

## Chapter 5

# Maximal Subsemigroups of the Baer-Levi Semigroups of Partial Transformations

In this chapter, we study maximal subsemigroups of  $PS(q)$ . In particular, in Section 5.1 we give necessary and sufficient conditions for the existence of maximal subsemigroups of  $PS(q)$  when  $p > q$ . We also determine some maximal subsemigroups of a subsemigroup  $S_r$  of  $PS(q)$  defined by

$$S_r = \{\alpha \in PS(q) : g(\alpha) \leq r\},$$

where  $q \leq r \leq p$ . Moreover, we extend  $M_A$ , a maximal subsemigroup of  $BL(q)$  (see Section 2.3.2) to determine maximal subsemigroups of  $PS(q)$ . In Section 5.2, we determine some maximal subsemigroups of  $PS(q)$  when  $p = q$ .

### 5.1 Maximal Subsemigroups of $PS(q)$ when $p > q$

The characterisation of maximal subsemigroups of a given semigroup is a natural topic to consider when studying its structure. Sometimes, it is difficult to describe all of them (see [2] and [9], for examples), but for a semigroup with some special properties, we can easily describe some of its maximal subsemigroups.

In what follows, if  $S$  is a semigroup and  $M$  is a subsemigroup of  $S$ , we sometimes use Lemma 2.1.4 to show the maximality of  $M$  in  $S$ . Also, recall that an ideal  $I$  of  $S$  is a prime ideal if its complement in  $S$  is a subsemigroup of  $S$ .

**Lemma 5.1.1.** *Suppose that  $S$  is a semigroup and  $I$  is a prime ideal of  $S$ . Let  $T = S \setminus I$ . Then,*

(i) for a maximal subsemigroup  $M$  of  $T$ ,  $M \cup I$  is a maximal subsemigroup of  $S$ ;

(ii) for a maximal subsemigroup  $N$  of  $S$  such that  $T \setminus N \neq \emptyset$  and  $T \cap N \neq \emptyset$ , the set  $T \cap N$  is a maximal subsemigroup of  $T$ .

**Proof.** To see that (i) holds, let  $M$  be a maximal subsemigroup of  $T$ . Since  $I$  is an ideal, we have  $M \cup I$  is a subsemigroup of  $S$ . Clearly,  $M \cup I \subsetneq T \cup I = S$ . If  $a \in S \setminus (M \cup I)$ , then  $a \in T \setminus M$  and thus  $T = \langle M \cup \{a\} \rangle \subseteq \langle M \cup I \cup \{a\} \rangle$ . Since  $\langle M \cup I \cup \{a\} \rangle$  contains  $I$ , we have  $S = T \cup I = \langle M \cup I \cup \{a\} \rangle$  and so  $M \cup I$  is maximal in  $S$  as required.

To prove (ii), let  $N$  be a maximal subsemigroup of  $S$  such that  $T \setminus N \neq \emptyset$  and  $T \cap N \neq \emptyset$ , and let  $a \in T \setminus N$ . Since  $N$  is maximal in  $S$ , we have  $\langle N \cup \{a\} \rangle = S$ . Thus, for each  $b \in T \setminus N$ ,  $b = c_1 c_2 \dots c_n$  for some natural  $n$  and some  $c_i \in N \cup \{a\}$  for all  $i = 1, 2, \dots, n$ . Since  $b \notin N$ , we have  $c_i = a$  for some  $i$ . Moreover, since  $b \notin I$ , we have  $c_j \in T \cap N$  for all  $j \neq i$ . It follows that  $T \setminus N \subseteq \langle (T \cap N) \cup \{a\} \rangle$ , therefore

$$T = (T \setminus N) \cup (T \cap N) \subseteq \langle (T \cap N) \cup \{a\} \rangle,$$

that is,  $T = \langle (T \cap N) \cup \{a\} \rangle$  and thus  $T \cap N$  is maximal in  $T$ . ■

From [12] p 95, for  $\aleph_0 \leq k \leq p$ , the authors showed that

$$S_k = \{\alpha \in PS(q) : g(\alpha) \leq k\}$$

is a subsemigroup of  $PS(q)$ . Also, when  $p > q$ , the proper ideals of  $PS(q)$  are precisely the sets:

$$T_s = \{\alpha \in PS(q) : g(\alpha) \geq s\}$$

where  $q < s \leq p$  (see Theorem 2.3.14). Thus, for any  $q \leq r < p$ , it is clear that

$$PS(q) = S_r \dot{\cup} T_{r'},$$

that is,  $PS(q)$  can be written as a disjoint union of the semigroup  $S_r$  and the ideal  $T_{r'}$ . In other words,  $T_{r'}$  is a prime ideal of  $PS(q)$ . Hence, the next result follows directly from Lemma 5.1.1(i).

**Corollary 5.1.2.** *Suppose that  $p > r > q \geq \aleph_0$ . If  $M$  is a maximal subsemigroup of  $S_r$ , then  $M \cup T_{r'}$  is a maximal subsemigroup of  $PS(q)$ .*

**Proof.** Let  $M$  be a maximal subsemigroup of  $S_r$ . Since  $PS(q) = S_r \dot{\cup} T_{r'}$  and  $T_{r'}$  is a proper prime ideal of  $PS(q)$ , we have  $M \cup T_{r'}$  is a maximal subsemigroup of  $PS(q)$  by Lemma 5.1.1(i). ■

**Lemma 5.1.3.** *Let  $p > q \geq \aleph_0$  and suppose that  $M$  is a maximal subsemigroup of  $PS(q)$ . Then,*

- (i)  $S_r \cap M \neq \emptyset$  for all  $q \leq r < p$  ;
- (ii) if there exists  $\alpha \notin M$  with  $g(\alpha) < p$ , then  $S_k \setminus M \neq \emptyset$  for some  $q \leq k < p$ .

**Proof.** To show that (i) holds, we first note that  $S_q$  is contained in  $S_r$  for all  $q \leq r < p$ . If  $S_q \cap M = \emptyset$ , then  $M \subseteq T_{q'} \subsetneq PS(q)$  and thus  $M = T_{q'}$  by the maximality of  $M$ . But  $T_{q'} \subsetneq T_{q'} \cup BL(q) \subsetneq PS(q)$  where  $T_{q'} \cup BL(q)$  is a subsemigroup of  $PS(q)$  (since  $T_{q'}$  is an ideal), this contradicts the maximality of  $T_{q'}$ . Therefore,  $\emptyset \neq S_q \cap M \subseteq S_r \cap M$  for all  $q \leq r < p$ .

To show that (ii) holds, suppose there is  $\alpha \notin M$  with  $g(\alpha) = k < p$ . If  $k < q$ , then  $\alpha \in S_r \setminus M$  for all  $q \leq r \leq p$ . Otherwise, if  $q \leq k$ , then  $\alpha \in S_k \setminus M$ . Hence (ii) holds. ■

In what follows, for any cardinal  $r \leq p$ , we let

$$G_r = \{\alpha \in PS(q) : g(\alpha) = r\}.$$

Then  $G_0 = BL(q)$  and  $G_q = R(q)$ . Moreover, if  $p > q$  and  $r > q$ , then  $G_r = S_r \cap T_{r'}$ , and so  $G_r$  is a subsemigroup of  $S_r$  (since it is the intersection of two semigroups). Also,  $G_r$  is bi-simple and idempotent-free, when  $p > q$  and  $r > q$  (see Corollary 2.3.16).

From Theorem 2.3.17, if  $p \geq q$ , then  $S_q = \alpha.R(q)$  for each  $\alpha \in BL(q)$ , and by Theorem 2.3.18,  $S_q = BL(q).\mu.BL(q)$  for each  $\mu \in R(q)$  when  $p \neq q$ . This motivates the following result.

**Lemma 5.1.4.** *Suppose that  $p \geq r > q \geq \aleph_0$ . Then  $G_r = BL(q).\alpha.BL(q)$  for each  $\alpha \in G_r$ .*

**Proof.** Let  $\alpha \in G_r$  and  $\beta, \gamma \in BL(q)$ . Since

$$X \setminus \text{dom } \alpha = [X\beta \cap (X \setminus \text{dom } \alpha)] \dot{\cup} [(X \setminus X\beta) \cap (X \setminus \text{dom } \alpha)]$$

where  $g(\alpha) = |X \setminus \text{dom } \alpha| = r > q$  and the second intersection on the right has cardinal at most  $q$  (since  $|X \setminus X\beta| = q$ ), we have  $|X\beta \cap (X \setminus \text{dom } \alpha)| = r$ . This means that

$$\begin{aligned} r &= |[X\beta \cap (X \setminus \text{dom } \alpha)]\beta^{-1}| \\ &= |(X\beta \setminus \text{dom } \alpha)\beta^{-1}| \\ &= |\text{dom } \beta \setminus \text{dom}(\beta\alpha)| \\ &= |X \setminus \text{dom}(\beta\alpha)|, \end{aligned}$$



that is  $g(\beta\alpha) = r$ . Since  $\text{dom } \gamma = X$ , we have  $\text{dom}(\beta\alpha\gamma) = \text{dom}(\beta\alpha)$ , and so  $g(\beta\alpha\gamma) = g(\beta\alpha) = r$ . Hence  $\beta\alpha\gamma \in G_r$  and therefore  $BL(q).\alpha.BL(q) \subseteq G_r$ .

For the converse, if  $\alpha, \beta \in G_r$ , then  $|X \setminus \text{dom } \alpha| = r = |X \setminus \text{dom } \beta|$ . Since  $p > q$ , every element in  $PS(q)$  has rank  $p$ , so we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix} \quad \text{where } |I| = p.$$

Now write  $X \setminus \{y_i\} = A \dot{\cup} B$  and  $X \setminus \{a_i\} = C \dot{\cup} D$  where  $|A| = |B| = |C| = q$  and  $|D| = r$  (note that this is possible since  $d(\beta) = q \geq \aleph_0$  and  $g(\alpha) = r > q \geq \aleph_0$ ).

Define

$$\delta = \begin{pmatrix} b_i & X \setminus \{b_i\} \\ a_i & D \end{pmatrix}, \epsilon = \begin{pmatrix} x_i & X \setminus \{x_i\} \\ y_i & A \end{pmatrix}$$

where  $\delta|(X \setminus \{b_i\})$  and  $\epsilon|(X \setminus \{x_i\})$  are bijections. Then  $\delta, \epsilon \in BL(q)$  and  $\beta = \delta\alpha\epsilon$ , that is,  $G_r \subseteq BL(q).\alpha.BL(q)$  and equality follows. ■

Now we can describe all maximal subsemigroups of  $PS(q)$  when  $p > q$ .

**Theorem 5.1.5.** *Suppose that  $p > q \geq \aleph_0$ . Then  $M$  is a maximal subsemigroup of  $PS(q)$  if and only if  $M$  equals one of the following sets:*

- (i)  $PS(q) \setminus G_p = \{\alpha \in PS(q) : g(\alpha) < p\}$ ;
- (ii)  $N \cup T_{r'}$ , where  $q \leq r < p$  and  $N$  is a maximal subsemigroup of  $S_r$ .

**Proof.** Let  $\alpha, \beta \in PS(q)$  be such that  $g(\alpha) < p$  and  $g(\beta) < p$ . Since

$$|X\alpha \setminus \text{dom } \beta| \leq |X \setminus \text{dom } \beta| = g(\beta) < p,$$

we have

$$\begin{aligned} |\text{dom } \alpha \setminus \text{dom}(\alpha\beta)| &= |[X\alpha \setminus (X\alpha \cap \text{dom } \beta)]\alpha^{-1}| \\ &= |(X\alpha \setminus \text{dom } \beta)\alpha^{-1}| \\ &= |X\alpha \setminus \text{dom } \beta| < p. \end{aligned}$$

Hence,

$$|X \setminus \text{dom}(\alpha\beta)| = |X \setminus \text{dom } \alpha| + |\text{dom } \alpha \setminus \text{dom}(\alpha\beta)| < p,$$

and this shows that  $PS(q) \setminus G_p$  is a subsemigroup of  $PS(q)$ . To show that  $PS(q) \setminus G_p$  is maximal in  $PS(q)$ , we let  $\alpha, \beta \in PS(q) \setminus (PS(q) \setminus G_p) = G_p$ . By Lemma 5.1.4,  $\alpha = \lambda\beta\mu$  for some  $\lambda, \mu \in BL(q) \subseteq PS(q) \setminus G_p$ . Thus,  $\alpha$  can be written as a finite product of elements in  $(PS(q) \setminus G_p) \cup \{\beta\}$ , and hence  $PS(q) \setminus G_p$  is maximal in  $PS(q)$  by Lemma 2.1.4. Also, if  $q \leq r < p$  and  $N$  is a maximal subsemigroup of  $S_r$ , then  $N \cup T_{r'}$  is maximal in  $PS(q)$  by Corollary 5.1.2.

Finally, suppose that  $M$  is a maximal subsemigroup of  $PS(q)$  such that  $M \neq PS(q) \setminus G_p$ . Then there exists  $\alpha \notin M$  with  $g(\alpha) < p$ . Thus, Lemma 5.1.3 implies that  $S_k \setminus M \neq \emptyset$  and  $S_k \cap M \neq \emptyset$  for some  $k$ , where  $q \leq k < p$ . Since  $T_{k'}$  is a proper prime ideal of  $PS(q)$  and  $S_k = PS(q) \setminus T_{k'}$ , Lemma 5.1.1(ii) implies that  $S_k \cap M$  is maximal in  $S_k$ . Since  $PS(q) = S_k \cup T_{k'}$ , we see that

$$M = (S_k \cap M) \cup (T_{k'} \cap M) \subseteq (S_k \cap M) \cup T_{k'},$$

where  $(S_k \cap M) \cup T_{k'}$  is maximal in  $PS(q)$  by Corollary 5.1.2. This means that  $M = (S_k \cap M) \cup T_{k'}$  by the maximality of  $M$ . ■

By the previous theorem, when  $p > q$ , most of the maximal subsemigroups of  $PS(q)$  are induced by maximal subsemigroups of  $S_r$  where  $q \leq r < p$ . Hence we now determine some maximal subsemigroups of  $S_r$ .

As mentioned in Section 2.3.2, for every non-empty subset  $A$  of  $X$  such that  $|X \setminus A| \geq q$ ,  $M_A$  is a maximal subsemigroup of  $BL(q)$ . Here we extend the definition of  $M_A$  and consider the set  $\overline{M}_A$  defined as

$$\overline{M}_A = \{\alpha \in PS(q) : A \not\subseteq X\alpha \text{ or } (A\alpha \subseteq A \subseteq \text{dom } \alpha \text{ or } |X\alpha \setminus A| < q)\},$$

that is,  $\alpha$  in  $PS(q)$  belongs to  $\overline{M}_A$  if and only if

- (i)  $A \not\subseteq X\alpha$ , or
- (ii)  $A \subseteq X\alpha$  and either  $A\alpha \subseteq A \subseteq \text{dom } \alpha$ , or  $|X\alpha \setminus A| < q$ .

The next result gives more details on  $\overline{M}_A$ .

**Lemma 5.1.6.** *Suppose that  $p \geq q \geq \aleph_0$  and let  $A$  be a non-empty subset of  $X$  such that  $|X \setminus A| \geq q$ . Then,*

- (i) *for any cardinal  $k$  such that  $0 \leq k \leq p$ , there exist  $\alpha, \beta \in PS(q)$  such that  $g(\alpha) = k = g(\beta)$  and  $\alpha \in \overline{M}_A, \beta \notin \overline{M}_A$ ;*
- (ii) *for each  $\gamma \notin \overline{M}_A$ ,  $|\text{dom } \gamma \setminus A\gamma^{-1}| = |X \setminus A| = |X\gamma \setminus A|$  and  $|A\gamma^{-1}| = |A|$ .*

**Proof.** To show that (i) holds, let  $|X \setminus A| = r \geq q$  and let  $k$  be a cardinal such that  $0 \leq k \leq p$ . We write  $X \setminus A = R \dot{\cup} Q$  where  $|R| = r$  and  $|Q| = q$ . If  $r = p$ , then  $|A \cup R| \geq r = p$ ; if  $r < p$ , then  $|X \setminus A| < p$ , and this implies  $|A| = p$ , and so  $|A \cup R| = p$ . Fix  $a \in A$  and let  $B = (A \setminus \{a\}) \cup R$ . Then,  $|B| = p$  and  $|X \setminus B| = |Q \cup \{a\}| = q$ . We write  $X = K \dot{\cup} L$  where  $|K| = k$  and  $|L| = p$ . Then there exists a bijection  $\alpha : L \rightarrow B$  and so  $g(\alpha) = k, d(\alpha) = q$ . Also, since  $A \not\subseteq B = X\alpha$ , we have  $\alpha \in \overline{M}_A$ .

To find  $\beta \in PS(q) \setminus \overline{M}_A$  with  $g(\beta) = k$ , we consider two cases. First, if  $r = p$ , we write  $X \setminus A = P \dot{\cup} Q \dot{\cup} K$  where  $|P| = p, |Q| = q, |K| = k$ . Fix  $a \in A$  and define

$$\beta = \begin{pmatrix} P \cup Q \cup \{a\} & A \setminus \{a\} \\ P \cup K \cup \{a\} & A \setminus \{a\} \end{pmatrix}$$

where  $\beta|(P \cup Q \cup \{a\})$  and  $\beta|(A \setminus \{a\})$  are bijections and  $a\beta \neq a$ . On the other hand, if  $r < p$ , then  $|A| = p$ . In this case we write  $A = A' \dot{\cup} K'$  and  $X \setminus A = R \dot{\cup} Q$  where  $|A'| = p$ ,  $|K'| = k$ ,  $|R| = r$  and  $|Q| = q$ . Fix  $a \in A'$  and re-define

$$\beta = \begin{pmatrix} (X \setminus A) \cup \{a\} & A' \setminus \{a\} \\ R \cup \{a\} & A \setminus \{a\} \end{pmatrix}$$

where  $\beta|((X \setminus A) \cup \{a\})$  and  $\beta|(A' \setminus \{a\})$  are bijections and  $a\beta \neq a$ . In both cases, we have  $d(\beta) = q$ ,  $g(\beta) = k$ ,  $A \subseteq X\beta$ ,  $A\beta \not\subseteq A$  and  $|X\beta \setminus A| \geq q$ , that is  $\beta \in PS(q) \setminus \overline{M}_A$ .

To see that (ii) holds, suppose that there is  $\gamma \notin \overline{M}_A$ , then  $A \subseteq X\gamma$  and  $|X\gamma \setminus A| \geq q$ . So  $|A\gamma^{-1}| = |A|$  since  $\gamma$  is injective. Also,

$$X \setminus A = (X \setminus X\gamma) \dot{\cup} (X\gamma \setminus A)$$

where  $|X \setminus X\gamma| = q$ . Since  $|X \setminus A| \geq q$  and  $|X\gamma \setminus A| \geq q$ , we have  $|X \setminus A| = |X\gamma \setminus A| = |(X\gamma \setminus A)\gamma^{-1}| = |\text{dom } \gamma \setminus A\gamma^{-1}|$  as required.  $\blacksquare$

In [9] Theorem 1, the authors proved that  $M_A$  is a maximal subsemigroup of  $BL(q)$  for every non-empty subset  $A$  of  $X$  such that  $|X \setminus A| \geq q$ . Using a similar argument, we can show that  $\overline{M}_A$  is a subsemigroup of  $PS(q)$ .

**Lemma 5.1.7.** *Suppose that  $p \geq q \geq \aleph_0$  and let  $A$  be a non-empty subset of  $X$  such that  $|X \setminus A| \geq q$ . Then  $\overline{M}_A$  is a proper subsemigroup of  $PS(q)$ .*

**Proof.** Let  $\alpha, \beta \in \overline{M}_A$ . If  $A \not\subseteq X\alpha\beta$ , then  $\alpha\beta \in \overline{M}_A$ . Now we suppose that  $A \subseteq X\alpha\beta$  and this implies  $A \subseteq X\beta$ . Since  $\beta \in \overline{M}_A$ , we either have

$$A\beta \subseteq A \subseteq \text{dom } \beta \quad \text{or} \quad |X\beta \setminus A| < q.$$

If  $|X\beta \setminus A| < q$ , then  $|X\alpha\beta \setminus A| \leq |X\beta \setminus A| < q$  and so  $\alpha\beta \in \overline{M}_A$ . If  $A\beta \subseteq A \subseteq \text{dom } \beta$ , then we have  $A\beta \subseteq A \subseteq X\alpha\beta$  and hence  $A \subseteq X\alpha$  since  $\beta$  is injective. Since  $\alpha \in \overline{M}_A$ , we either have  $A\alpha \subseteq A \subseteq \text{dom } \alpha$ , or  $|X\alpha \setminus A| < q$ . If the latter occurs, then

$$|X\alpha\beta \setminus A| \leq |X\alpha\beta \setminus A\beta| = |(X\alpha \setminus A)\beta| \leq |X\alpha \setminus A| < q,$$

therefore  $\alpha\beta \in \overline{M}_A$ . On the other hand, if  $A\alpha \subseteq A \subseteq \text{dom } \alpha$ , we have  $A\alpha\beta \subseteq A\beta \subseteq A$ . Moreover,  $A\alpha \subseteq X\alpha \cap \text{dom } \beta$ , that is,  $A \subseteq (X\alpha \cap \text{dom } \beta)\alpha^{-1} = \text{dom } \alpha\beta$ . It follows that  $A\alpha\beta \subseteq A \subseteq \text{dom } \alpha\beta$ . Therefore  $\alpha\beta \in \overline{M}_A$ , and hence  $\overline{M}_A$  is a subsemigroup of  $PS(q)$ . Finally, this subsemigroup is properly contained in  $PS(q)$  by Lemma 5.1.6(i). ■

**Remark 5.1.8.** By Lemma 5.1.6(i), for any cardinal  $r$  such that  $q \leq r \leq p$ ,  $S_r \cap \overline{M}_A$  is always non-empty and properly contained in  $S_r$ . Therefore, it is a proper subsemigroup of  $S_r$ , but it is not maximal when  $q < r$ . To see this, suppose that  $S_r \cap \overline{M}_A$  is maximal and choose  $\alpha, \beta \notin \overline{M}_A$  such that  $g(\alpha) = r$  and  $g(\beta) = 0$  (possible by Lemma 5.1.6(i)). Then  $\alpha, \beta \in S_r \setminus \overline{M}_A$  where  $\text{dom } \beta = X$ . Moreover  $\langle (S_r \cap \overline{M}_A) \cup \{\alpha\} \rangle = S_r$ , and so

$$\beta = \gamma_1\gamma_2 \dots \gamma_n \alpha \lambda_1 \lambda_2 \dots \lambda_m$$

for some  $n, m \in \mathbb{N}_0$  and  $\gamma_i, \lambda_j \in (S_r \cap \overline{M}_A) \cup \{\alpha\}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . If  $n = 0$  or  $\gamma_1 = \alpha$ , then  $\text{dom } \beta \subseteq \text{dom } \alpha$  and so  $g(\alpha) = 0$ , a contradiction. Thus,  $n \neq 0$  and  $\gamma_1 \neq \alpha$ . Since  $X = \text{dom } \beta \subseteq \text{dom}(\gamma_1\gamma_2 \dots \gamma_n)$ , it follows that  $\gamma = \gamma_1\gamma_2 \dots \gamma_n \in BL(q)$ . Moreover,  $X\gamma \subseteq \text{dom } \alpha$ , and this implies,

$$q \leq r = |X \setminus \text{dom } \alpha| \leq |X \setminus X\gamma| = q,$$

and hence  $r = q$ .

Since  $M_A$  is maximal in  $BL(q)$ , a subsemigroup of  $PS(q)$ , it is natural to think that  $\overline{M}_A$  is maximal in  $PS(q)$ . But when  $p > q$ , by taking  $r = p$ , the above observation shows that this claim is false since  $S_p = PS(q)$ . Thus,  $\overline{M}_A$  is not always a maximal subsemigroup of  $PS(q)$  when  $p > q$ .

The proof of the next result follows some ideas from [9] Theorem 1.

**Theorem 5.1.9.** *Suppose that  $p \geq r \geq q \geq \aleph_0$  and let  $A$  be a non-empty subset of  $X$  such that  $|X \setminus A| \geq q$ . Then  $S_r \cap \overline{M}_A$  is a maximal subsemigroup of  $S_r$  precisely when  $r = q$ .*

**Proof.** In Remark 5.1.8, we have shown that  $S_r \cap \overline{M}_A$  is not maximal in  $S_r$  when  $r > q$ . It remains to show  $S_q \cap \overline{M}_A$  is maximal in  $S_q$ . Let  $\alpha, \beta \in S_q \setminus \overline{M}_A$ . Then  $g(\alpha) \leq q, g(\beta) \leq q$  and Lemma 5.1.6(ii) implies that

$$|A\alpha^{-1}| = |A| = |A\beta^{-1}|, \text{ and}$$

$$|\text{dom } \alpha \setminus A\alpha^{-1}| = |\text{dom } \beta \setminus A\beta^{-1}| = |X\beta \setminus A| = |X\alpha \setminus A| = |X \setminus A| = s \text{ (say)} \geq q.$$

We also have  $A\beta \not\subseteq A$  or  $A \not\subseteq \text{dom } \beta$ . In the case that  $A\beta \not\subseteq A$ , we have  $A\beta \cap (X \setminus A) \neq \emptyset$ . Thus, there exists  $y \in A \cap (X \setminus A)\beta^{-1}$ , so  $y \notin A\beta^{-1}$ . Since  $|\text{dom } \beta \setminus (A\beta^{-1} \cup \{y\})| = s$ , we can write

$$\text{dom } \beta \setminus (A\beta^{-1} \cup \{y\}) = \{c_j\} \dot{\cup} \{d_k\}$$

where  $|J| = s$  and  $|K| = q$ . Also, since  $\alpha, \beta \notin \overline{M}_A$ , we have  $A \subseteq X\alpha$  and  $A \subseteq X\beta$ . Thus, for convenience, write  $A = \{a_i\}$ , let  $y_i, z_i \in X$  be such that  $y_i\alpha = a_i = z_i\beta$  for each  $i$ , and let  $\text{dom } \alpha \setminus A\alpha^{-1} = \{b_j\}$ . Hence, we can write

$$\beta = \begin{pmatrix} z_i & c_j & d_k & y \\ a_i & c_j\beta & d_k\beta & y\beta \end{pmatrix}.$$

Now define  $\gamma \in P(X)$  by

$$\gamma = \begin{pmatrix} y_i & b_j \\ z_i & c_j \end{pmatrix}.$$

Then,  $d(\gamma) = |X \setminus (\{z_i\} \cup \{c_j\})| = |\{d_k\} \cup \{y\} \cup (X \setminus \text{dom } \beta)| = |\{d_k\} \cup \{y\}| + g(\beta) = q$ , that is,  $\gamma \in PS(q)$ . Also, since  $\text{dom } \gamma = \text{dom } \alpha$ , we have  $g(\gamma) = g(\alpha) \leq q$  and so  $\gamma \in S_q$ . Moreover, since  $y \in A$  and  $y \notin X\gamma$ , we have  $A \not\subseteq X\gamma$ , that is,  $\gamma \in \overline{M}_A$ . Also, since  $d(\alpha) = q$ , we can write  $X \setminus X\alpha = \{m_k\} \dot{\cup} \{n_k\} \dot{\cup} \{z\}$  and define  $\mu$  in  $P(X)$  by

$$\mu = \begin{pmatrix} a_i & c_j\beta & d_k\beta & y\beta \\ a_i & b_j\alpha & m_k & z \end{pmatrix}.$$

Then  $d(\mu) = |X \setminus (\{a_i\} \cup \{b_j\alpha\} \cup \{m_k\} \cup \{z\})| = |X \setminus (X\alpha \cup \{m_k\} \cup \{z\})| = |\{n_k\}| = q = d(\beta) = g(\mu)$ , that is,  $\mu \in S_q$ . Moreover,  $\mu \in \overline{M}_A$  since  $A\mu = A \subseteq \text{dom } \mu$ . Finally, we can see that  $\alpha = \gamma\beta\mu$  where  $\gamma, \mu \in S_q \cap \overline{M}_A$ .

On the other hand, if  $A \not\subseteq \text{dom } \beta$ , then there exists  $w \in A \setminus \text{dom } \beta$ . In this case, we rewrite

$$\text{dom } \beta \setminus A\beta^{-1} = \{c_j\} \dot{\cup} \{d_k\} \quad \text{and} \quad X \setminus X\alpha = \{m_k\} \dot{\cup} \{n_k\}$$

where  $|J| = s, |K| = q$ . Like before, we write  $A = \{a_i\}$  and  $\text{dom } \alpha = \{y_i\} \dot{\cup} \{b_j\}$  where  $y_i\alpha = a_i = z_i\beta$  and  $\{b_j\} = \text{dom } \alpha \setminus A\alpha^{-1}$ , then

$$\beta = \begin{pmatrix} z_i & c_j & d_k \\ a_i & c_j\beta & d_k\beta \end{pmatrix}.$$

Define  $\gamma, \mu \in P(X)$  by

$$\gamma = \begin{pmatrix} y_i & b_j \\ z_i & c_j \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i & c_j\beta & d_k\beta \\ a_i & b_j\alpha & m_k \end{pmatrix}.$$

Then,  $d(\gamma) = |\{d_k\}| + g(\beta) = q$ ,  $g(\gamma) = g(\alpha) \leq q$ ,  $d(\mu) = |\{n_k\}| = q = d(\beta) = g(\mu)$ , and so  $\gamma, \mu \in S_q$ . Also,  $\gamma, \mu \in \overline{M}_A$  since  $A \not\subseteq X\gamma$  (note that  $w \in A \setminus \text{dom } \beta \subseteq A \setminus X\gamma$ ) and  $A\mu = A \subseteq \text{dom } \mu$ . Moreover,  $\alpha = \gamma\beta\mu$ . In other words, we have shown that for every  $\alpha, \beta \in S_q \setminus \overline{M}_A$ ,  $\alpha$  can be written as a finite product of elements in  $(S_q \cap \overline{M}_A) \cup \{\beta\}$ . Therefore,  $S_q \cap \overline{M}_A$  is maximal in  $S_q$ . ■

We now determine some other classes of maximal subsemigroups of  $S_r$ .

**Lemma 5.1.10.** *Suppose that  $p \geq r \geq q \geq \aleph_0$ . Let  $k$  be a cardinal such that  $k = 0$  or  $q \leq k \leq r$ . Then*

$$S_r \setminus G_k = \{\alpha \in PS(q) : k \neq g(\alpha) \leq r\}$$

*is a proper subsemigroup of  $S_r$ .*

**Proof.** Since  $k \leq r$ , we have  $S_r \setminus G_k \subsetneq S_r$ . If  $k = 0$ , then  $S_r \setminus G_0 = S_r \setminus BL(q)$  and this is a subsemigroup of  $S_r$  since, for  $\alpha, \beta \in S_r \setminus BL(q)$ ,  $\text{dom}(\alpha\beta) \subseteq \text{dom } \alpha \subsetneq X$ , and this implies  $\alpha\beta \in S_r \setminus BL(q)$ . Now suppose  $q \leq k \leq r$  and let  $\alpha, \beta \in S_r$  be such that  $g(\alpha\beta) = k$ . We claim that  $g(\alpha) = k$  or  $g(\beta) = k$ . To see this, assume that  $g(\alpha) \neq k$ . Since

$$k = |X \setminus \text{dom}(\alpha\beta)| = |X \setminus \text{dom } \alpha| + |\text{dom } \alpha \setminus \text{dom}(\alpha\beta)|,$$

we have  $|X \setminus \text{dom } \alpha| < k$ , thus

$$k = |\text{dom } \alpha \setminus \text{dom } (\alpha\beta)| = |[X\alpha \setminus (X\alpha \cap \text{dom } \beta)]\alpha^{-1}| = |(X\alpha \setminus \text{dom } \beta)\alpha^{-1}| = |X\alpha \setminus \text{dom } \beta|.$$

Note that

$$X \setminus \text{dom } \beta = [X\alpha \setminus \text{dom } \beta] \dot{\cup} [(X \setminus X\alpha) \cap (X \setminus \text{dom } \beta)]$$

where the intersection on the right has cardinal at most  $q$ . Hence

$$g(\beta) = |X \setminus \text{dom } \beta| = k$$

since  $k \geq q$ , and we have shown that  $S_r \setminus G_k$  is a subsemigroup of  $S_r$ . ■

**Remark 5.1.11.** Observe that, if  $0 < k < q$  then  $S_r \setminus G_k$  is not a semigroup for all  $q \leq r \leq p$ . To see this, let  $\alpha \in BL(q)$  and  $\beta = \text{id}_{X\alpha \setminus K}$  for some subset  $K$  of  $X\alpha$  such that  $|K| = k$  (possible since  $|X\alpha| = p > k$ ), then  $\alpha, \beta \in PS(q)$  since  $d(\beta) = d(\alpha) + k = q$ . Moreover, since  $g(\alpha) = 0$  and  $g(\beta) = q \neq k$ , we have  $\alpha, \beta \in S_r \setminus G_k$ . But

$$\text{dom}(\alpha\beta) = (X\alpha \cap \text{dom } \beta)\alpha^{-1} = (X\alpha \setminus K)\alpha^{-1} = X \setminus K\alpha^{-1},$$

thus  $g(\alpha\beta) = |K\alpha^{-1}| = k$ , that is,  $\alpha\beta \in G_k$ .

**Theorem 5.1.12.** *Suppose that  $p \geq r \geq q \geq \aleph_0$ . Then the following statements hold:*

(i)  $S_r \setminus G_0$  is a maximal subsemigroup of  $S_r$ ;

(ii) if  $p > q$ , then for each cardinal  $k$  such that  $q \leq k \leq r$ ,  $S_r \setminus G_k$  is a maximal subsemigroup of  $S_r$ .

**Proof.** By Lemma 5.1.10,  $S_r \setminus G_0$  is a subsemigroup of  $S_r$ . To see that it is maximal in  $S_r$ , let  $\alpha, \beta \in S_r \setminus (S_r \setminus G_0) = G_0 = BL(q) \subseteq S_q$ . By Theorem 2.3.17,  $S_q = \beta.R(q)$ , and this implies that  $\alpha = \beta\gamma$  for some  $\gamma \in R(q) \subseteq S_r \setminus G_0$ . Hence  $S_r \setminus G_0$  is maximal in  $S_r$ .

Now suppose that  $p > q$  and  $q \leq k \leq r$ . Let  $\alpha, \beta \in S_r \setminus (S_r \setminus G_k) = G_k$ . If  $k = q$ , then  $G_k = R(q) \subseteq S_q$  and, by Theorem 2.3.18,  $S_q = BL(q).\beta.BL(q)$ . If  $k > q$ , then  $G_k = BL(q).\beta.BL(q)$  (by Lemma 5.1.4). In both cases, we have  $\alpha = \gamma\beta\mu$  for some  $\gamma, \mu \in BL(q) \subseteq S_r \setminus G_k$ , and so  $S_r \setminus G_k$  is maximal in  $S_r$ . ■

**Corollary 5.1.13.** *Suppose that  $p > q \geq \aleph_0$  and let  $A$  be a non-empty subset of  $X$  such that  $|X \setminus A| \geq q$ . Then the following sets are maximal subsemigroups of  $PS(q)$ :*

(i)  $\overline{M}_A \cup T_{q'}$ ;

(ii)  $N_k = \{\alpha \in PS(q) : g(\alpha) \neq k\}$  where  $k = 0$  or  $q \leq k \leq p$ .

**Proof.** By Theorem 5.1.9,  $S_q \cap \overline{M}_A$  is maximal in  $S_q$ . Then Corollary 5.1.2 implies that  $(S_q \cap \overline{M}_A) \cup T_{q'}$  is maximal in  $PS(q)$ . But

$$(S_q \cap \overline{M}_A) \cup T_{q'} = (S_q \cup T_{q'}) \cap (\overline{M}_A \cup T_{q'}) = PS(q) \cap (\overline{M}_A \cup T_{q'}) = \overline{M}_A \cup T_{q'},$$

and so (i) holds. To show that (ii) holds, let  $r = p$  in Theorem 5.1.12. Then  $S_p = PS(q)$  and thus  $N_k = S_p \setminus G_k$  is maximal in  $PS(q)$ . ■

**Corollary 5.1.14.** *Suppose that  $p > q \geq \aleph_0$  and  $k$  equals 0 or  $q$ . Let  $A$  be a non-empty subset of  $X$  such that  $|X \setminus A| \geq q$ . Then the two classes of maximal subsemigroups  $S_q \cap \overline{M}_A$  and  $S_q \setminus G_k$  of  $S_q$  are always disjoint.*

**Proof.** By Theorem 5.1.9 and Theorem 5.1.12,  $S_q \cap \overline{M}_A$  and  $S_q \setminus G_k$  are maximal subsemigroups of  $S_q$ . By Lemma 5.1.6(i), there exists  $\alpha \in \overline{M}_A$  with  $g(\alpha) = k$ . Then  $\alpha \in S_k \cap \overline{M}_A \subseteq S_q \cap \overline{M}_A$  but  $\alpha \notin S_q \setminus G_k$ , that is,  $S_q \cap \overline{M}_A \not\subseteq S_q \setminus G_k$ . Also,  $S_q \setminus G_k \not\subseteq S_q \cap \overline{M}_A$  by the maximality of  $S_q \cap \overline{M}_A$  and  $S_q \setminus G_k$ . Therefore  $S_q \cap \overline{M}_A$  is not equal to  $S_q \setminus G_k$ . ■

## 5.2 Maximal Subsemigroups of $PS(q)$ when $p = q$

We first recall that, when  $p = q$ , the empty transformation  $\emptyset$  belongs to  $PS(q)$  since  $d(\emptyset) = p = q$ . In this case, the ideals of  $PS(q)$  are precisely the sets:

$$J_r = \{\alpha \in PS(q) : r(\alpha) < r\}$$

where  $1 \leq r \leq p'$  (see Theorem 2.3.15). Clearly,  $J_{p'} = PS(q)$  and

$$J_p = \{\alpha \in PS(q) : r(\alpha) < p\}$$

is the largest proper ideal. In this case,  $J_r$  is not a prime ideal of  $PS(q)$ . To see this, write  $X = A \dot{\cup} B \dot{\cup} C$  where  $|A| = p$  and  $|B| = r = |C|$ . Then  $\text{id}_B, \text{id}_C \in PS(q) \setminus J_r$  whereas  $\text{id}_B \cdot \text{id}_C = \emptyset \in J_r$ . Hence, unlike what was done in Section 5.1, we cannot use Lemma 5.1.1 to find maximal subsemigroups of  $PS(q)$  when  $p = q$ . In this section, we determine some maximal subsemigroups of  $PS(q)$ , for  $p = q$ , using a different approach. We first describe some properties of each maximal subsemigroup in this case.

**Lemma 5.2.1.** *Suppose that  $p = q \geq \aleph_0$  and  $M$  is a maximal subsemigroup of  $PS(q)$ . Then the following statements hold:*

- (i)  $M$  contains all  $\alpha \in PS(q)$  with  $r(\alpha) < p$ ,
- (ii) if  $R(q) \subseteq M$ , then  $M \cap BL(q) = \emptyset$ .

**Proof.** Let  $\alpha \in PS(q)$  with  $r(\alpha) = k < p$ . Then  $g(\alpha) = p$  and we write in the usual way

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}.$$

Also, write  $X \setminus \{a_i\} = P \dot{\cup} Q$  and  $X \setminus \{x_i\} = R \dot{\cup} S$  where  $|P| = |Q| = p = |R| = |S|$ , and define  $\beta, \gamma$  in  $P(X)$  by

$$\beta = \begin{pmatrix} a_i & P \\ a_i & P \end{pmatrix}, \gamma = \begin{pmatrix} a_i & Q \\ x_i & R \end{pmatrix}$$

where  $\beta|_P$  and  $\gamma|_Q$  are bijections. Then  $\beta, \gamma \in PS(q)$  and also,

$$\alpha = \beta \cdot \alpha \cdot \text{id}_{X \setminus \alpha} \in PS(q) \cdot \alpha \cdot PS(q).$$

If  $M = M \cup (PS(q) \cdot \alpha \cdot PS(q))$ , then  $\alpha \in M$  and we have completed the proof. Otherwise,  $M \subsetneq M \cup (PS(q) \cdot \alpha \cdot PS(q))$  where  $M \cup (PS(q) \cdot \alpha \cdot PS(q))$  is a subsemigroup of  $PS(q)$ . This means that  $M \cup (PS(q) \cdot \alpha \cdot PS(q)) = PS(q)$  by the maximality of  $M$ . Since all mappings in  $PS(q) \cdot \alpha \cdot PS(q)$  have rank at most  $k$ , it follows that  $M$

contains all mappings with rank greater than  $k$ . Therefore  $\beta, \gamma \in M$  and thus  $\alpha = \beta\gamma \in M$  as required.

To show that (ii) holds, suppose that  $R(q) \subseteq M$ . If there exists  $\alpha \in M \cap BL(q)$ , then Theorem 2.3.17 implies that  $PS(q) = \alpha.R(q) \subseteq M$  (note that  $S_q = PS(q)$  when  $p = q$ ), so  $M = PS(q)$ , contrary to the maximality of  $M$ . Thus  $M \cap BL(q) = \emptyset$ . ■

**Remark 5.2.2.** The maximality properties in Lemma 5.2.1 hold for  $PS(q)$  precisely when  $p = q$ . If  $p > q$ , then every  $\alpha \in PS(q)$  has rank  $p$ . This contrast with Lemma 5.2.1(i). Also, by Corollary 5.1.13, if  $p > q$  and  $q < k \leq p$ ,  $N_k$  is a maximal subsemigroup of  $PS(q)$  containing  $R(q) \cup BL(q)$ , this contrast with Lemma 5.2.1(ii).

As in Section 5.1, for any cardinal  $k$ , we let

$$N_k = \{\alpha \in PS(q) : g(\alpha) \neq k\}.$$

By Lemma 5.1.10 and Remark 5.1.11, if  $p = q$ , then  $N_k$  is a subsemigroup of  $PS(q)$  precisely when  $k = 0$  or  $k = p$ . From Corollary 5.1.13(ii), when  $p > q$ ,  $N_p$  is a maximal subsemigroup of  $PS(q)$ , but when  $p = q$ , Lemma 5.2.1(i) implies that  $N_p$  is not maximal since  $\emptyset \notin N_p$ . Moreover, Lemma 5.2.1(i) implies that every maximal subsemigroup of  $PS(q)$  must contain the largest proper ideal

$$J_p = \{\alpha \in PS(q) : r(\alpha) < p\}.$$

Note that  $J_p$  itself is a subsemigroup of  $PS(q)$  and it is contained in  $R(q)$  since, in case  $p = q$ ,  $r(\alpha) < p$  implies  $g(\alpha) = p = q$ . Moreover, this containment is always proper. For example, write  $X = A \dot{\cup} B$  where  $|A| = p = |B|$  and let  $\alpha : A \rightarrow B$  be a bijection, thus  $\alpha \in R(q) \setminus J_p$ . Hence  $J_p$  is not maximal in  $PS(q)$ .

**Theorem 5.2.3.** *Suppose that  $p = q \geq \aleph_0$  and let  $A$  be a non-empty subset of  $X$  such that  $|X \setminus A| \geq q$ . The following are maximal subsemigroups of  $PS(q)$ :*

- (i)  $\overline{M}_A$ ;
- (ii)  $N_0$ ;
- (iii)  $N_p \cup J_p$ .

**Proof.** If  $p = q$ , then  $S_q = PS(q)$ , and so (i) holds by Theorem 5.1.9. Also, by taking  $r = p$  in Theorem 5.1.12(i), we see that (ii) holds. To show that (iii) holds, take  $r = p = k$  in Lemma 5.1.10, we have  $N_p = S_p \setminus G_p$  is a subsemigroup of  $PS(q)$ . Moreover,  $N_p \cup J_p$  is also a subsemigroup of  $PS(q)$  since  $J_p$  is an ideal. To show the maximality of  $N_p \cup J_p$ , let  $\alpha, \beta \in PS(q) \setminus (N_p \cup J_p)$ . Then  $g(\alpha) = g(\beta) = p = r(\alpha) = r(\beta)$ . Write in the usual way

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix}$$

where  $|I| = p$ , and let

$$X \setminus \{a_i\} = A \dot{\cup} B \quad \text{and} \quad X \setminus \{y_i\} = C \dot{\cup} D$$

where  $|A| = |B| = |C| = |D| = p$ . Then define  $\gamma, \mu \in P(X)$  by

$$\gamma = \begin{pmatrix} b_i & X \setminus \{b_i\} \\ a_i & A \end{pmatrix}, \quad \mu = \begin{pmatrix} x_i & X \setminus \{x_i\} \\ y_i & C \end{pmatrix}$$

where  $\gamma|(X \setminus \{b_i\})$  and  $\mu|(X \setminus \{x_i\})$  are bijections. Thus  $\gamma, \mu \in PS(q)$  since  $d(\gamma) = |B| = p = |D| = d(\mu)$ . Moreover  $\gamma, \mu \in N_p \cup J_p$  since  $g(\gamma) = g(\mu) = 0 < p$ . It is clear that  $\beta = \gamma\alpha\mu$  and therefore  $N_p \cup J_p$  is maximal in  $PS(q)$ . ■

**Remark 5.2.4.** When  $p = q$ , if  $M$  is a maximal subsemigroup of  $PS(q)$  containing  $R(q)$ , then

$$M \subseteq (PS(q) \setminus BL(q)) = N_0$$

by Lemma 5.2.1(ii). Thus,  $M = N_0$  by the maximality of  $M$ . So we conclude that  $N_0$  is the only maximal subsemigroup of  $PS(q)$  containing  $R(q)$  when  $p = q$ .