Chapter 2

Main Results

Throughout this report, $F(T)$ denotes the set of fixed points of a mapping T, that is, $F(T) = \{x : x = Tx\}$. Let us summarize all results of this research in the following sections.

2.1 Xu-Ori's implicit iterations

We published two papers in this topic (see Appendices **A1** and **A5**). Based on the implicit iteration introduced by Xu and Ori, we consider the following iteration:

For a countable family of nonexpansive mappings $\{T_n : C \to C\}_{n=1}^{\infty}$, where C is a closed convex subset of a Banach space E, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ in C iteratively:

$$\begin{cases} x_1 \in C \text{ is arbitrarily chosen} \\ x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$. (The sequence above is well defined by Banach's contraction principle.) Suppose in addition that $\{T_n\}_{n=1}^{\infty}$ satisfies some hypothesis and E is a uniformly Banach space and one of the following holds:

- E has Opial's condition;

- the dual space E^* of E has Kadec–Klee property.

Then $\{x_n\}_{n=1}^{\infty}$ converges *weakly* to an element in $\bigcap_{n=1}^{\infty} F(T_n)$. To obtain strong convergence, we have to assume that some addition conditions. However, in Hilbert space setting, we can adapt Nakajo and Takahashi's idea to modify this iteration and conclude strong convergence. In fact, we consider the following iteration:

$$\begin{cases} x_0 \in C \text{ is taken arbitrary,} \\ y_n = \alpha_n x_n + (1-\alpha_n) T_n y_n; \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}; \\ Q_n = \{z \in C : \langle x_n - x_0, z - x_n \rangle \geq 0\}; \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

Here P_K denotes the metric projection from a Hilbert space onto its closed convex subset K.

2.2 Convergence theorems for a family of quasi-Lipschitzian mappings

We are interested in a more general class of mappings that includes all nonexpansive mappings. Recall that a mapping $T : C \to C$, where C is a subset of a Banach space E, is *Lipschitzian* if there exists an $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. In this case, we also say that T is *L-Lipschitzian*. A 1-Lipschitzian mapping is called nonexpansive. Suppose that $F(T) \neq \varnothing$. A mapping T is *quasi-L-Lipschitzian* if

$$\|Tx - p\| \leq L\|x - p\|$$

for all $x \in C$ and $p \in F(T)$. In our study, we always assume that T is L-Lipschitzian or quasi-L-Lipschitzian with $L \geq 1$.

We obtain several convergence theorems for a countable family of mappings $\{T_n\}_{n=1}^{\infty}$ where each T_n is L_n-Lipschitzian (or quasi-L_n-Lipschitzian) and $\sum_{n=1}^{\infty}(L_n - 1) < \infty$. See Appendices **A2**, **A3** and **A4**. We consider the following (explicit) Mann-type iteration:

$$\begin{cases} x_1 \in C \text{ is arbitrarily chosen} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n, \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$. The scheme above can conclude only weak convergence. In order to obtain strong convergence, we adapt Kim and Xu's idea. We consider the following iteration in Hilbert space setting:

$$\begin{cases} x_0 \in C \text{ is taken arbitrary,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T_n x_n; \\ C_n = \Big\{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|x_n - Tx_n\|^2 \\ \qquad +(1-\alpha_n)(L_n^2 - 1)\sup\{\|x_n - z\|^2 : z \in \bigcap_{n=1}^{\infty} F(T_n)\}\Big\}; \\ Q_n = \{z \in C : \langle x_n - x_0, z - x_n\rangle \geq 0\}; \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

2.3 Implicit iteration for pseudocontractive mappings

We also study the implicit type iteration for pseudocontractive mappings and we obtain some interesting results (see Appendix **A6**). More precisely, we give a necessary and sufficient condition for the strong convergence of an implicit iteration to a common fixed point of a countable of continuous pseudocontractive mappings. Recall that, for a subset C of a Banach space E, a mapping $T : C \to C$ is *pseudocontractive* if for each $x, y \in C$ there exists $j \in J(x-y)$ such that

$$\langle Tx - Ty, j\rangle \leq \|x - y\|^2.$$

We also restrict ourselve to the special subclass of continuous pseudocontractive mappings, namely a class of strictly pseudocontractive mappings. A mapping T is *strictly pseudocontractive* if there exist a constant $\lambda > 0$ such that for each $x, y \in C$ there exists $j \in J(x-y)$ such that

$$\langle Tx - Ty, j \rangle \leq \|x-y\|^2 - \lambda \|x - Tx - (y - Ty)\|^2.$$

We can show that the common fixed-point set of a countable family of strictly pseudocontractive mappings is identical to that of a single strictly pseudocontractive mapping provided that the former set is nonempty.

2.4 Relatively nonexpansive mappings

To extend several convergence theorems for nonexpansive mappings in Hilbert space setting to Banach space setting, it is not always possible. It is known that many natural generalizations of nice mappings in Hilbert spaces are no longer nonexpansive in Banach space setting. One way to overcome this is to change the concept of nonexpansiveness in the sense of the norm to the sense of Lyaponov functionals. To more precisely, let us give some definitions.

Let E be a smooth, strictly convex and reflexive Banach space. The *Lyaponov functional* $\varphi : E \times E \to [0, \infty)$ is defined by

$$\varphi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for all } x, y \in E.$$

Following Matsushita and Takahashi, a mapping T from a subset $C \subset E$ into E is called a *relatively nonexpansive* mapping if the following conditions are satisfied:

(R1) $F(T) \neq \varnothing$;

(R2) $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x, \in C$ and $p \in F(T)$;

(R3) If $\{x_n\}$ is a sequence in C such that $x_n - Tx_n \to 0$ and it converges weakly to $p \in E$, then $p \in F(T)$.

If only conditions (R1) and (R2) are satisfied, T is called a *relatively quasi-nonexpansive* mapping.

In this setting in a Banach space, we published one paper (see Appendix **A7**). We introduce the following two general iterative schemes for finding a common fixed point of a countable family of relatively nonexpansive mappings in a Banach space: Let $\{T_n\}_{n=1}^{\infty} : E \to C$ be a family of relatively quasi-nonexpansive mappings and let $\{S_i\}_{i=1}^{N} : C \to C$ be a family of relatively quasi-nonexpansive mappings such that

$$F := \bigcap_{n=1}^{\infty} F(T_n) \cap \bigcap_{i=1}^{N} F(S_i) \neq \varnothing.$$

- Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_1 \in C \\ x_{n+1} = T_n J^{-1} \sum_{i=1}^{N} \omega_{n,i}(\alpha_{n,i} J x_n + (1-\alpha_{n,i}) J S_i x_n) \end{cases}$$

- Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_1 \in C \text{ taken arbitrary}, \\ y_n = T_n J^{-1} \sum_{i=1}^{N} \omega_{n,i}(\alpha_{n,i} J x_n + (1-\alpha_{n,i}) J S_i x_n); \\ C_n = \{z \in C : \varphi(z, y_n) \leq \varphi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_1) \end{cases}$$

where $\{\omega_{n,i}\}$, $\{\alpha_{n,i}\}$ are sequences in $[0, 1]$.

Under suitable setting, we not only obtain several convergence theorems announced by many authors but also prove them under weaker assumptions. Applications to the problem of finding a common element of the fixed point set of a relatively nonexpansive mapping and the solution set of an equilibrium problem are also discussed.