

## Chapter 4

### Automorphisms and Isomorphisms of $R(q)$

In [13] Theorem 3, Sullivan showed that  $\text{Aut } PS(q)$  and  $G(X)$  are isomorphic when  $p = q$ . Later, in [12] Theorem 2, Pinto and Sullivan showed that this is also true when  $p > q$ . Here, we first consider the problem of describing all automorphisms of

$$R(q) = \{\alpha \in PS(q) : g(\alpha) = q\},$$

the largest regular (and also inverse) subsemigroup of  $PS(q)$ .

#### 4.1 Automorphisms

From Example 2.4.3,  $PS(q)$  is  $G(X)$ -normal and consequently the same is true for  $R(q)$ .

**Lemma 4.1.1.**  $R(q)$  is  $G(X)$ -normal.

**Proof.** Let  $h \in G(X)$ ,  $\alpha \in R(q)$ . Since  $R(q) \subseteq PS(q)$  and  $PS(q)$  is  $G(X)$ -normal, we have  $h\alpha h^{-1} \in PS(q)$ . To show that  $h\alpha h^{-1} \in R(q)$ , consider  $\text{dom } h\alpha h^{-1} = \text{dom } h\alpha = (Xh \cap \text{dom } \alpha)h^{-1} = (\text{dom } \alpha)h^{-1}$ . So

$$g(h\alpha h^{-1}) = |X \setminus (\text{dom } \alpha)h^{-1}| = |(X \setminus \text{dom } \alpha)h^{-1}| = q$$

since  $\alpha \in R(q)$ . Therefore  $h\alpha h^{-1} \in R(q)$ , that is,  $R(q)$  is  $G(X)$ -normal. ■

Levi showed in [6] that every automorphism of a  $G(X)$ -normal subsemigroup of  $P(X)$  is inner. Then by lemma 4.1.1, we have  $\varphi$  is inner for all  $\varphi \in \text{Aut } R(q)$ , that is there exists  $g \in G(X)$  such that  $\alpha\varphi = g\alpha g^{-1}$  for all  $\alpha \in R(q)$ . The next result gives more details on  $\text{Aut } R(q)$ .

**Lemma 4.1.2.** *For each  $\varphi \in \text{Aut } R(q)$ , there exists a unique  $\gamma \in G(X)$  such that  $\alpha\varphi = \gamma^{-1}\alpha\gamma$  for all  $\alpha \in R(q)$  and, in this event, we write  $\gamma = \gamma_\varphi$ .*

**Proof.** Let  $\varphi \in \text{Aut } R(q)$ . Then  $\varphi$  is inner, so there exists  $\gamma \in G(X)$  such that  $\alpha\varphi = \gamma^{-1}\alpha\gamma$  for all  $\alpha \in R(q)$ . Suppose there exists  $\mu \in G(X)$  such that  $\gamma^{-1}\alpha\gamma = \alpha\varphi = \mu^{-1}\alpha\mu$  for all  $\alpha \in R(q)$ . Let  $x \in X$  and write  $X = A \dot{\cup} B \dot{\cup} \{x\}$  where  $|A| = p$  and  $|B| = q$ . If  $\alpha = \text{id}_A$  and  $\beta = \text{id}_{A \dot{\cup} \{x\}}$ , then  $\alpha, \beta \in R(q)$ . This implies that

$$A\gamma = X\gamma^{-1}\alpha\gamma = X\mu^{-1}\alpha\mu = A\mu$$

and

$$(A \dot{\cup} \{x\})\gamma = X\gamma^{-1}\beta\gamma = X\mu^{-1}\beta\mu = (A \dot{\cup} \{x\})\mu.$$

Since  $\gamma$  and  $\mu$  are injective, we have

$$A\gamma \dot{\cup} \{x\gamma\} = A\mu \dot{\cup} \{x\mu\}$$

where  $A\gamma = A\mu$ . Thus  $x\gamma = x\mu$  for all  $x \in X$ , that is,  $\gamma = \mu$ . ■

The proof of the next result is similar to that for [12] Theorem 2.

**Theorem 4.1.3.**  *$\text{Aut } R(q)$  and  $G(X)$  are isomorphic.*

**Proof.** Define  $\theta : \text{Aut } R(q) \rightarrow G(X)$  by  $\varphi \mapsto \gamma_\varphi$  the unique permutation on  $X$  such that  $\alpha\varphi = \gamma_\varphi^{-1}\alpha\gamma_\varphi$  for all  $\alpha \in R(q)$  (possible by Lemma 4.1.2). To show  $\theta$  is a homomorphism, let  $\varphi, \psi \in \text{Aut } R(q)$ . Then for all  $\alpha \in R(q)$ , we have

$$\alpha(\varphi\psi) = \alpha(\varphi)\psi = (\gamma_\varphi^{-1}\alpha\gamma_\varphi)\psi = \gamma_\psi^{-1}(\gamma_\varphi^{-1}\alpha\gamma_\varphi)\gamma_\psi = (\gamma_\varphi\gamma_\psi)^{-1}\alpha(\gamma_\varphi\gamma_\psi).$$

Thus  $\gamma_{\varphi\psi} = \gamma_\varphi\gamma_\psi$  by the uniqueness of  $\gamma_{\varphi\psi}$  (Lemma 4.1.2). So  $\theta$  is a homomorphism. To show  $\theta$  is surjective, let  $\lambda \in G(X)$  and define

$$\varphi : R(q) \rightarrow R(q) \text{ by } \alpha \mapsto \lambda^{-1}\alpha\lambda.$$

Since  $R(q)$  is  $G(X)$ -normal, we have  $\varphi$  is a well-defined automorphism of  $R(q)$ . Thus  $\gamma_\varphi = \lambda$ , so  $(\varphi)\theta = \gamma_\varphi = \lambda$ , that is,  $\theta$  is onto. Finally, if  $\gamma_\varphi = \gamma_\psi$ , then  $\alpha\varphi = \gamma_\varphi^{-1}\alpha\gamma_\varphi = \gamma_\psi^{-1}\alpha\gamma_\psi = \alpha\psi$  for all  $\alpha \in R(q)$ , that is,  $\varphi = \psi$  and therefore  $\theta$  is one-to-one. ■

## 4.2 Isomorphisms

In what follows, we sometimes write  $R(X, p, q)$  in place of  $R(q)$  to highlight the underlying set  $X$  and its cardinal  $p$ . Since  $R(X, p, q)$  played an important role in both [4] and [12], it is natural to ask whether any of the semigroups  $R(X, p, q)$  are isomorphic for different cardinals  $p$  and  $q$ . To answer this question, we first need a result for  $R(q)$  which corresponds to [12] Lemma 1 for  $PS(q)$ .

**Lemma 4.2.1.** *If  $\alpha, \beta \in R(q)$  then the following are equivalent.*

- (i)  $X\alpha \subseteq X\beta$ ,
- (ii) for each  $\gamma \in R(q)$ ,  $\beta\gamma = \beta$  implies  $\alpha\gamma = \alpha$ .

**Proof.** If  $X\alpha \subseteq X\beta$  and  $\beta\gamma = \beta$  for some  $\gamma \in R(q)$ , then  $X\alpha \subseteq X\beta \subseteq \text{dom } \gamma$  and  $\gamma|_{X\beta} = \text{id}_{X\beta}$ . Hence  $(x\alpha)\gamma = x\alpha$  for each  $x\alpha \in X\alpha \subseteq X\beta$ , so  $\alpha\gamma = \alpha$ .

Conversely, suppose there exists  $y = x\alpha \notin X\beta = B$  say. Then  $\text{id}_B \in R(q)$  and  $\beta \circ \text{id}_B = \beta$  but  $y \text{id}_B \neq y$ ; that is,  $\alpha \circ \text{id}_B \neq \alpha$  and hence the condition does not hold. ■

**Corollary 4.2.2.** *Suppose that  $|X| = p \geq q \geq \aleph_0$  and  $|Y| = r \geq s \geq \aleph_0$ . If  $\varphi : R(X, p, q) \rightarrow R(Y, r, s)$  is an isomorphism then, for each  $\alpha, \beta \in R(X, p, q)$ ,  $X\alpha \subseteq X\beta$  if and only if  $Y(\alpha\varphi) \subseteq Y(\beta\varphi)$ .*

**Proof.** Suppose that  $\alpha, \beta \in R(X, p, q)$ . Then, since  $\varphi$  is an isomorphism, Lemma 4.2.1 provides the following equivalences.

$$\begin{aligned}
 X\alpha \subseteq X\beta &\iff \text{for each } \gamma \in R(X, p, q), \beta\gamma = \beta \text{ implies } \alpha\gamma = \alpha, \\
 &\iff \text{for each } \gamma \in R(X, p, q), \beta\varphi.\gamma\varphi = \beta\varphi \text{ implies } \alpha\varphi.\gamma\varphi = \alpha\varphi, \\
 &\iff \text{for each } \gamma' \in R(Y, r, s), \beta\varphi.\gamma' = \beta\varphi \text{ implies } \alpha\varphi.\gamma' = \alpha\varphi, \\
 &\iff Y(\alpha\varphi) \subseteq Y(\beta\varphi).
 \end{aligned}$$

Therefore we have proved the corollary. ■

**Theorem 4.2.3.** *The semigroups  $R(X, p, q)$  and  $R(Y, r, s)$  are isomorphic if and only if  $p = r$  and  $q = s$ . Moreover, for each isomorphism  $\varphi : R(X, p, q) \rightarrow R(Y, r, s)$ , there is a bijection  $\gamma : X \rightarrow Y$  such that  $\alpha\varphi = \gamma^{-1}\alpha\gamma$  for each  $\alpha \in R(X, p, q)$ .*

**Proof.** We assume that there is an isomorphism  $\varphi : R(X, p, q) \rightarrow R(Y, r, s)$  and write

$$U = \{X\alpha : \alpha \in R(X, p, q)\}, \quad V = \{Y\beta : \beta \in R(Y, r, s)\}.$$

Let  $\Gamma : U \rightarrow V$  be defined by  $(X\alpha)\Gamma = Y(\alpha\varphi)$ . Then, by Corollary 4.2.2,  $\Gamma$  is an order-monomorphism: that is,  $\Gamma$  is injective and  $A \subseteq B$  if and only if  $A\Gamma \subseteq B\Gamma$  for all  $A, B \in U$ . Next, if  $C = Y\beta$  for some  $\beta \in R(Y, r, s)$ , then  $\beta = \alpha\varphi$  for some  $\alpha \in R(X, p, q)$  (since  $\varphi$  is onto). Thus  $(X\alpha)\Gamma = Y(\alpha\varphi) = Y\beta = C$ , so  $\Gamma$  is onto. In fact, if

$$\mathcal{B}(X, q) = \{A \subseteq X : |X \setminus A| = q\}, \quad \mathcal{B}(Y, s) = \{B \subseteq Y : |Y \setminus B| = s\}$$

then  $U = \mathcal{B}(X, q)$  and  $V = \mathcal{B}(Y, s)$ , since  $\text{id}_A \in R(X, p, q)$  and  $\text{id}_B \in R(Y, r, s)$  for all  $A \in \mathcal{B}(X, q)$  and  $B \in \mathcal{B}(Y, s)$ . That is,  $\Gamma$  is an order-isomorphism from  $\mathcal{B}(X, q)$  onto  $\mathcal{B}(Y, s)$ . Thus by Lemma 2.4.8, there exists a bijection  $\gamma : X \rightarrow Y$  such that  $A\Gamma = A\gamma$  for all  $A \in \mathcal{B}(X, q)$ , so  $p = r$ . Now we aim to show  $\alpha\varphi = \gamma^{-1}\alpha\gamma$  for each  $\alpha \in R(X, p, q)$ . Clearly this holds if  $\alpha = \emptyset$  (in this case,  $p = q$ ). So suppose  $\alpha \neq \emptyset$  and note that  $\text{dom } \alpha\gamma = \text{dom } \alpha$  since  $\text{dom } \gamma = X$ . Let  $x \in \text{dom } \alpha$  and write  $X = C \dot{\cup} D \dot{\cup} \{x\}$  where  $|C| = p$  and  $|D| = q$ . Then  $\beta = \text{id}_C, \lambda = \text{id}_{C \cup \{x\}} \in R(X, p, q)$ . Let  $A = X\beta$  and  $B = X\lambda$ , we have  $A, B \in \mathcal{B}(X, q)$ ,  $B \setminus A = X\lambda \setminus X\beta = \{x\}$  and

$$\begin{aligned} Y((\lambda\alpha)\varphi) \setminus Y((\beta\alpha)\varphi) &= Y((\lambda\varphi)(\alpha\varphi)) \setminus Y((\beta\varphi)(\alpha\varphi)) \\ &= (Y(\lambda\varphi) \setminus Y(\beta\varphi))(\alpha\varphi) \\ &= ((X\lambda)\Gamma \setminus (X\beta)\Gamma)(\alpha\varphi) \\ &= (B\Gamma \setminus A\Gamma)(\alpha\varphi) \\ &= (B\gamma \setminus A\gamma)(\alpha\varphi) \\ &= (B \setminus A)\gamma(\alpha\varphi) \\ &= \{x\}\gamma(\alpha\varphi) \end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
Y((\lambda\alpha)\varphi) \setminus Y((\beta\alpha)\varphi) &= (X(\lambda\alpha))\Gamma \setminus (X(\beta\alpha))\Gamma \\
&= (X(\lambda\alpha))\gamma \setminus (X(\beta\alpha))\gamma \\
&= (X\lambda \setminus X\beta)\alpha\gamma \\
&= \{x\}\alpha\gamma.
\end{aligned}$$

Thus  $\{x\}\gamma(\alpha\varphi) = \{x\}\alpha\gamma$  for all  $x \in \text{dom } \alpha$ . We now aim to show that  $\text{dom}(\gamma(\alpha\varphi)) = \text{dom}(\alpha\gamma)$ . To do this, we first note that,  $\alpha \in R(X, p, q)$  implies  $\alpha^{-1} \in R(X, p, q)$  since  $d(\alpha^{-1}) = g(\alpha) = q$  and  $g(\alpha^{-1}) = d(\alpha) = p$ . It follows that  $\text{dom } \alpha = X\alpha^{-1} \in U = \mathcal{B}(X, q)$ . Also, since  $\varphi$  is an isomorphism and  $R(X, p, q)$  and  $R(Y, r, s)$  are inverse semigroups, we have  $(\alpha\varphi)^{-1} = \alpha^{-1}\varphi$ . Thus,  $(\text{dom } \alpha)\gamma = (X\alpha^{-1})\gamma = (X\alpha^{-1})\Gamma = Y(\alpha^{-1}\varphi) = \text{dom}(\alpha\varphi)$ , that is,  $\text{dom } \alpha = (\text{dom}(\alpha\varphi))\gamma^{-1}$ . Consequently, together with  $\text{dom } \gamma = X$  and  $X\gamma = Y$ , we have

$$\text{dom}(\alpha\gamma) = \text{dom } \alpha = (\text{dom}(\alpha\varphi))\gamma^{-1} = (X\gamma \cap \text{dom}(\alpha\varphi))\gamma^{-1} = \text{dom}(\gamma(\alpha\varphi)).$$

Therefore  $\gamma(\alpha\varphi) = \alpha\gamma$  and so  $\alpha\varphi = \gamma^{-1}\alpha\gamma$ . Finally, since  $\alpha\varphi \in R(Y, r, s)$ , we have  $s = |Y \setminus Y(\alpha\varphi)| = |Y \setminus Y\gamma^{-1}\alpha\gamma| = |X\gamma \setminus X\alpha\gamma| = |(X \setminus X\alpha)\gamma| = q$ .

Conversely, if  $p = r$  and  $q = s$ , then Theorem 2.4.7 implies that, there is an isomorphism  $\varphi : PS(X, p, q) \rightarrow PS(Y, r, s)$ . Recall that every elements in  $R(X, p, q)$  and  $R(Y, r, s)$  are all regular, thus, when restrict  $\varphi$  to  $R(X, p, q)$  we have  $(R(X, p, q))\varphi = R(Y, r, s)$  since  $\varphi$  and  $\varphi^{-1}$  preserve the regularity. Therefore  $R(X, p, q)$  and  $R(Y, r, s)$  are isomorphic. ■