq 3r$. Consequently, for $i = 1, 2, \dots, N$,

$$\|\alpha_i Jz + (1 - \alpha_i)JS_i z\|^2 \leq \alpha_i \|Jz\|^2 + (1 - \alpha_i)\|JS_i z\|^2 - \alpha_i(1 - \alpha_i)g^*(\|Jz - JS_i z\|). \quad (1.15)$$

Then

$$\begin{aligned} \phi(u, Uz) &\leq \phi\left(u, J^{-1} \sum_{i=1}^N \omega_i (\alpha_i Jz + (1 - \alpha_i)JS_i z)\right) \\ &= \|u\|^2 - 2 \left\langle u, \sum_{i=1}^N \omega_i (\alpha_i Jz + (1 - \alpha_i)JS_i z) \right\rangle + \left\| \sum_{i=1}^N \omega_i (\alpha_i Jz + (1 - \alpha_i)JS_i z) \right\|^2 \\ &\leq \sum_{i=1}^N \omega_i \left(\|u\|^2 - 2 \langle u, \alpha_i Jz + (1 - \alpha_i)JS_i z \rangle + \|\alpha_i Jz + (1 - \alpha_i)JS_i z\|^2 \right) \\ &\leq \sum_{i=1}^N \omega_i \left(\|u\|^2 - 2 \langle u, \alpha_i Jz + (1 - \alpha_i)JS_i z \rangle + \alpha_i \|Jz\|^2 + (1 - \alpha_i)\|JS_i z\|^2 \right. \\ &\quad \left. - \alpha_i(1 - \alpha_i)g^*(\|Jz - JS_i z\|) \right) \\ &= \sum_{i=1}^N \omega_i (\alpha_i \phi(u, z) + (1 - \alpha_i)\phi(u, S_i z) - \alpha_i(1 - \alpha_i)g^*(\|Jz - JS_i z\|)) \\ &\leq \phi(u, z) - \sum_{i=1}^N \omega_i \alpha_i (1 - \alpha_i)g^*(\|Jz - JS_i z\|). \end{aligned} \quad (1.16)$$

Thus

$$\sum_{i=1}^N \omega_i \alpha_i (1 - \alpha_i)g^*(\|Jz - JS_i z\|) \leq \phi(u, z) - \phi(u, Uz). \quad (1.17)$$

□

Lemma 1.9. *Let C be a closed convex subset of a uniformly smooth and strictly convex Banach space E . Let T be a relatively quasi-nonexpansive mapping from E into E and let $\{S_i\}_{i=1}^N$ be a family of relatively quasi-nonexpansive mappings from C into itself such that $F(T) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. The mapping $U : C \rightarrow E$ is defined by*

$$Ux = TJ^{-1} \sum_{i=1}^N \omega_i (\alpha_i Jx + (1 - \alpha_i)JS_i x) \quad (1.18)$$

for all $x \in C$ and $\{\omega_i\}, \{\alpha_i\} \subset (0, 1)$, $i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \omega_i = 1$. Then, the following hold:

- (1) $F(U) = F(T) \cap \bigcap_{i=1}^N F(S_i)$,
- (2) U is relatively quasi-nonexpansive.

Proof. (1) Clearly, $F(T) \cap \bigcap_{i=1}^N F(S_i) \subset F(U)$. We want to show the reverse inclusion. Let $z \in F(U)$ and $u \in F(T) \cap \bigcap_{i=1}^N F(S_i)$. Choose

$$r := \max\{\|u\|, \|z\|, \|S_1 z\|, \|S_2 z\|, \dots, \|S_m z\|\}. \quad (1.19)$$

From Lemma 1.8, we have

$$\sum_{i=1}^N \omega_i \alpha_i (1 - \alpha_i) g^*(\|Jz - JS_i z\|) = 0. \quad (1.20)$$

From $\omega_i \alpha_i (1 - \alpha_i) > 0$ for all $i = 1, 2, \dots, N$ and by the properties of g^* , we have

$$Jz = JS_i z \quad (1.21)$$

for all $i = 1, 2, \dots, N$. From J is one to one, we have

$$z = S_i z \quad (1.22)$$

for all $i = 1, 2, \dots, N$. Consider

$$z = Uz = TJ^{-1} \sum_{i=1}^N \omega_i (\alpha_i Jz + (1 - \alpha_i) JS_i z) = Tz. \quad (1.23)$$

Thus $z \in F(T) \cap \bigcap_{i=1}^N F(S_i)$.

(2) It follows directly from the above discussion. \square

2. Weak Convergence Theorem

Theorem 2.1. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $\{T_n\}_{n=1}^{\infty} : E \rightarrow C$ be a family of relatively quasi-nonexpansive mappings and let $\{S_i\}_{i=1}^N : C \rightarrow C$ be a family of relatively quasi-nonexpansive mappings such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$,*

$$x_{n+1} = T_n J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JS_i x_n) \quad (2.1)$$

for any $n \in \mathbb{N}$, $\{\omega_{n,i}\}, \{\alpha_{n,i}\} \subset [0, 1]$ for all $n \in \mathbb{N}$, $i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \omega_{n,i} = 1$ for all $n \in \mathbb{N}$. Then $\{\Pi_F x_n\}$ converges strongly to $z \in F$, where Π_F is the generalized projection of C onto F .

Proof. Let $u \in \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^N F(S_i)$. Put

$$U_n = T_n J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J + (1 - \alpha_{n,i}) J S_i). \quad (2.2)$$

From Lemma 1.7, we have

$$\phi(u, x_{n+1}) = \phi(u, U_n x_n) \leq \phi(u, x_n). \quad (2.3)$$

Therefore $\lim_{n \rightarrow \infty} \phi(u, x_n)$ exists. This implies that $\{\phi(u, x_n)\}$, $\{x_n\}$ and $\{S_i x_n\}$ are bounded for all $i = 1, 2, \dots, N$.

Let $y_n \equiv \Pi_F x_n$. From (2.3) and $m \in \mathbb{N}$, we have

$$\phi(y_n, x_{n+m}) \leq \phi(y_n, x_n). \quad (2.4)$$

Consequently,

$$\phi(y_n, y_{n+m}) + \phi(y_{n+m}, x_{n+m}) \leq \phi(y_n, x_{n+m}) \leq \phi(y_n, x_n). \quad (2.5)$$

In particular,

$$\phi(y_{n+1}, x_{n+1}) \leq \phi(y_n, x_n). \quad (2.6)$$

This implies that $\lim_{n \rightarrow \infty} \phi(y_n, x_n)$ exists. This together with the boundedness of $\{x_n\}$ gives $r := \sup_{n \in \mathbb{N}} \|y_n\| < \infty$. Using Lemma 1.3, there exists a strictly increasing, continuous, and convex function $h : [0, 2r] \rightarrow \mathbb{R}$ such that $h(0) = 0$ and

$$h(\|y_n - y_{n+m}\|) \leq \phi(y_n, y_{n+m}) \leq \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m}). \quad (2.7)$$

Since $\{\phi(y_n, x_n)\}$ is a convergent sequence, it follows from the properties of g that $\{y_n\}$ is a Cauchy sequence. Since F is closed, there exists $z \in F$ such that $y_n \rightarrow z$. \square

We first establish weak convergence theorem for finding a common fixed point of a countable family of relatively quasi-nonexpansive mappings. Recall that, for a family of mappings $\{T_n\}_{n=1}^{\infty} : C \rightarrow E$ with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, we say that $\{T_n\}$ satisfies the NST-condition [19] if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0 \text{ implies } \omega_w \{z_n\} \subset \bigcap_{n=1}^{\infty} F(T_n), \quad (2.8)$$

where $\omega_w \{z_n\}$ denotes the set of all weak subsequential limits of a sequence $\{z_n\}$.

Theorem 2.2. Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $\{T_n\}_{n=1}^{\infty} : E \rightarrow C$ be a family of relatively quasi-nonexpansive mappings satisfying NST-condition and let $\{S_i\}_{i=1}^N : C \rightarrow C$ be a family of relatively nonexpansive mappings such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$ and suppose that

$$\phi(u, T_n x) + \phi(T_n x, x) \leq \phi(u, x) \quad (2.9)$$

for all $u \in \bigcap_{n=1}^{\infty} F(T_n)$, $n \in \mathbb{N}$ and $x \in E$. Let the sequence $\{x_n\}$ be generated by $x_1 \in C$,

$$x_{n+1} = T_n J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J S_i x_n) \quad (2.10)$$

for any $n \in \mathbb{N}$, $\{\omega_{n,i}\}, \{\alpha_{n,i}\} \subset [0, 1]$ for all $n \in \mathbb{N}$, $i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \omega_{n,i} = 1$ for all $n \in \mathbb{N}$, $\liminf_{n \rightarrow \infty} \omega_{n,i} \alpha_{n,i} (1 - \alpha_{n,i}) > 0$ for all $i = 1, 2, \dots, N$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in F$, where $z = \lim_{n \rightarrow \infty} \Pi_F x_n$.

Proof. Let $u \in F$. From Theorem 2.1, $\lim_{n \rightarrow \infty} \phi(u, x_n)$ exists and hence $\{x_n\}$ and $\{S_i x_n\}$ are bounded for all $i = 1, 2, \dots, N$. Let

$$r = \sup_{n \in \mathbb{N}} \{\|x_n\|, \|S_1 x_n\|, \|S_2 x_n\|, \dots, \|S_N x_n\|\}. \quad (2.11)$$

By Lemma 1.8, there exists a strictly increasing, continuous, and convex function $g^* : [0, 2r] \rightarrow \mathbb{R}$ such that $g^*(0) = 0$ and

$$\sum_{i=1}^N \omega_{n,i} \alpha_{n,i} (1 - \alpha_{n,i}) g^*(\|J x_n - J S_i x_n\|) \leq \phi(u, x_n) - \phi(u, x_{n+1}). \quad (2.12)$$

In particular, for all $i = 1, 2, \dots, N$,

$$\omega_{n,i} \alpha_{n,i} (1 - \alpha_{n,i}) g^*(\|J x_n - J S_i x_n\|) \leq \phi(u, x_n) - \phi(u, x_{n+1}). \quad (2.13)$$

Hence,

$$\sum_{n=1}^{\infty} \omega_{n,i} \alpha_{n,i} (1 - \alpha_{n,i}) g^*(\|J x_n - J S_i x_n\|) < \infty \quad (2.14)$$

for all $i = 1, 2, \dots, N$. Since $\liminf_{n \rightarrow \infty} \omega_{n,i} \alpha_{n,i} (1 - \alpha_{n,i}) > 0$ for all $i = 1, 2, \dots, N$ and the properties of g , we have

$$\lim_{n \rightarrow \infty} \|J x_n - J S_i x_n\| = 0 \quad (2.15)$$

for all $i = 1, 2, \dots, N$. Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = 0 \quad (2.16)$$

for all $i = 1, 2, \dots, N$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow z \in C$. Since S_i is relatively nonexpansive, $z \in \widehat{F}(S_i) = F(S_i)$ for all $i = 1, 2, \dots, N$.

We show that $z \in \bigcap_{n=1}^{\infty} F(T_n)$. Let

$$y_n = J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J S_i x_n). \quad (2.17)$$

We note from (2.15) that

$$\left\| \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J S_i x_n) - J x_n \right\| \leq \sum_{i=1}^N \omega_{n,i} (1 - \alpha_{n,i}) \|J S_i x_n - J x_n\| \rightarrow 0. \quad (2.18)$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, it follows that

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = \lim_{n \rightarrow \infty} \left\| J^{-1} \left(\sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J S_i x_n) \right) - J^{-1} J x_n \right\| = 0. \quad (2.19)$$

Moreover, by (2.9) and the existence of $\lim_{n \rightarrow \infty} \phi(u, x_n)$, we have

$$\begin{aligned} \phi(T_n y_n, y_n) &\leq \phi(u, y_n) - \phi(u, T_n y_n) \\ &= \phi \left(u, J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J S_i x_n) \right) - \phi(u, x_{n+1}) \\ &\leq \phi(u, x_n) - \phi(u, x_{n+1}) \rightarrow 0. \end{aligned} \quad (2.20)$$

It follows from Lemma 1.4 that $\lim_{n \rightarrow \infty} \|T_n y_n - y_n\| = 0$. From (2.19) and $x_{n_k} \rightarrow z$, we have $y_{n_k} \rightarrow z$. Since $\{T_n\}$ satisfies NST-condition, we have $z \in \bigcap_{n=1}^{\infty} F(T_n)$. Hence $z \in F$.

Let $z_n = \Pi_F x_n$. From Lemma 1.5 and $z \in F$, we have

$$\langle z_{n_k} - z, J x_{n_k} - J z_{n_k} \rangle \geq 0. \quad (2.21)$$

From Theorem 2.1, we know that $z_n \rightarrow z' \in F$. Since J is weakly sequentially continuous, we have

$$\langle z' - z, J z - J z' \rangle \geq 0. \quad (2.22)$$

Moreover, since J is monotone,

$$\langle z' - z, Jz - Jz' \rangle \leq 0. \quad (2.23)$$

Then

$$\langle z' - z, Jz - Jz' \rangle = 0. \quad (2.24)$$

Since E is strictly convex, $z' = z$. This implies that $\omega_w\{x_n\} = \{z'\}$ and hence $x_n \rightarrow z' = \lim_{n \rightarrow \infty} \Pi_F x_n$. \square

We next apply our result for finding a common element of a fixed point set of a relatively nonexpansive mapping and the solution set of an equilibrium problem. This problem is extensively studied in [11, 14–16]. Let C be a subset of a Banach space E and let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction. The equilibrium problem for a bifunction f is to find $x \in C$ such that $f(x, y) \geq 0$ for all $y \in C$. The set of solutions above is denoted by $EP(f)$, that is

$$x \in EP(f) \quad \text{iff} \quad f(x, y) \geq 0 \quad \forall y \in C. \quad (2.25)$$

To solve the equilibrium problem, we usually assume that a bifunction f satisfies the following conditions (C is closed and convex):

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$;
- (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemma gives a characterization of a solution of an equilibrium problem.

Lemma 2.3. *Let C be a nonempty closed convex subset of a Banach space E . Let f be a bifunction from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). Suppose that $p \in C$. Then $p \in EP(f)$ if and only if $f(y, p) \leq 0$ for all $y \in C$.*

Proof. Let $p \in EP(f)$, then $f(p, y) \geq 0$ for all $y \in C$. From (A2), we get that $f(y, p) \leq -f(p, y) \leq 0$ for all $y \in C$.

Conversely, assume that $f(y, p) \leq 0$ for all $y \in C$. For any $y \in C$, let

$$x_t = ty + (1-t)p, \quad \text{for } t \in (0, 1]. \quad (2.26)$$

Then $f(x_t, p) \leq 0$ and hence

$$0 = f(x_t, x_t) \leq tf(x_t, y) + (1-t)f(x_t, p) \leq tf(x_t, y). \quad (2.27)$$

So $f(x_t, y) \geq 0$ for all $t \in (0, 1]$. From (A3), we have

$$0 \leq \limsup_{t \downarrow 0} f(ty + (1-t)p, y) \leq f(p, y) \quad \forall y \in C. \quad (2.28)$$

Hence $p \in EP(f)$. □

Takahashi and Zembayashi proved the following important result.

Lemma 2.4 (see [15, Lemma 2.8]). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E . Let f be a bifunction from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0 \quad \forall y \in C \right\} \quad (2.29)$$

for all $x \in E$. Then, the following hold:

- (1) T_r is single-valued;
- (2) T_r is a firmly nonexpansive-type mapping [20], that is, for all $x, y \in E$

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle; \quad (2.30)$$

- (3) $F(T_r) = EP(f)$;
- (4) $EP(f)$ is closed and convex.

We now deduce Takahashi and Zembayashi's recent result from Theorem 2.2.

Corollary 2.5 (see [15, Theorem 4.1]). *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)–(A4) and let S be a relatively nonexpansive mapping from C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by $u_1 \in E$,*

$$\begin{aligned} x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle &\geq 0 \quad \forall y \in C, \\ u_{n+1} &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n) \end{aligned} \quad (2.31)$$

for every $n \in \mathbb{N}$, $\{\alpha_n\} \subset [0, 1]$ satisfying $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in \Pi_{F(S) \cap EP(f)}$, where $z = \lim_{n \rightarrow \infty} \Pi_{F(S) \cap EP(f)} x_n$.

Proof. Put $T_n \equiv T_{r_n}$ where T_{r_n} is defined by Lemma 2.4. Then $\bigcap_{n=1}^{\infty} F(T_n) = EP(f)$. By reindexing the sequences $\{x_n\}$ and $\{u_n\}$ of this iteration, we can apply Theorem 2.2 by showing that the family $\{T_n\}$ satisfies the condition (2.9) and NST-condition. It is proved in [15, Lemma 2.9] that

$$\phi(u, T_n x) + \phi(T_n x, x) \leq \phi(u, x) \quad \forall x \in E, u \in \bigcap_{n=1}^{\infty} F(T_n). \quad (2.32)$$

To see that $\{T_n\}$ satisfies NST-condition, let $\{z_n\}$ be a bounded sequence in C such that $\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$ and $p \in \omega_w\{z_n\}$. Suppose that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup p$. Then $T_{n_k} z_{n_k} \rightharpoonup p \in C$. Since J is uniformly continuous on bounded sets and $r_{n_k} \geq a$, we have

$$\lim_{k \rightarrow \infty} \frac{1}{r_{n_k}} \|Jz_{n_k} - JT_{n_k} z_{n_k}\| = 0. \quad (2.33)$$

From the definition of $T_{r_{n_k}}$, we have

$$f(T_{n_k} z_{n_k}, y) + \frac{1}{r_{n_k}} \langle y - T_{n_k} z_{n_k}, JT_{n_k} z_{n_k} - Jz_{n_k} \rangle \geq 0 \quad \forall y \in C. \quad (2.34)$$

Since

$$\begin{aligned} f(y, T_{n_k} z_{n_k}) &\leq -f(T_{n_k} z_{n_k}, y) \\ &\leq \frac{1}{r_{n_k}} \langle y - T_{n_k} z_{n_k}, JT_{n_k} z_{n_k} - Jz_{n_k} \rangle \\ &\leq \frac{1}{r_{n_k}} \|y - T_{n_k} z_{n_k}\| \|JT_{n_k} z_{n_k} - Jz_{n_k}\| \end{aligned} \quad (2.35)$$

and f is lower semicontinuous and convex in the second variable, we have

$$f(y, p) \leq \liminf_{k \rightarrow \infty} f(y, T_{n_k} z_{n_k}) \leq 0. \quad (2.36)$$

Thus $f(y, p) \leq 0$ for all $y \in C$. From Lemma 2.3, we have $p \in EP(f)$. Then $\{T_n\}$ satisfies the NST-condition. From Theorem 2.2 where $N = 1$, $\{x_n\}$ converges weakly to $z \in F(T_n) \cap F(S) = EP(f) \cap F(S)$, where $z = \lim_{n \rightarrow \infty} \Pi_{EP(f) \cap F(S)} x_n$. \square

Using the same proof as above, we have the following result.

Corollary 2.6 (see [11, Theorem 3.5]). *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E . Let f be a bifunction from $C \times C$ to \mathbb{R} satisfies (A1)–(A4) and let $T, S : C \rightarrow C$ be two relatively nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the following manner:*

$$\begin{aligned}
 x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle &\geq 0 \quad \forall y \in C, \\
 u_{n+1} &= J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n JS x_n) \quad \forall n \geq 1.
 \end{aligned}
 \tag{2.37}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following restrictions:

- (a) $\alpha_n + \beta_n + \gamma_n = 1$;
- (b) $\liminf_{n \rightarrow \infty} \alpha_n \beta_n > 0$, $\liminf_{n \rightarrow \infty} \alpha_n \gamma_n > 0$;
- (c) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in F$, where $z = \lim_{n \rightarrow \infty} \Pi_F x_n$.

The following result also follows from Theorem 2.2.

Corollary 2.7 (see [9, Theorem 5.3]). *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E . Let $\{S_i\}_{i=1}^N$ be a finite family of relatively nonexpansive mappings from C into itself such that $F = \bigcap_{i=1}^N F(S_i)$ is a nonempty and let $\{\alpha_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq N\} \subset [0, 1]$ and $\{\omega_{n,i} : n, i \in \mathbb{N}, 1 \leq i \leq N\} \subset [0, 1]$ be sequences such that $\liminf_{n \rightarrow \infty} \alpha_{n,i} (1 - \alpha_{n,i}) > 0$ and $\liminf_{n \rightarrow \infty} \omega_{n,i} > 0$ for all $i \in \{1, 2, \dots, N\}$ and $\sum_{i=1}^N \omega_{n,i} = 1$ for all $n \in \mathbb{N}$. Let U_n be a sequence of mappings defined by*

$$U_n x = \Pi_C J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} Jx + (1 - \alpha_{n,i}) JS_i x)
 \tag{2.38}$$

for all $x \in C$ and let the sequence $\{x_n\}$ be generated by $x_1 = x \in C$ and

$$x_{n+1} = U_n x_n \quad (n = 1, 2, \dots).
 \tag{2.39}$$

Then the following hold:

- (1) the sequence $\{x_n\}$ is bounded and each weak subsequential limit of $\{x_n\}$ belongs to $\bigcap_{i=1}^N F(S_i)$;
- (2) if the duality mapping J from E into E^* is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the strong limit of $\{\Pi_F x_n\}$.

Proof. Since Π_C is relatively nonexpansive, the family $\{\Pi_C\}$ satisfies the NST-condition. Moreover, $F(\Pi_C) = C$ and

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \forall y \in E, x \in C.
 \tag{2.40}$$

Thus the conclusions of this corollary follow. □

3. Strong Convergence Theorem

In this section, we prove strong convergence of an iterative sequence generated by the hybrid method in mathematical programming. We start with the following useful common tools.

Lemma 3.1. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $\{T_n\}_{n=1}^{\infty} : E \rightarrow E$ and $\{S_i\}_{i=1}^N : C \rightarrow C$ be families of relatively quasi-nonexpansive mappings such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$, and*

$$\phi(u, T_n x) + \phi(T_n x, x) \leq \phi(u, x) \quad (3.1)$$

for all $u \in \bigcap_{n=1}^{\infty} F(T_n)$, $n \in \mathbb{N}$ and $x \in E$. Let $\{x_n\} \subset C$ be such that $\{x_n\}$ and $\{S_i x_n\}$ are bounded for all $i = 1, 2, \dots, N$, and

$$\begin{aligned} y_n &= J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J S_i x_n), \\ u_n &= T_n y_n, \end{aligned} \quad (3.2)$$

where $\{\omega_{n,i}\}, \{\alpha_{n,i}\} \subset [0, 1]$ for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$ satisfy $\sum_{i=1}^N \omega_{n,i} = 1$ for all $n \in \mathbb{N}$, $\liminf_{n \rightarrow \infty} \omega_{n,i} (1 - \alpha_{n,i}) > 0$ for all $i = 1, 2, \dots, N$ and $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$. Then the following statements hold:

- (1) $\lim_{n \rightarrow \infty} (\phi(u, x_n) - \phi(u, u_n)) = 0$ for all $u \in C$,
- (2) $\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0$,
- (3) $\omega_w \{x_n\} = \omega_w \{y_n\}$,
- (4) if $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, then $\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = 0$ for all $i = 1, 2, \dots, N$,
- (5) if $x_n \rightarrow z$, then $u_n \rightarrow z$ and $y_n \rightarrow z$.

Proof. (1) Since $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$ and J is uniformly norm-to-norm continuous on bounded sets,

$$\lim_{n \rightarrow \infty} \|J x_n - J u_n\| = 0. \quad (3.3)$$

We note here that $\{u_n\}$ is also bounded. For any $u \in C$, we have

$$\begin{aligned} |\phi(u, x_n) - \phi(u, u_n)| &= \left| \|x_n\|^2 - \|u_n\|^2 - 2\langle u, J u_n - J x_n \rangle \right| \\ &\leq \left| \|x_n\|^2 - \|u\|^2 \right| + 2|\langle u, J u_n - J x_n \rangle| \\ &\leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + 2\|u\| \|J u_n - J x_n\| \rightarrow 0. \end{aligned} \quad (3.4)$$

(2) Let $u \in F$. Using (3.1) and the relative quasi-nonexpansiveness of each T_n , we have

$$\phi(u_n, y_n) = \phi(T_n y_n, y_n) \leq \phi(u, y_n) - \phi(u, T_n y_n) \leq \phi(u, x_n) - \phi(u, u_n) \rightarrow 0. \quad (3.5)$$

By Lemma 1.4 and the boundedness of $\{u_n\}$, we have

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \tag{3.6}$$

(3) Since

$$\|x_n - y_n\| \leq \|x_n - u_n\| + \|u_n - y_n\| = \|x_n - u_n\| + \|T_n y_n - y_n\| \rightarrow 0, \tag{3.7}$$

we have $\omega_w\{x_n\} = \omega_w\{y_n\}$.

(4) Assume that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. From $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$, we get that $\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - Jx_n\| = \lim_{n \rightarrow \infty} \|Jx_{n+1} - Jy_n\| = 0. \tag{3.8}$$

So,

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \left\| Jx_{n+1} - \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JS_i x_n) \right\| \\ &\geq \sum_{i=1}^N (\omega_{n,i} (1 - \alpha_{n,i}) \|Jx_{n+1} - JS_i x_n\| - \omega_{n,i} \alpha_{n,i} \|Jx_{n+1} - Jx_n\|). \end{aligned} \tag{3.9}$$

From (3.8), we have

$$\sum_{i=1}^N \omega_{n,i} (1 - \alpha_{n,i}) \|Jx_{n+1} - JS_i x_n\| \leq \|Jx_{n+1} - Jy_n\| + \sum_{i=1}^N \omega_{n,i} \alpha_{n,i} \|Jx_{n+1} - Jx_n\| \rightarrow 0. \tag{3.10}$$

It follows from $\liminf_{n \rightarrow \infty} \omega_{n,i} (1 - \alpha_{n,i}) > 0$ for all $i = 1, 2, \dots, N$ that

$$\lim_{n \rightarrow \infty} \|Jx_{n+1} - JS_i x_n\| = 0 \tag{3.11}$$

for all $i = 1, 2, \dots, N$. Since J^{-1} is uniformly norm-to-norm continuous on bounded sets and $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, we have

$$\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = 0 \tag{3.12}$$

for all $i = 1, 2, \dots, N$, as desired.

(5) Assume that $x_n \rightarrow z$. From the assumption and (2), we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = \lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \tag{3.13}$$

Hence $u_n \rightarrow z$ and $y_n \rightarrow z$. □

Lemma 3.2 (see [21, Lemma 2.4]). *Let F be a closed convex subset of a strictly convex, smooth and reflexive Banach space E satisfying Kadec-Klee property. Let $x \in E$ and $\{x_n\}$ be a sequence in E such that $\omega_w\{x_n\} \subset F$ and $\phi(x_n, x) \leq \phi(\Pi_F x, x)$ for all $n \in \mathbb{N}$. Then $x_n \rightarrow z = \Pi_F x$.*

Recall that a Banach space E satisfies Kadec-Klee property if whenever $\{u_n\}$ is a sequence in E with $x_n \rightarrow x$ and $\|x_n\| \rightarrow \|x\|$, it follows that $x_n \rightarrow x$.

3.1. The CQ-Method

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $\{T_n\}_{i=1}^\infty : E \rightarrow E$ be a family of relatively quasi-nonexpansive mappings satisfying NST-condition and let $\{S_i\}_{i=1}^N : C \rightarrow C$ be a family of relatively nonexpansive mappings such that $F := \bigcap_{n=1}^\infty F(T_n) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$, and*

$$\phi(u, T_n x) + \phi(T_n x, x) \leq \phi(u, x) \quad (3.14)$$

for all $u \in \bigcap_{n=1}^\infty F(T_n)$, $n \in \mathbb{N}$ and $x \in E$. Let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} x_1 &= x \in C, \\ u_n &= T_n J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JS_i x_n), \\ C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x \end{aligned} \quad (3.15)$$

for every $n \in \mathbb{N}$, $\{\omega_{n,i}\}, \{\alpha_{n,i}\} \subset [0, 1]$ for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$ satisfying $\sum_{i=1}^N \omega_{n,i} = 1$ for all $n \in \mathbb{N}$, $\liminf_{n \rightarrow \infty} \omega_{n,i} (1 - \alpha_{n,i}) > 0$ for all $i = 1, 2, \dots, N$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.

Proof. The proof is broken into 4 steps.

Step 1 ($\{x_n\}$ is well defined). First, we show that $C_n \cap Q_n$ is closed and convex. Clearly, Q_n is closed and convex. Since

$$\phi(z, u_n) \leq \phi(z, x_n) \iff \|u_n\|^2 - \|x_n\|^2 - 2\langle z, Ju_n - Jx_n \rangle \leq 0, \quad (3.16)$$

then C_n is closed and convex. Thus $C_n \cap Q_n$ is closed and convex.

We next show that $F \subset C_n \cap Q_n$. Let $u \in F$. Then, from Lemma 1.7,

$$\phi(u, u_n) \leq \phi(u, x_n). \quad (3.17)$$

Thus $u \in C_n$. Hence $F \subset C_n$ for all $n \in \mathbb{N}$.

Next, we show by induction that $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. Since $Q_1 = C$, we have

$$F \subset C_1 \cap Q_1. \quad (3.18)$$

Suppose that $F \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = \Pi_{C_k \cap Q_k} x \in C_k \cap Q_k$ and the definition of the generalized projection, we have

$$\langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0 \quad (3.19)$$

for all $z \in C_k \cap Q_k$. From $F \subset C_k \cap Q_k$,

$$\langle x_{k+1} - p, Jx - Jx_{k+1} \rangle \geq 0 \quad (3.20)$$

for all $p \in F$. Hence $F \subset Q_{k+1}$, and we also have $F \subset C_{k+1} \cap Q_{k+1}$. So, we have $\emptyset \neq F \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$ and hence the sequence $\{x_n\}$ is well defined.

Step 2 ($\omega_w\{x_n\} \subset \bigcap_{i=1}^N F(S_i)$). From the definition of Q_n , we have $x_n = \Pi_{Q_n} x$. Using Lemma 1.2, we get

$$\phi(x_n, x) = \phi(\Pi_{Q_n} x, x) \leq \phi(u, x) - \phi(u, \Pi_{Q_n} x) \leq \phi(u, x) \quad (3.21)$$

for all $u \in Q_n$. In particular, since $x_{n+1} \in Q_n$ and $\Pi_F x \in F \subset Q_n$,

$$\phi(x_n, x) \leq \phi(x_{n+1}, x), \quad (3.22)$$

$$\phi(x_n, x) \leq \phi(\Pi_F x, x) \quad (3.23)$$

for all $n \in \mathbb{N}$. This implies that $\lim_{n \rightarrow \infty} \phi(x_n, x)$ exists and $\{x_n\}$ is bounded. Moreover, from (3.21) and $x_{n+1} \in Q_n$,

$$\phi(x_n, x) \leq \phi(x_{n+1}, x) - \phi(x_{n+1}, x_n). \quad (3.24)$$

Hence

$$\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x) - \phi(x_n, x) \rightarrow 0. \quad (3.25)$$

It follows from $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n$ that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \rightarrow 0. \quad (3.26)$$

From (3.25), (3.26), and Lemma 1.4, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 = \lim_{n \rightarrow \infty} \|x_{n+1} - u_n\|. \quad (3.27)$$

So $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$. Using Lemma 3.1(4), we get that

$$\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = 0 \quad (3.28)$$

for all $i = 1, 2, \dots, N$. Since each S_i is relatively nonexpansive,

$$\omega_w \{x_n\} \subset \bigcap_{i=1}^N \widehat{F}(S_i) = \bigcap_{i=1}^N F(S_i). \quad (3.29)$$

Step 3 ($\omega_w \{x_n\} \subset \bigcap_{n=1}^{\infty} F(T_n)$). Let $y_n = J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JS_i x_n)$. From Lemma 3.1(2), we have

$$\lim_{n \rightarrow \infty} \|T_n y_n - y_n\| = 0, \quad (3.30)$$

and $\omega_w \{x_n\} = \omega_w \{y_n\}$. It follows from NST-condition that $\omega_w \{x_n\} = \omega_w \{y_n\} \subset \bigcap_{n=1}^{\infty} F(T_n)$.

Step 4 ($x_n \rightarrow \Pi_F x$). From Steps 2 and 3, we have $\omega_w \{x_n\} \subset F$. The conclusion follows by Lemma 3.2 and (3.23). \square

We apply Theorem 3.3 and the proof of Corollary 2.5 and then obtain the following result.

Corollary 3.4. *Let C, E, f, S be as in Corollary 2.5. Let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JSx_n), \\ u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0 \quad \forall y \in C, \\ C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x \end{aligned} \quad (3.31)$$

for every $n \in \mathbb{N}$, $\{\alpha_n\} \subset [0, 1]$ satisfying $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\{r_n\} \subset [a, \infty]$ for some $a > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap EP(f)} x$, where $\Pi_{F(S) \cap EP(f)}$ is the generalized projection of E onto $F(S) \cap EP(f)$.

Remark 3.5. Corollary 3.4 improves the restriction on $\{\alpha_n\}$ of [15, Theorem 3.1]. In fact, it is assumed in [15, Theorem 3.1] that $\liminf_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$.

3.2. The Monotone CQ-Method

Let C be a closed subset of a Banach space E . Recall that a mapping $T : C \rightarrow C$ is closed if for each $\{x_n\}$ in C , if $x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$. A family of mappings $\{T_n\} : C \rightarrow E$ with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ is said to satisfy the $(*)$ -condition if for each bounded sequence $\{z_n\}$ in C ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0, \quad z_n \rightarrow z \text{ imply } z \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.32}$$

Remark 3.6. (1) If $\{T_n\}$ satisfies NST-condition, then $\{T_n\}$ satisfies $(*)$ -condition.
 (2) If $T_n \equiv T$ and T is closed, then $\{T_n\}$ satisfies $(*)$ -condition.

Theorem 3.7. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E . Let $\{T_n\}_{n=1}^{\infty} : E \rightarrow E$ be a family of relatively quasi-nonexpansive mappings satisfying $(*)$ -condition and let $\{S_i\}_{i=1}^N : C \rightarrow C$ be a family of closed relatively quasi-nonexpansive mappings such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$, and*

$$\phi(u, T_n x) + \phi(T_n x, x) \leq \phi(u, x) \tag{3.33}$$

for all $u \in \bigcap_{n=1}^{\infty} F(T_n)$, $n \in \mathbb{N}$, and $x \in E$. Let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} x_0 &= x \in C, & Q_0 &= C, \\ u_n &= T_n J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JS_i x_n), \\ C_0 &= \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\}, \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x \end{aligned} \tag{3.34}$$

for every $n \in \mathbb{N}$, $\{\omega_{n,i}\}, \{\alpha_{n,i}\} \subset [0, 1]$ satisfying $\sum_{i=1}^N \omega_{n,i} = 1$ and $\liminf_{n \rightarrow \infty} \omega_{n,i} (1 - \alpha_{n,i}) > 0$ for all $i = 1, 2, \dots, N$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.

Proof.

Step 1 ($\{x_n\}$ is well defined). This step is almost the same as Step 1 of the proof of Theorem 3.3, so it is omitted.

Step 2 ($\{x_n\}$ is a Cauchy sequence in C). We can follow the proof of Theorem 3.3 and conclude that

$$\lim_{n \rightarrow \infty} \phi(x_n, x) \tag{3.35}$$

exists. Moreover, as $x_{n+m} \in Q_n$ for all n, m and $x_n = \Pi_{Q_n} x$,

$$\begin{aligned}\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{Q_n} x) \\ &\leq \phi(x_{n+m}, x) - \phi(\Pi_{Q_n} x, x) \\ &= \phi(x_{n+m}, x) - \phi(x_n, x).\end{aligned}\tag{3.36}$$

Since $\{x_n\}$ is bounded, it follows from Lemma 1.3 that there exists a strictly increasing, continuous, and convex function h such that $h(0) = 0$ and

$$h(\|x_{n+m} - x_n\|) \leq \phi(x_{n+m}, x) - \phi(x_n, x).\tag{3.37}$$

Since $\lim_{n \rightarrow \infty} \phi(x_n, x)$ exists, we have that $\{x_n\}$ is a Cauchy sequence. Therefore, $x_n \rightarrow z$ for some $z \in C$.

Step 3 ($z \in \bigcap_{i=1}^N F(S_i)$). Since $x_{n+1} = \Pi_{C_n \cap Q_n} x \in C_n$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \rightarrow \phi(z, z) = 0.\tag{3.38}$$

By Lemma 1.4 and the boundedness of $\{x_n\}$, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0.\tag{3.39}$$

So, we have $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$. Using Lemma 3.1(4), we get that

$$\lim_{n \rightarrow \infty} \|x_n - S_i x_n\| = 0\tag{3.40}$$

for all $i = 1, 2, \dots, N$. Since each S_i is closed, $z \in \bigcap_{i=1}^N F(S_i)$.

Step 4 ($z \in \bigcap_{n=1}^{\infty} F(T_n)$). Let $y_n = J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} Jx_n + (1 - \alpha_{n,i}) JS_i x_n)$. From Lemma 3.1(2), we have $\lim_{n \rightarrow \infty} \|y_n - T_n y_n\| = 0$ and $y_n \rightarrow z$. It follows from (*)-condition that $z \in \bigcap_{n=1}^{\infty} F(T_n)$.

Step 5 ($x_n \rightarrow \Pi_F x$). From Steps 3 and 4, we have $\omega_w \{x_n\} \subset F$. The conclusion follows by Lemma 3.2 and (3.23). \square

Letting $T_n = \text{identity}$ and $S_1 = S_2 = \dots = S_N$ yield the following result.

Corollary 3.8 (see [12, Theorem 3.1]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth real Banach space E . Let $T : C \rightarrow C$ be a closed relatively quasi-nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
 x_0 &\in C \text{ chosen arbitrarily,} \\
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\
 C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
 C_0 &= \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\
 Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
 Q_0 &= C, \\
 x_{n+1} &= \Pi_{C_n \cap Q_n} x_0.
 \end{aligned} \tag{3.41}$$

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.

Letting $T_n = \text{identity}$ and $N = 2$ yield the following result.

Corollary 3.9 (see [13, Theorem 3.1]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth real Banach space E . Let T, S be two closed relatively quasi-nonexpansive mappings from C into itself such that $F := F(T) \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
 x_0 &\in C \text{ chosen arbitrarily,} \\
 z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT x_n + \beta_n^{(3)} JS x_n), \\
 y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\
 C_0 &= \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\
 C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
 Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
 Q_0 &= C, \\
 x_{n+1} &= \Pi_{C_n \cap Q_n} x_0
 \end{aligned} \tag{3.42}$$

with the conditions: $\beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \in [0, 1]$ with $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ and

- (1) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (2) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (3) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.10. Using Theorem 3.7, we can show that the conclusion of Corollary 3.9 remains true under the more general restrictions on $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, and $\{\beta_n^{(3)}\}$:

- (1) $\alpha_n, \beta_n^{(1)} \in [0, 1]$ are arbitrary;
- (2) $\liminf_{n \rightarrow \infty} \beta_n^{(2)} > 0$ and $\liminf_{n \rightarrow \infty} \beta_n^{(3)} > 0$.

3.3. The Shrinking Projection Method

Theorem 3.11. Let C , E , $\{T_n\}_{n=1}^\infty$, $\{S_i\}_{i=1}^N$ be as in Theorem 3.7. Let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} x_0 &\in E \text{ chosen arbitrarily,} \\ C_1 &= C, \\ x_1 &= \Pi_{C_1} x_0, \\ u_n &= T_n J^{-1} \sum_{i=1}^N \omega_{n,i} (\alpha_{n,i} J x_n + (1 - \alpha_{n,i}) J S_i x_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0 \end{aligned} \tag{3.43}$$

for every $n \in \mathbb{N}$, $\{\omega_{n,i}\}$, $\{\alpha_{n,i}\} \subset [0, 1]$ for all $n \in \mathbb{N}$ and $i = 1, 2, \dots, N$ satisfies $\sum_{i=1}^N \omega_{n,i} = 1$ for all $n \in \mathbb{N}$, $\liminf_{n \rightarrow \infty} \omega_{n,i} (1 - \alpha_{n,i}) > 0$ for all $i = 1, 2, \dots, N$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.

Proof. The proof is almost the same as the proofs of Theorems 3.3 and 3.7; so it is omitted. \square

In particular, applying Theorem 3.11 gives the following result.

Corollary 3.12. Let C , E , f , S be as in Corollary 2.5. Let the sequence $\{x_n\}$ be generated by $x_0 = x \in C$, $C_0 = C$ and

$$\begin{aligned} y_n &= J^{-1} (\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0 \quad \forall y \in C, \\ C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0 \end{aligned} \tag{3.44}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E . Assume that $\{\alpha_n\} \subset [0, 1]$ satisfies $\limsup_{n \rightarrow \infty} \alpha_n < 1$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap EP(f)} x$, where $\Pi_{F(S) \cap EP(f)}$ is the generalized projection of E onto $F(S) \cap EP(f)$.

Remark 3.13. Corollary 3.12 improves the restriction on $\{\alpha_n\}$ of [16, Theorem 3.1]. In fact, it is assumed in [16, Theorem 3.1] that $\liminf_{n \rightarrow \infty} \alpha_n (1 - \alpha_n) > 0$.

Corollary 3.14 (see [11, Theorem 3.1]). *Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E . Let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)–(A4) and let $T, S : C \rightarrow C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP(f) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by the following manner:*

$$\begin{aligned}
 x_0 &\in E \text{ chosen arbitrarily,} \\
 C_1 &= C, \\
 x_1 &= \Pi_{C_1} x_0, \\
 y_n &= J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n JS x_n), \\
 u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0 \quad \forall y \in C, \\
 C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
 x_{n+1} &= \Pi_{C_{n+1}} x_0.
 \end{aligned} \tag{3.45}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the restrictions:

- (a) $\alpha_n + \beta_n + \gamma_n = 1$;
- (b) $\liminf_{n \rightarrow \infty} \alpha_n \beta_n > 0$, $\liminf_{n \rightarrow \infty} \alpha_n \gamma_n > 0$;
- (c) $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.15. The conclusion of Corollary 3.14 remains true under the more general assumption; that is, we can replace (b) by the following one:

- (b') $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n > 0$.

We also deduce the following result.

Corollary 3.16 (see [14, Theorem 3.1]). *Let C , E , f , T , S be as in Corollary 3.14. Let the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be generated by the following:*

$$\begin{aligned}
 x_0 &\in E \text{ chosen arbitrarily,} \\
 C_1 &= C, \\
 x_1 &= \Pi_{C_1} x_0, \\
 y_n &= J^{-1}(\delta_n Jx_n + (1 - \delta_n) Jz_n), \\
 z_n &= J^{-1}(\alpha_n Jx_n + \beta_n JT x_n + \gamma_n JS x_n), \\
 u_n &\in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jz_n \rangle \geq 0 \quad \forall y \in C, \\
 C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
 x_{n+1} &= \Pi_{C_{n+1}} x_0.
 \end{aligned} \tag{3.46}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following restrictions:

- (a) $\alpha_n + \beta_n + \gamma_n = 1$;
- (b) $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (c) $\liminf_{n \rightarrow \infty} \alpha_n \beta_n > 0$, $\liminf_{n \rightarrow \infty} \alpha_n \gamma_n > 0$;
- (d) $\{\tau_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $\Pi_F x_0$.

Remark 3.17. The conclusion of Corollary 3.16 remains true under the more general restrictions; that is, we replace (b) and (c) by the following one:

- (b') $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n > 0$.

Corollary 3.18 (see [10, Theorem 3.1]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E . Let $\{T_i\}_{i=1}^N : C \rightarrow C$ be a family of relatively nonexpansive mappings such that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $x_0 \in E$. For $C_1 = C$ and $x_1 = \Pi_{C_1} x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{aligned} y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ z_n &= J^{-1}\left(\beta_n^{(1)} Jx_n + \sum_{i=1}^N \beta_n^{(i+1)} JT_i x_n\right), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \end{aligned} \tag{3.47}$$

where $\{\alpha_n\}$, $\{\beta_n^{(i)}\} \subset [0, 1]$ satisfies the following restrictions:

- (i) $0 \leq \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$;
- (ii) $0 \leq \beta_n^{(i)} \leq 1$ for all $i = 1, 2, \dots, N+1$, $\sum_{i=1}^{N+1} \beta_n^{(i)} = 1$ for all $n \in \mathbb{N} \cup \{0\}$. If
 - (a) either $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(i+1)} > 0$ for all $i = 1, 2, \dots, N$ or
 - (b) $\lim_{n \rightarrow \infty} \beta_n^{(1)} = 0$ and $\liminf_{n \rightarrow \infty} \beta_n^{(k+1)} \beta_n^{(l+1)} > 0$ for all $i \neq j$, $k, l = 1, 2, \dots, N$.

then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 3.19. The conclusion of Corollary 3.18 remains true under the more general restrictions on $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$:

- (1) $\alpha_n, \beta_n^{(1)} \in [0, 1]$ are arbitrary.
- (2) $\liminf_{n \rightarrow \infty} \beta_n^{(i)} > 0$ for all $i = 2, \dots, N$.

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