

## Chapter 3

# Partial Orders on the Baer-Levi Semigroups of Partial Transformations

Throughout this thesis,  $|X| = p \geq q \geq \aleph_0$ . Also,  $Y = A \dot{\cup} B$  means that  $Y$  is a *disjoint* union of  $A$  and  $B$ . We modify the convention introduced in [1] vol 2, p 241: namely, if  $\alpha \in I(X)$  is non-zero then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript  $i$  belongs to some (unmentioned) index set  $I$ , that the abbreviation  $\{x_i\}$  denotes  $\{x_i : i \in I\}$ , and that  $\text{ran } \alpha = \{x_i\}$ ,  $x_i \alpha^{-1} = \{a_i\}$  and  $\text{dom } \alpha = \{a_i : i \in I\}$ . For simplicity, we often write  $X\alpha$  in place of  $\text{ran } \alpha$ , in which case  $X\alpha^{-1} = \text{ran } \alpha^{-1} = \text{dom } \alpha$ .

In this chapter, we consider the natural partial order  $\leq$ , the containment order  $\subseteq$  and other partial orders defined on  $I(X)$  and  $PS(q)$ .

### 3.1 Partial Orders

In order to characterise  $\leq$  on  $PS(q)$ , we first define the relation  $\mathbb{L}$  on  $PS(q)$  by

$$(\alpha, \beta) \in \mathbb{L} \quad \text{if and only if} \quad PS(q)^1 \alpha \subseteq PS(q)^1 \beta.$$

It is easy to see that  $\mathbb{L}$  is reflexive and transitive. However, in general, it is not anti-symmetric. For example, Let  $X = A \dot{\cup} B \dot{\cup} \{c, d, e\}$  where  $|A| = p$  and  $|B| = q$ , and define  $\alpha, \beta, \lambda, \mu \in PS(q)$  by

$$\alpha = \text{id}_A \cup \begin{pmatrix} d \\ c \end{pmatrix}, \beta = \text{id}_A \cup \begin{pmatrix} e \\ c \end{pmatrix}, \lambda = \text{id}_A \cup \begin{pmatrix} d \\ e \end{pmatrix}, \mu = \text{id}_A \cup \begin{pmatrix} e \\ d \end{pmatrix}.$$

Then  $\alpha = \lambda\beta$  and  $\beta = \mu\alpha$ , so  $(\alpha, \beta) \in \mathbb{L}$  and  $(\beta, \alpha) \in \mathbb{L}$ , but  $\alpha \neq \beta$ .

Nonetheless, if  $\rho$  is any partial order on  $PS(q)$ , then  $\rho \cap \mathbb{L}$  is also a partial order on  $PS(q)$ . This idea leads to a simple description of  $\leq$  on  $PS(q)$ .

**Theorem 3.1.1.** *When restricted to  $PS(q)$ ,  $\leq$  equals  $\subseteq \cap \mathbb{L}$ . Moreover,  $\leq$  is properly contained in  $\subseteq$ .*

**Proof.** Suppose that  $\alpha, \beta \in PS(q)$  are distinct and  $\alpha \leq \beta$  in  $PS(q)$ . Then  $\alpha = \lambda\beta = \beta\mu$  and  $\alpha = \alpha\mu$  for some  $\lambda, \mu \in PS(q)$ , and so

$$PS(q)^1\alpha = PS(q)^1\lambda\beta \subseteq PS(q)^1\beta,$$

that is,  $(\alpha, \beta) \in \mathbb{L}$ . Also,  $\alpha = \lambda\beta$  implies  $X\alpha \subseteq X\beta$  and  $\alpha = \alpha\mu$  implies  $X\alpha \subseteq \text{dom } \mu$ . Hence

$$\alpha\alpha^{-1} = \alpha\mu(\beta\mu)^{-1} = \alpha(\mu\mu^{-1})\beta^{-1} = \alpha\beta^{-1},$$

and so  $\alpha \subseteq \beta$  by Theorem 2.5.6. Therefore,  $\leq$  is a subset of  $\subseteq \cap \mathbb{L}$ .

Conversely, suppose that  $(\alpha, \beta) \in \subseteq \cap \mathbb{L}$  and  $\alpha \neq \beta$ . Then  $PS(q)^1\alpha \subseteq PS(q)^1\beta$  and so  $\alpha = \lambda\beta$  for some  $\lambda \in PS(q)$ . Moreover, since  $\alpha \subseteq \beta$ , we can write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \beta = \begin{pmatrix} a_i & a_j \\ x_i & x_j \end{pmatrix}, \mu = \begin{pmatrix} x_i \\ x_i \end{pmatrix},$$

where  $d(\mu) = d(\alpha) = q$ . Hence  $\mu \in PS(q)$  and clearly  $\alpha = \beta\mu$  and  $\alpha = \alpha\mu$ . Therefore,  $\alpha \leq \beta$  in  $PS(q)$ .

Now we deduce that  $\leq$  is a subset of  $\subseteq$  on  $PS(q)$  and we assert that this containment is always proper on  $PS(q)$ . To see this, we suppose  $X = A \dot{\cup} B \dot{\cup} \{c\}$  where  $|A| = p$  and  $|B| = q$ , and let  $\alpha : A \cup B \rightarrow A$  be a bijection. Then  $d(\alpha) = |B \cup \{c\}| = q$  and so  $\alpha \in PS(q)$ . Likewise, if  $\beta \in T(X)$  equals  $\alpha$  on  $A \cup B$  and satisfies  $c\beta = c$ , then  $\beta \in PS(q)$  and  $\alpha \subseteq \beta$ . But  $g(\beta) = 0 < q$  and  $|X\beta \setminus X\alpha| = 1 < q$ , hence there is no  $\lambda \in PS(q)$  such that  $\alpha = \lambda\beta$  by Theorem 2.3.10. This follows that  $(\alpha, \beta) \notin \mathbb{L}$  and so  $\alpha \not\leq \beta$ . ■

In [11] p 384 and Lemma 1(x), Mitsch observed that, if  $S$  is an inverse semi-group, then the natural partial order on  $S$  equals the order  $\preceq$  defined on  $S$  by

$$a \preceq b \quad \text{if and only if} \quad a = eb \quad \text{for some idempotent } e \in S.$$

Moreover, from [3] Proposition V.2.3, we know that  $\preceq$  equals  $\subseteq$  on  $I(X)$ , and thus  $\leq = \subseteq$  on  $I(X)$ .

We recall from Chapter 2, Section 2.5 that  $\Omega'$  and  $\Omega$  are partial orders defined on  $P(X)$  by

$$\begin{aligned} (\alpha, \beta) \in \Omega' & \text{ if and only if } X\alpha \subseteq X\beta, & \text{dom } \alpha \subseteq \text{dom } \beta \text{ and} \\ & \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}, \\ (\alpha, \beta) \in \Omega & \text{ if and only if } (\alpha, \beta) \in \Omega' & \text{ and } \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}. \end{aligned}$$

In [10], the authors showed that  $\Omega'$  is an upper bound for  $\leq$  and  $\subseteq$ , and that  $\Omega = \leq \vee \subseteq = \subseteq \circ \leq$  on  $P(X)$ . Clearly  $\Omega \subseteq \Omega'$  and these are also partial orders on  $I(X)$ , a semigroup in which  $\leq = \subseteq$ . Therefore, we get the following result.

**Theorem 3.1.2.**  $\Omega = \Omega'$  on  $I(X)$  and  $PS(q)$ .

**Proof.** Suppose that  $\alpha, \beta \in I(X)$  and  $(\alpha, \beta) \in \Omega'$ . Then  $\text{dom } \alpha \subseteq \text{dom } \beta$  and  $\beta\beta^{-1} = \text{id}_{\text{dom } \beta}$ , so

$$\beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \text{id}_{\text{dom } \alpha} = \alpha\alpha^{-1}.$$

Hence  $(\alpha, \beta) \in \Omega$ , and thus  $\Omega' \subseteq \Omega$  on  $I(X)$  as required. When  $\alpha, \beta \in PS(q)$ , we can use the same proof for  $\alpha, \beta \in I(X)$  to obtain  $\Omega' \subseteq \Omega$  on  $PS(q)$ . ■

Since  $\leq = \subseteq$  and  $\Omega = \Omega'$  on  $I(X)$ , it is natural to ask whether all four orders are equal on  $I(X)$ . In [10], the authors showed that  $\subseteq$  is contained in  $\Omega'$  in  $P(X)$ . The next result shows that this also holds in  $I(X)$  and  $PS(q)$ . We first note that  $\Omega = \subseteq$  on  $I(X)$  when  $|X| = 1$ .

**Theorem 3.1.3.** If  $|X| > 1$ , then  $\subseteq$  is properly contained in  $\Omega$  on  $I(X)$ .

**Proof.** Assume that  $|X| > 1$ . If  $\alpha, \beta$  are injective and  $\alpha \subseteq \beta$ , then  $\alpha\beta^{-1} = \alpha\alpha^{-1}$ . So  $X\alpha \subseteq X\beta$ ,  $\text{dom } \alpha \subseteq \text{dom } \beta$  and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \alpha\alpha^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \alpha\alpha^{-1}.$$

That is,  $(\alpha, \beta) \in \Omega' = \Omega$ . Since  $|X| > 1$ , we can choose distinct  $x, y \in X$  and define  $\alpha, \beta \in I(X)$  by

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \beta = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

Then  $X\alpha \subseteq X\beta$ ,  $\text{dom } \alpha \subseteq \text{dom } \beta$  and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \emptyset \subseteq \alpha\alpha^{-1}.$$

Hence  $(\alpha, \beta) \in \Omega' = \Omega$  but  $\alpha \not\subseteq \beta$ , so  $\subseteq$  is properly contained in  $\Omega$  on  $I(X)$ . ■

**Theorem 3.1.4.**  $\subseteq$  is properly contained in  $\Omega$  on  $PS(q)$ .

**Proof.** From the first part of the proof in Theorem 3.1.3, we see that  $\subseteq$  is also contained in  $\Omega$  on  $PS(q)$ . Suppose that  $X = A \dot{\cup} B \dot{\cup} \{x\} \dot{\cup} \{y\}$  where  $|A| = p$  and  $|B| = q$ , and let  $\theta : A \cup B \rightarrow A$  be a bijection. Define  $\alpha, \beta \in PS(q)$  by

$$\alpha = \begin{pmatrix} A \cup B & x \\ A & x \end{pmatrix}, \beta = \begin{pmatrix} A \cup B & x & y \\ A & y & x \end{pmatrix}$$

where  $\alpha|(A \cup B) = \theta = \beta|(A \cup B)$ , we see that  $\alpha \not\subseteq \beta$ . Since  $y \notin \text{dom } \alpha$ , we have

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \text{id}_{A \cup B} \subseteq \text{id}_{\text{dom } \alpha} = \alpha\alpha^{-1},$$

that is,  $(\alpha, \beta) \in \Omega$ . Therefore  $\subseteq$  is always properly contained in  $\Omega$ . ■

From [10],  $\Omega$  is the join of  $\subseteq$  and  $\leq$  on  $P(X)$  and it equals  $\subseteq \circ \leq$ . But,  $\Omega \neq \subseteq \circ \leq$  on  $PS(q)$ : otherwise,  $\Omega$  is contained in  $\subseteq \circ \subseteq$  (since  $\leq$  is contained in  $\subseteq$ ), so  $\Omega$  is contained in  $\subseteq$ , which is a contradiction.

From Theorem 3.1.1 and Theorem 3.1.4, on  $PS(q)$  we always have:

$$\leq = \subseteq \cap \mathbb{L} \quad \not\subseteq \quad \subseteq \quad \not\subseteq \quad \Omega.$$

## 3.2 Compatible Partial Orders

We first note that  $\subseteq$  is left and right compatible on  $P(X)$ . Therefore, it is also left and right compatible on  $PS(q)$  since  $PS(q)$  is contained in  $P(X)$ . In

this section, since we know that  $\Omega = \Omega'$  on  $PS(q)$ , we will only characterize the compatibility of  $\leq$  and  $\Omega$  on  $PS(q)$ .

**Theorem 3.2.1.** *Suppose that  $\gamma \in PS(q)$ .*

- (i)  $\gamma$  is left compatible with  $\leq$  on  $PS(q)$  if and only if  $q \leq g(\gamma)$ ,
- (ii)  $\leq$  is right compatible on  $PS(q)$ .

**Proof.** To prove (i), suppose that  $\gamma$  is left compatible with  $\leq$ . If  $\gamma = \emptyset$  (in case  $p = q$ ), then  $g(\gamma) = p = q$ . If  $\gamma \neq \emptyset$ , we choose  $x \in X\gamma$  and let  $\alpha = \text{id}_{X\gamma \setminus \{x\}}$  and  $\beta = \text{id}_{X\gamma}$ . Then  $\alpha \subseteq \beta$  and  $g(\beta) = d(\beta) = d(\gamma) = q$  and  $g(\alpha) = d(\alpha) = q + 1 = q$  (since  $q \geq \aleph_0$ ). Hence  $\alpha, \beta \in PS(q)$  and

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) = q = \max(g(\alpha), q).$$

Then  $\alpha = \lambda\beta$  for some  $\lambda \in PS(q)$  by Theorem 2.3.10 and so  $PS(q)^1\alpha \subseteq PS(q)^1\beta$ , that is,  $(\alpha, \beta) \in \mathbb{L}$ . It follows that  $\alpha \leq \beta$  by Theorem 3.1.1. Since  $\gamma$  is left compatible, we have  $\gamma\alpha \leq \gamma\beta$  and so  $(\gamma\alpha, \gamma\beta) \in \mathbb{L}$  (since  $\leq = \subseteq \cap \mathbb{L}$ ) where  $\gamma\alpha \neq \gamma\beta = \gamma$ . Then Theorem 2.3.10 implies that

$$q \leq \max(g(\gamma\beta), |X\gamma\beta \setminus X\gamma\alpha|).$$

But, since  $|X\gamma\beta \setminus X\gamma\alpha| = 1 < q$ , this implies  $q \leq g(\gamma\beta) = g(\gamma)$ .

Conversely, suppose that  $q \leq g(\gamma)$ . If  $\alpha, \beta \in PS(q)$  and  $\alpha \leq \beta$ , then  $\alpha \subseteq \beta$  and  $(\alpha, \beta) \in \mathbb{L}$  by Theorem 3.1.1. Since  $\subseteq$  is left compatible, we have  $\gamma\alpha \subseteq \gamma\beta$ . Also,  $\text{dom } \gamma\beta \subseteq \text{dom } \gamma$  implies  $q \leq g(\gamma) \leq g(\gamma\beta)$ . By the definition of  $\leq$ , we have  $\alpha = \beta\mu$  for some  $\mu \in PS(q)^1$ . If  $\mu = 1$ , then  $\alpha = \beta$  and so  $\gamma\alpha \leq \gamma\beta$ . If  $\mu \in PS(q)$ , then  $\gamma\alpha = (\gamma\beta)\mu$  and hence  $g(\gamma\beta) \leq g(\gamma\alpha)$ . Moreover, since  $\gamma\alpha \in PS(q)$ ,

$$|X\gamma\beta \setminus X\gamma\alpha| = |X\gamma\beta \cap (X \setminus X\gamma\alpha)| \leq q$$

and so

$$q \leq g(\gamma\beta) = \max(g(\gamma\beta), |X\gamma\beta \setminus X\gamma\alpha|) \leq g(\gamma\alpha) = \max(g(\gamma\alpha), q),$$

that is,  $(\gamma\alpha, \gamma\beta) \in \mathbb{L}$ . Since  $\leq = \subseteq \cap \mathbb{L}$ , we have  $\gamma\alpha \leq \gamma\beta$ . In both cases we deduce that  $\gamma$  is left compatible with respect to  $\leq$  as required. Finally, note that

$\subseteq$  is right compatible, and  $PS(q)^1\alpha \subseteq PS(q)^1\beta$  implies  $PS(q)^1\alpha\gamma \subseteq PS(q)^1\beta\gamma$  for any  $\gamma \in PS(q)$ , that is,  $\mathbb{L}$  is also right compatible on  $PS(q)$ . Hence, (ii) follows from Theorem 3.1.1.  $\blacksquare$

Here, for simplicity, we write  $x_y$  for the  $\alpha \in I(X)$  with domain  $\{x\}$  and range  $\{y\}$ .

**Theorem 3.2.2.** *Suppose that  $p = q$  and let  $\gamma \in PS(q)$ . Then*

- (i)  $\emptyset$  is the only element of  $PS(q)$  which is left compatible with  $\Omega$ ,
- (ii)  $\gamma$  is right compatible with  $\Omega$  if and only if  $\gamma = \emptyset$  or  $\text{dom } \gamma = X$ .

**Proof.** Clearly  $\emptyset \in PS(q)$  and it is left and right compatible with  $\Omega$ . Let  $\gamma$  be a non-zero element in  $PS(q)$ . If we choose  $x \in X\gamma$ ,  $y \in X \setminus X\gamma$  and define

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

then  $\alpha, \beta \in PS(q)$  (since  $p = q$ ) and it is easy to check that  $(\alpha, \beta) \in \Omega$ . However, since  $X\gamma\alpha = \{x\} \not\subseteq \{y\} = X\gamma\beta$ , we have  $(\gamma\alpha, \gamma\beta) \notin \Omega$  (by definition) and so  $\gamma$  is not left compatible with  $\Omega$ . Therefore (i) holds.

Next, suppose that  $\gamma \in PS(q)$  is non-empty and right compatible with  $\Omega$ . If  $a \in \text{dom } \gamma$ ,  $x \in X \setminus \text{dom } \gamma$  and  $Y = \{a, x\}$  then  $x_a, \text{id}_Y \in PS(q)$  and  $(x_a, \text{id}_Y) \in \Omega$  (note that  $x_a \cdot \text{id}_Y^{-1} \cap \{(x, x)\} = \emptyset$ ). Hence  $(x_a \cdot \gamma, \text{id}_Y \cdot \gamma) \in \Omega$  and so  $\text{dom}(x_a \cdot \gamma) = \{x\} \subseteq \text{dom}(\text{id}_Y \cdot \gamma) = \{a\}$ , a contradiction. Thus, we have shown that  $\text{dom } \gamma = X$ . Therefore, to prove (ii), it remains to show that, if  $\text{dom } \gamma = X$ , then  $\gamma$  is right compatible with  $\Omega$ . To do this, let  $\alpha, \beta \in PS(q)$  and  $(\alpha, \beta) \in \Omega$ . Then, since  $\Omega = \Omega'$ , we have  $X\alpha \subseteq X\beta$ ,  $\text{dom } \alpha \subseteq \text{dom } \beta$  and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

Clearly  $X\alpha\gamma \subseteq X\beta\gamma$  and, since  $\text{dom } \gamma = X$ ,  $\text{dom } \alpha\gamma = \text{dom } \alpha \subseteq \text{dom } \beta = \text{dom } \beta\gamma$ . Also  $\gamma\gamma^{-1} = \text{id}_X$  (but note that  $\text{id}_X \notin PS(q)$ ), and hence

$$\alpha\gamma(\beta\gamma)^{-1} \cap (\text{dom } \alpha\gamma \times \text{dom } \alpha\gamma) = \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1} = \alpha\gamma(\alpha\gamma)^{-1},$$

from which it follows that  $(\alpha\gamma, \beta\gamma) \in \Omega$ .  $\blacksquare$

**Theorem 3.2.3.** *Suppose that  $p > q$  and let  $\gamma \in PS(q)$ . Then*

- (i) *no element of  $PS(q)$  is left compatible with  $\Omega$ ,*
- (ii)  *$\gamma$  is right compatible with  $\Omega$  if and only if  $\text{dom } \gamma = X$ .*

**Proof.** To prove (i), let  $\theta \in PS(q)$ , choose  $x \in X\theta$ ,  $y \in X \setminus X\theta$  and define

$$\alpha = \text{id}_{X\theta}, \quad \beta = \begin{pmatrix} X\theta \setminus \{x\} & x & y \\ X\theta \setminus \{x\} & y & x \end{pmatrix},$$

where  $z\beta = z$  for all  $z \in X\theta \setminus \{x\}$ . Then  $\alpha, \beta \in PS(q)$  and  $(\alpha, \beta) \in \Omega$ . Since  $x \in X\theta\alpha \setminus X\theta\beta$ , we have  $(\theta\alpha, \theta\beta) \notin \Omega$  (by definition). That is,  $\theta$  is not left compatible with  $\Omega$ . The proof of (ii) is the same as that for Theorem 3.2.2(ii), except that now  $\emptyset \notin PS(q)$ . ■

For completeness, we note the following result for  $\Omega$  on  $I(X)$ .

**Theorem 3.2.4.** *If  $\gamma \in I(X)$  is non-zero, then*

- (i)  *$\gamma$  is left compatible with  $\Omega$  on  $I(X)$  if and only if  $X\gamma = X$ ,*
- (ii)  *$\gamma$  is right compatible with  $\Omega$  on  $I(X)$  if and only if  $\text{dom } \gamma = X$ .*

**Proof.** Suppose that  $X\gamma \neq X$ . Then, as in the proof of Theorem 3.2.3(i), there exists  $(\alpha, \beta) \in \Omega$  on  $I(X)$  but  $(\gamma\alpha, \gamma\beta) \notin \Omega$ . Therefore  $\gamma$  is not left compatible with  $\Omega$ . For the converse of (i), if  $\gamma \in PS(q)$  is surjective, then Theorem 2.5.7 (i) implies that it is left compatible with  $\Omega$  on  $P(X)$ , and so the same is true for  $I(X)$ . To see (ii), suppose that  $\gamma \in I(X)$  is non-empty and right compatible with  $\Omega$ . If  $a \in \text{dom } \gamma$ ,  $x \in X \setminus \text{dom } \gamma$  and  $Y = \{a, x\}$  then  $x_a, \text{id}_Y \in I(X)$  and  $(x_a, \text{id}_Y) \in \Omega$  (note that  $x_a \cdot \text{id}_Y^{-1} \cap \{(x, x)\} = \emptyset$ ). Hence  $(x_a \cdot \gamma, \text{id}_Y \cdot \gamma) \in \Omega$  and so  $\text{dom}(x_a \cdot \gamma) = \{x\} \subseteq \text{dom}(\text{id}_Y \cdot \gamma) = \{a\}$ , a contradiction. Thus, we have shown that  $\text{dom } \gamma = X$ . It remains to show that, if  $\text{dom } \gamma = X$ , then  $\gamma$  is right compatible with  $\Omega$ . To do this, let  $\alpha, \beta \in I(X)$  and  $(\alpha, \beta) \in \Omega$ . Then, since  $\Omega = \Omega'$ , we have  $X\alpha \subseteq X\beta$ ,  $\text{dom } \alpha \subseteq \text{dom } \beta$  and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

Clearly  $X\alpha\gamma \subseteq X\beta\gamma$  and, since  $\text{dom } \gamma = X$ ,  $\text{dom } \alpha\gamma = \text{dom } \alpha \subseteq \text{dom } \beta = \text{dom } \beta\gamma$ .

Also  $\gamma\gamma^{-1} = \text{id}_X$ , and hence

$$\alpha\gamma(\beta\gamma)^{-1} \cap (\text{dom } \alpha\gamma \times \text{dom } \alpha\gamma) = \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1} = \alpha\gamma(\alpha\gamma)^{-1}.$$

Therefore  $(\alpha\gamma, \beta\gamma) \in \Omega$ . ■

### 3.3 Minimal and Maximal Elements

In this section, we consider the existence of minimal (maximal) elements in  $PS(q)$  with respect to each of the orders  $\leq$ ,  $\subseteq$  and  $\Omega$ .

First, recall that, if  $\preceq$  is any partial order on a set  $T$ , and if  $x \in S \subseteq T$  is minimal (maximal) in  $T$ , then  $x$  is minimal (maximal) in  $S$ . Similarly, suppose  $<_1$  and  $<_2$  are partial orders on a set  $S$  such that  $<_2$  contains  $<_1$ . Clearly, if  $x \in S$  is minimal (maximal) with respect to  $<_2$ , then  $x$  is minimal (maximal) with respect to  $<_1$ . Consequently, under the same supposition, if  $x$  is a minimum (maximum) with respect to  $<_1$ , then  $x$  is a minimum (maximum) with respect to  $<_2$ .

**Theorem 3.3.1.**  *$PS(q)$  has no maximum element with respect to  $\leq$ ,  $\subseteq$  or  $\Omega$ .*

**Proof.** Write  $X = A \dot{\cup} B \dot{\cup} C$  where  $|A| = p$  and  $|B| = q = |C|$ . Clearly, if  $\alpha = \text{id}_{A \cup B}$  and  $\beta = \text{id}_{A \cup C}$ , then  $\alpha, \beta \in PS(q)$ . If  $\gamma \in PS(q)$  is a maximum with respect to  $\Omega$ , then  $(\alpha, \gamma) \in \Omega$  and  $(\beta, \gamma) \in \Omega$ . Consequently  $X\alpha \subseteq X\gamma$  and  $X\beta \subseteq X\gamma$ , hence  $X\alpha \cup X\beta \subseteq X\gamma$  and so  $X\gamma = X$ , which contradicts  $d(\gamma) = q$ . Therefore  $PS(q)$  has no maximum element with respect to  $\Omega$ . Next recall that  $\leq$  is properly contained in  $\subseteq$  which is properly contained in  $\Omega$  on  $PS(q)$ . So, if  $\alpha$  is a maximum under  $\subseteq$ , then it is also a maximum under  $\Omega$ , a contradiction. Likewise, there is no maximum under  $\leq$ . ■

**Theorem 3.3.2.** *The following are equivalent for  $\alpha \in PS(q)$ .*

- (i)  $\alpha$  is maximal with respect to  $\Omega$ ,
- (ii)  $\alpha$  is maximal with respect to  $\subseteq$ ,
- (iii)  $\text{dom } \alpha = X$ .

**Proof.** (i) implies (ii) since  $\subseteq$  is contained in  $\Omega$ . To show (ii) implies (iii), suppose that (ii) holds and assume  $\text{dom } \alpha \subsetneq X$ . Choose  $x \in X \setminus \text{dom } \alpha$  and  $y \in X \setminus X\alpha$  (recall that  $d(\alpha) = q$ ) and let  $\beta$  be the mapping such that  $\text{dom } \beta = \text{dom } \alpha \cup \{x\}$ ,  $\beta|_{\text{dom } \alpha} = \alpha$  and  $x\beta = y$ . Then  $\beta \in PS(q)$  and  $\alpha \subseteq \beta$  with  $\alpha \neq \beta$ , contradicting our supposition.

Finally, to show (iii) implies (i), suppose that  $\text{dom } \alpha = X$  and let  $\beta \in PS(q)$  satisfy  $(\alpha, \beta) \in \Omega$ . Then by Theorem 3.1.2, we have

$$\text{dom } \alpha \subseteq \text{dom } \beta, X\alpha \subseteq X\beta \text{ and } \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

So  $\text{dom } \beta = X$ . For each  $x \in X$ , if  $x\alpha = y\beta$  for some  $y \in X$ , then  $x\alpha\beta^{-1} = y$  and thus

$$(x, y) \in \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1} = \text{id}_X.$$

This follows that  $x = y$ , that is,  $x\alpha = x\beta$  and hence  $\alpha = \beta$ . This shows that (i) holds. ■

The corresponding result for  $\leq$  is substantially different.

**Theorem 3.3.3.** *Let  $\alpha \in PS(q)$ . Then  $\alpha$  is maximal with respect to  $\leq$  if and only if  $g(\alpha) < q$ .*

**Proof.** Suppose that  $g(\alpha) \geq q$ . Then  $X \setminus \text{dom } \alpha \neq \emptyset$ . Choose  $x \in X \setminus \text{dom } \alpha$  and  $y \in X \setminus X\alpha$  (recall that  $d(\alpha) = q$ ) and let  $\beta$  be the mapping such that  $\text{dom } \beta = \text{dom } \alpha \cup \{x\}$ ,  $\beta|_{\text{dom } \alpha} = \alpha$  and  $x\beta = y$ . Then  $\beta \in PS(q)$  and  $\alpha \subseteq \beta$  with  $\alpha \neq \beta$ ,  $X\beta = X\alpha \cup \{y\}$  and  $g(\alpha) = g(\beta)$ . So  $|X\beta \setminus X\alpha| = 1$  and hence the inequation in Theorem 2.3.10 is satisfied, that is,  $\alpha = \lambda\beta$  for some  $\lambda \in PS(q)$ . Then  $(\alpha, \beta) \in \mathbb{L}$  and it follows that  $\alpha \leq \beta$  by Theorem 3.1.1, but  $\alpha \neq \beta$ , so  $\alpha$  is not maximal. Conversely, suppose that  $g(\alpha) < q$  and assume that  $\alpha < \beta$  for some  $\beta \in PS(q)$ . Thus, by Theorem 3.1.1, we have  $\alpha \subsetneq \beta$  and, by Theorem 2.3.10

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q) = q.$$

Therefore,  $g(\beta) \leq g(\alpha) < q$  and so  $|X\beta \setminus X\alpha| = q$ . Consequently, since  $X\alpha \subseteq X\beta$ , then

$$q = |(X\beta \setminus X\alpha)\beta^{-1}| = |\text{dom } \beta \setminus \text{dom } \alpha| \leq g(\alpha) < q,$$

a contradiction. This shows that  $\alpha$  is maximal. ■

As in many algebraic settings, it is interesting to know when  $\alpha \in PS(q)$  lies below some maximal element of  $PS(q)$ .

**Theorem 3.3.4.** *The following are equivalent for  $\alpha \in PS(q)$ .*

- (i)  $g(\alpha) \leq q$ ,
- (ii)  $\alpha \leq \beta$  for some  $\beta \in PS(q)$  maximal with respect to  $\leq$ ,
- (iii)  $\alpha \subseteq \beta$  for some  $\beta \in PS(q)$  maximal with respect to  $\subseteq$ ,
- (iv)  $(\alpha, \beta) \in \Omega$  for some  $\beta \in PS(q)$  maximal with respect to  $\Omega$ .

**Proof.** Suppose that (i) holds. If  $g(\alpha) < q$ , then  $\alpha \leq \alpha$  and  $\alpha$  is maximal under  $\leq$  by Theorem 3.3.3. Therefore, suppose that  $g(\alpha) = q$ . Since  $d(\alpha) = q$ , we can write  $X \setminus X\alpha = A \dot{\cup} B$  where  $|A| = |B| = q$ . Let  $\theta : X \setminus \text{dom } \alpha \rightarrow A$  be any bijection and define  $\beta \in PS(q)$  by letting  $\text{dom } \beta = X$ ,  $\beta|_{\text{dom } \alpha} = \alpha$  and  $\beta|(X \setminus \text{dom } \alpha) = \theta$ . Then  $g(\beta) = 0$  and  $X\beta = X\alpha \dot{\cup} A$ , so

$$q = |A| = \max(g(\beta), |X\beta \setminus X\alpha|) = \max(g(\alpha), q).$$

That is,  $(\alpha, \beta) \in \mathbb{L}$  and clearly  $\alpha \not\subseteq \beta$ . Hence  $\alpha < \beta$  where  $\beta$  is maximal with respect to  $\leq$  since  $g(\beta) = 0 < q$ .

Now suppose that (ii) holds: namely, suppose  $\alpha \leq \beta$  where  $g(\beta) = r < q$ . Then  $\alpha \subseteq \beta$  and  $d(\beta) = q$ , so we can write  $X \setminus X\beta = A \dot{\cup} B$  where  $|A| = r$  and  $|B| = q$ . Let  $\theta : X \setminus \text{dom } \beta \rightarrow A$  be any bijection and define  $\beta^+ \in PS(q)$  by letting  $\text{dom } \beta^+ = X$ ,  $\beta^+|_{\text{dom } \beta} = \beta$  and  $\beta^+|(X \setminus \text{dom } \beta) = \theta$ . Then  $\alpha \subseteq \beta \subseteq \beta^+$  where  $\beta^+$  is maximal with respect to  $\subseteq$ : that is, (iii) holds by Theorem 3.3.2.

Next, suppose that (iii) holds. Then  $\alpha \subseteq \beta$  for some  $\beta \in PS(q)$  maximal with respect to  $\subseteq$ . By Theorem 3.3.2,  $\beta$  is maximal with respect to  $\Omega$ . Since  $\subseteq$  is contained in  $\Omega$ , we deduce that (iv) also holds.

Finally, suppose that (iv) holds: that is, suppose  $(\alpha, \beta) \in \Omega$  where  $\text{dom } \beta = X$ , and write

$$\begin{aligned} A &= \{x \in \text{dom } \alpha : x\alpha\beta^{-1} = x\}, \\ B &= \{x \in \text{dom } \alpha : x\alpha\beta^{-1} \notin \text{dom } \alpha\}. \end{aligned}$$

By the definition of  $\Omega$ , if  $x \in \text{dom } \alpha$  and  $x\alpha = y\beta$  (possible since  $X\alpha \subseteq X\beta$ ) then either  $y \in \text{dom } \alpha$  (so  $y = x$  and  $x \in A$ ) or  $y \notin \text{dom } \alpha$  (so  $x \in B$ ). It follows that  $\text{dom } \alpha = A \dot{\cup} B$ ,  $A\alpha = A\beta$  and  $B\alpha = C\beta$  for some  $C \subseteq \text{dom } \beta \setminus \text{dom } \alpha$ . Note that  $X\alpha = (A \cup C)\beta$  and  $(A \cup C) \cap B = \emptyset$ . Therefore  $(A \cup C)\beta \cap B\beta = \emptyset$  (since  $\beta$  is injective). This follows that  $X\alpha \cap B\beta = \emptyset$ , that is,  $B\beta \subseteq X \setminus X\alpha$ . So, since  $\text{dom } \beta = X$ ,

$$|B| = |B\alpha| = |B\beta| \leq |X \setminus X\alpha| = q.$$

Next let  $D = X \setminus (A \cup B \cup C)$  and observe that  $D\beta \cap X\alpha = D\beta \cap (A \cup C)\beta = \emptyset$ . Therefore

$$|D\beta| \leq |X \setminus X\alpha| = q.$$

Now  $X\beta = A\beta \dot{\cup} B\beta \dot{\cup} C\beta \dot{\cup} D\beta$  and thus

$$(X \setminus \text{dom } \alpha)\beta = (X \setminus (A \cup B))\beta = X\beta \setminus (A \cup B)\beta = C\beta \cup D\beta.$$

Consequently

$$g(\alpha) = |X \setminus \text{dom } \alpha| = |(X \setminus \text{dom } \alpha)\beta| = |C\beta| + |D\beta| \leq |B\alpha| + q = q,$$

and so (i) holds. ■

Observe that, if  $p = q$ , then  $g(\alpha) \leq q$  for all  $\alpha \in PS(q)$ . Hence, in this case, every  $\alpha \in PS(q)$  is contained in some maximal element.

**Theorem 3.3.5.** *If  $p > q$ , then  $PS(q)$  has no minimal element with respect to  $\leq, \subseteq$  or  $\Omega$ , and hence also no minimum element.*

**Proof.** Suppose that  $p > q$  and let  $\alpha \in PS(q)$ . Since  $|X \setminus X\alpha| = q < p$ , we have  $p = |X\alpha| = |\text{dom } \alpha|$  and we can write  $\text{dom } \alpha = A \dot{\cup} B$  where  $|A| = p$  and  $|B| = q$ . If  $\gamma = \alpha|_A$ , then  $d(\gamma) = |B\alpha| + d(\alpha) = q$ , thus  $\gamma \in PS(q)$  and clearly  $\gamma \subsetneq \alpha$ . Also, let  $C = X \setminus \text{dom } \alpha$  and  $\lambda = \text{id}_{A \cup C}$ , then  $d(\lambda) = |B| = q$ , so  $\lambda \in PS(q)$  and  $\gamma = \lambda\alpha$ . Consequently,  $(\gamma, \alpha) \in \mathbb{L}$  and so  $\gamma < \alpha$  by Theorem 3.1.1. Therefore, there is no minimal element under  $\leq$ , and hence none for  $\subseteq$  and  $\Omega$  (due to them containing  $\leq$ ). Hence, there is also no minimum element under each of these orders. ■

When  $p = q$ , it is easy to see that  $\emptyset$  is the minimum under  $\leq, \subseteq$  and  $\Omega$ . In this case, we say  $\alpha \in PS(q)$  is *non-zero minimal* with respect to an order  $\preceq$  on  $PS(q)$  if  $\alpha$  is minimal among the non-zero elements of  $PS(q)$  under  $\preceq$ .

**Theorem 3.3.6.** *If  $p = q$ , then the following are equivalent for  $\alpha \in PS(q)$ .*

- (i)  $\alpha$  is non-zero minimal with respect to  $\Omega$ ,
- (ii)  $\alpha$  is non-zero minimal with respect to  $\subseteq$ ,
- (iii)  $\alpha$  is non-zero minimal with respect to  $\leq$ ,
- (iv)  $|\text{dom } \alpha| = 1$ .

**Proof.** Since  $\Omega$  contains  $\subseteq$ , and  $\subseteq$  contains  $\leq$ , we have (i) implies (ii), and (ii) implies (iii). To show that (iii) implies (iv), suppose that (iii) holds and assume that  $|\text{dom } \alpha| > 1$ . Now, as in the proof of Theorem 3.3.5, if  $|\text{dom } \alpha| = p$ , then we can write  $\text{dom } \alpha = A \dot{\cup} B$  where  $|A| = p$  and  $|B| = q$ . If  $\gamma = \alpha|_A$ , then  $d(\gamma) = |B\alpha| + d(\alpha) = q$ , thus  $\gamma \in PS(q)$  and clearly  $\gamma \subsetneq \alpha$ . Also, if  $X = A \dot{\cup} B \dot{\cup} C$  and  $\lambda = \text{id}_{A \cup C}$ , then  $d(\lambda) = |B| = q$ , so  $\lambda \in PS(q)$  and  $\gamma = \lambda\alpha$  (since  $C = X \setminus \text{dom } \alpha$ ). Consequently,  $(\gamma, \alpha) \in \mathbb{L}$  and so  $\emptyset < \gamma < \alpha$  by Theorem 3.1.1, contradicting (iii). On the other hand, if  $|\text{dom } \alpha| < p$  then  $g(\alpha) = p$ . In this case, choose  $a \in \text{dom } \alpha$  and write  $C = \text{dom } \alpha \setminus \{a\}$  (which is non-empty by assumption). If  $\beta = \alpha|_C$  and  $\lambda = \text{id}_C$  then  $\beta, \lambda \in PS(q)$  and  $\beta = \lambda\alpha$ . Therefore,  $(\beta, \alpha) \in \mathbb{L}$  and clearly  $\beta \subsetneq \alpha$ . That is,  $\emptyset < \beta < \alpha$ , contradicting (iii) again.

Finally, to show (iv) implies (i), suppose that  $|\text{dom } \alpha| = 1$ , say  $\text{dom } \alpha = \{x\}$ . Since  $\Omega = \Omega'$  and by the definition of  $\Omega'$ , if there exists  $\beta \neq \emptyset$  such that  $(\beta, \alpha) \in \Omega$ , then  $\text{dom } \beta = \{x\}$  and  $X\beta = \{x\alpha\}$ . Hence  $\alpha = \beta$  and so  $\alpha$  is non-zero minimal under  $\Omega$ . ■

Recall that  $\leq$  is properly contained in  $\subseteq$ , and  $\subseteq$  is properly contained in  $\Omega$  on  $PS(q)$ . Thus, it is interesting to consider the following problems: Are there other partial orders lie between  $\leq$  and  $\subseteq$ , and between  $\subseteq$  and  $\Omega$  on  $PS(q)$ ? The following two theorems answer this question.

**Theorem 3.3.7.** *Let  $\alpha, \beta \in PS(q)$ . For any  $(\alpha, \beta)$  in  $\subseteq \setminus \leq$ ,*

$$\rho_{\alpha, \beta} = \leq \cup \{(\alpha, \beta)\}$$

is a minimal partial order on  $PS(q)$  containing  $\leq$ .

**Proof.** It is clear that a pair  $(\alpha, \beta)$  in  $\subseteq \setminus \leq$  always exists since  $\leq$  is properly contained in  $\subseteq$  on  $PS(q)$ . Since  $\leq$  is a partial order contained in  $\rho_{\alpha, \beta}$ , we have  $\rho_{\alpha, \beta}$  is reflexive. To see that  $\rho_{\alpha, \beta}$  is anti-symmetric, we let  $(\gamma, \lambda), (\lambda, \gamma) \in \rho_{\alpha, \beta}$ . We consider only the case that  $\gamma \leq \lambda, (\lambda, \gamma) = (\alpha, \beta)$ , otherwise, it is easy to see that  $\gamma = \lambda$ . In this case we have  $\lambda = \alpha, \gamma = \beta$ , and so  $\beta \leq \alpha$ . It follows that  $\beta \subseteq \alpha$  (since  $\subseteq$  contains  $\leq$ ), and hence  $\alpha = \beta$  (since  $\alpha \subseteq \beta$ ). Therefore  $\lambda = \gamma$ . To see that  $\rho_{\alpha, \beta}$  is transitive, let  $(\gamma, \lambda), (\lambda, \mu) \in \rho_{\alpha, \beta}$ . It is clear that  $(\gamma, \mu) \in \rho_{\alpha, \beta}$  when  $\gamma \leq \lambda, \lambda \leq \mu$ , and if  $(\gamma, \lambda), (\lambda, \mu) \in \{(\alpha, \beta)\}$ , then  $\gamma = \mu$  and thus  $(\gamma, \mu) \in \rho_{\alpha, \beta}$  since  $\rho_{\alpha, \beta}$  is reflexive. For the rest, we have either

$$\gamma \leq \lambda, \lambda = \alpha, \mu = \beta \quad \text{or} \quad \lambda \leq \mu, \gamma = \alpha, \lambda = \beta.$$

In the first case, we have  $\gamma \leq \alpha$ . If  $g(\gamma) < q$ , then  $\gamma$  is maximal under  $\leq$ , and hence  $\alpha = \gamma$ . This implies that  $(\gamma, \mu) = (\alpha, \beta) \in \rho_{\alpha, \beta}$ . Otherwise,  $q \leq g(\gamma)$  where  $\gamma \subseteq \alpha \subseteq \beta$  (since  $\subseteq$  contains  $\leq$ ). Consequently, since

$$X \setminus \text{dom } \gamma = (X \setminus \text{dom } \beta) \dot{\cup} (\text{dom } \beta \setminus \text{dom } \gamma),$$

we have

$$q \leq |X \setminus \text{dom } \beta| = g(\beta) \quad \text{or} \quad q \leq |\text{dom } \beta \setminus \text{dom } \gamma| = |X\beta \setminus X\gamma|,$$

that is,  $q \leq \max(g(\beta), |X\beta \setminus X\gamma|)$ . Also, since  $g(\beta) \leq g(\gamma)$  and  $|X\beta \setminus X\gamma| \leq d(\gamma) = q$ , we have

$$q \leq \max(g(\beta), |X\beta \setminus X\gamma|) \leq \max(g(\gamma), q),$$

that is  $(\gamma, \beta) \in \mathbb{L}$ , and so  $\gamma \leq \beta = \mu$ . Therefore  $(\gamma, \mu) \in \rho_{\alpha, \beta}$ . For the latter, we have  $\beta \leq \mu$  and hence  $\alpha \subseteq \beta \subseteq \mu$ . Like in the first case, if  $g(\beta) < q$ , then  $\beta = \mu$  and so  $(\gamma, \mu) = (\alpha, \beta) \in \rho_{\alpha, \beta}$ . Otherwise, we have  $q \leq g(\beta) \leq g(\alpha)$ . Consequently, since

$$X \setminus \text{dom } \alpha = (X \setminus \text{dom } \mu) \dot{\cup} (\text{dom } \mu \setminus \text{dom } \alpha),$$

we have

$$q \leq |X \setminus \text{dom } \mu| = g(\mu) \quad \text{or} \quad q \leq |\text{dom } \mu \setminus \text{dom } \alpha| = |X\mu \setminus X\alpha|,$$

that is,  $q \leq \max(g(\mu), |X\mu \setminus X\alpha|)$ . Also, since  $g(\mu) \leq g(\alpha)$  and  $|X\mu \setminus X\alpha| \leq d(\alpha) = q$ , we have

$$q \leq \max(g(\mu), |X\mu \setminus X\alpha|) \leq \max(g(\alpha), q),$$

that is  $(\alpha, \mu) \in \mathbb{L}$ , and so  $\alpha \leq \mu$ . Therefore  $(\gamma, \mu) = (\alpha, \mu) \in \rho_{\alpha, \beta}$  as required. By the definition of  $\rho_{\alpha, \beta}$ , it is clear that

$$\leq \subsetneq \rho_{\alpha, \beta} \subsetneq \subseteq,$$

and obviously, there is no other partial orders lie between  $\leq$  and  $\rho_{\alpha, \beta}$ . Therefore  $\rho_{\alpha, \beta}$  is a minimal partial order containing  $\leq$ .  $\blacksquare$

For the following result, we sometimes write  $\alpha \sim_{\Omega} \beta$  instead of  $(\alpha, \beta) \in \Omega$  for convenience.

**Theorem 3.3.8.** *For distinct  $x, y \in X$ , there is a partial order  $\delta_{x, y}$  on  $PS(q)$  lies strictly between  $\subseteq$  and  $\Omega$ .*

**Proof.** For distinct  $x, y$  in  $X$ , we write  $X = A \dot{\cup} B \dot{\cup} \{x\} \dot{\cup} \{y\}$  where  $|A| = p$  and  $|B| = q$ . Let  $\theta : A \cup B \rightarrow A$  be a bijection and define  $\alpha, \beta \in PS(q)$  by

$$\alpha = \begin{pmatrix} A \cup B & x \\ A & x \end{pmatrix}, \quad \beta = \begin{pmatrix} A \cup B & x & y \\ A & y & x \end{pmatrix}$$

where  $\alpha|(A \cup B) = \theta = \beta|(A \cup B)$ . Since  $y \notin \text{dom } \alpha$ , we have

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \text{id}_{A \cup B} \subseteq \text{id}_{\text{dom } \alpha} = \alpha\alpha^{-1}.$$

That is,  $(\alpha, \beta) \in \Omega$ , but  $\alpha \not\subseteq \beta$  since  $x\alpha \neq x\beta$ . Let

$$T_{x, y} = \{(\gamma, \beta) : \gamma \in PS(q) \text{ and } \gamma \subseteq \alpha\} \quad \text{and} \quad \delta_{x, y} = \subseteq \cup T_{x, y}.$$

Since  $(\alpha, \beta) \in T_{x, y} \setminus \subseteq$ , we have  $T_{x, y} \neq \emptyset$  and  $\subseteq$  is properly contained in  $\delta_{x, y}$ . Moreover, for each pair  $(\gamma, \beta)$  in  $T_{x, y}$ ,  $\gamma \subseteq \alpha$ . It follows that  $\gamma \sim_{\Omega} \alpha \sim_{\Omega} \beta$

(since  $\Omega$  contains  $\subseteq$ ), that is,  $\Omega$  contains  $T_{x,y}$ . For  $y' \in X, x \neq y' \neq y$ , we write  $X = A' \dot{\cup} B' \dot{\cup} \{x\} \dot{\cup} \{y'\}$  where  $|A'| = p$  and  $|B'| = q$ . Then define  $\alpha', \beta' \in PS(q)$  in the same way as  $\alpha$  and  $\beta$ . We have  $(\alpha', \beta') \in \Omega \setminus \subseteq$  and also,  $(\alpha', \beta') \notin T_{x,y}$  since  $\beta \neq \beta'$ . It follows that  $\subseteq \cup T_{x,y} \subsetneq \Omega$ , that is  $\delta_{x,y}$  is properly contained in  $\Omega$ . To see that  $\delta_{x,y}$  is a partial order, we observe that  $\delta_{x,y}$  contains a partial order  $\subseteq$ , then it is reflexive. To show that  $\delta_{x,y}$  is anti-symmetric, let  $(\lambda, \mu), (\mu, \lambda) \in \delta_{x,y}$ . If both of these pairs belong to  $T_{x,y}$ , then  $\beta \subseteq \alpha$ , a contradiction since  $\text{dom } \beta \not\subseteq \text{dom } \alpha$ . Also, if  $\lambda \subseteq \mu$  and  $(\mu, \lambda) \in T_{x,y}$ , then  $\beta \subseteq \mu \subseteq \alpha$  and we get a contradiction again and, similarly, this also happen when  $\mu \subseteq \lambda$  and  $(\lambda, \mu) \in T_{x,y}$ . It follows that both  $(\lambda, \mu), (\mu, \lambda)$  belong to  $\subseteq$ , and hence  $\lambda = \mu$ . To show  $\delta_{x,y}$  is transitive, let  $(\lambda, \mu), (\mu, \theta) \in \delta_{x,y}$ . Like before, if  $(\lambda, \mu), (\mu, \theta) \in T_{x,y}$ , then  $\beta \subseteq \alpha$ , that is, we get a contradiction again. So we consider only the following three cases. If both  $(\lambda, \mu), (\mu, \theta)$  belong to  $\subseteq$ , then  $\lambda \subseteq \theta$  and so  $(\lambda, \theta) \in \delta_{x,y}$ . If  $\lambda \subseteq \mu$  and  $(\mu, \theta) \in T_{x,y}$ , then  $\lambda \subseteq \mu \subseteq \alpha$  and  $\beta = \theta$ . Thus  $(\lambda, \theta) \in T_{x,y} \subseteq \delta_{x,y}$ . Finally, if  $\mu \subseteq \theta$  and  $(\lambda, \mu) \in T_{x,y}$ , then  $\lambda \subseteq \alpha$  and  $\beta = \mu \subseteq \theta$ . Since  $\text{dom } \beta = X$ , we have  $\beta$  is maximal under  $\subseteq$ , so  $\beta = \theta$  and this implies that  $(\lambda, \theta) \in T_{x,y} \subseteq \delta_{x,y}$  as required. ■

### 3.4 Meets and Joins

In this section, we study the existence of a meet  $\alpha \wedge \beta$  and a join  $\alpha \vee \beta$  for  $\alpha, \beta$  in the semigroups  $I(X)$ ,  $PS(q)$  and  $R(q)$  for each of the orders  $\leq$  and  $\subseteq$ . To do this, we first define the *equaliser* of  $\alpha, \beta \in I(X)$  (compare [14] p 416 for linear transformations) as follows:

$$E(\alpha, \beta) = \{x \in \text{dom } \alpha \cap \text{dom } \beta : x\alpha = x\beta\}.$$

The next result may be well-known, but we do not know a reference in the literature (recall that  $\subseteq$  equals  $\leq$  on  $I(X)$ ).

**Theorem 3.4.1.** *Let  $\alpha, \beta \in I(X)$  and  $E = E(\alpha, \beta)$ . Then, under  $\subseteq$ ,  $\alpha \wedge \beta = \alpha|E = \beta|E$ .*

**Proof.** Clearly  $\alpha|E = \beta|E \subseteq \alpha, \beta$ . If  $\gamma \subseteq \alpha, \beta$ , then for each  $x \in \text{dom } \gamma$ ,  $x\alpha = x\gamma = x\beta$  and this follows that  $\text{dom } \gamma \subseteq E$ . Also,  $\gamma \subseteq \alpha|E$ , then we have  $\alpha \wedge \beta = \alpha|E$ . ■

**Theorem 3.4.2.** *Let  $\alpha, \beta \in PS(q)$  and  $E = E(\alpha, \beta)$ . Then  $\gamma \subseteq \alpha, \beta$  for some non-empty  $\gamma \in PS(q)$  if and only if*

- (i)  $E \neq \emptyset$ , and
- (ii)  $\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$ .

Moreover, when this occurs,  $\alpha|E$  (equals  $\beta|E$ ) is the non-empty meet of  $\alpha, \beta$  under  $\subseteq$ .

**Proof.** Suppose that  $\emptyset \neq \gamma \subseteq \alpha, \beta$  in  $PS(q)$ . Then  $\emptyset \neq \text{dom } \gamma \subseteq \text{dom } \alpha \cap \text{dom } \beta$  and  $x\alpha = x\gamma = x\beta$  for all  $x \in \text{dom } \gamma$ . That is,  $\emptyset \neq \text{dom } \gamma \subseteq E$  and this implies  $X\gamma = E\gamma$ . Now  $E\gamma = (E \cap \text{dom } \gamma)\gamma \subseteq E\alpha \subseteq X\alpha$  and so

$$|X\alpha \setminus E\alpha| \leq |X\alpha \setminus E\gamma| = |X\alpha \setminus X\gamma| \leq |X \setminus X\gamma| = q.$$

Similarly,  $|X\beta \setminus E\beta| \leq q$  and hence the conditions hold. Conversely, if the conditions hold then  $\gamma = \alpha|E = \beta|E$  is a non-empty element of  $I(X)$  with domain  $E = E(\alpha, \beta)$  and  $\gamma \subseteq \alpha, \beta$ . Moreover, since  $X\gamma = E\gamma = E\alpha \subseteq X\alpha$ , we have

$$X \setminus X\gamma = (X \setminus X\alpha) \dot{\cup} (X\alpha \setminus E\alpha)$$

and it follows that  $d(\gamma) = q$  since  $d(\alpha) = q$  and  $|X\alpha \setminus E\alpha| \leq q$ . That is,  $\gamma \in PS(q)$ . Finally, as shown in the proof of Theorem 3.4.1, we have  $\alpha \wedge \beta = \alpha|E$ . ■

**Remark 3.4.3.** Suppose  $S$  is any inverse subsemigroup of  $I(X)$ . If  $\alpha \leq \beta$  in  $S$ , then  $\alpha = \gamma\beta$  for some  $\gamma \in E(S)$ . That is,  $\alpha = \text{id}_A \circ \beta$  for some  $A \subseteq X$  and we deduce that  $\alpha \subseteq \beta$ . On the other hand, if  $\alpha \subseteq \beta$  in the inverse semigroup  $R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}$ , then  $\alpha = \text{id}_{\text{dom } \alpha} \circ \beta$ , where  $\text{id}_{\text{dom } \alpha}$  is an idempotent in  $R(q)$ , and so  $\alpha \leq \beta$  in  $R(q)$ . That is,  $\leq = \subseteq$  on  $R(q)$ .

Of course, when we turn to  $R(q)$ , we expect a further condition to be needed in order to characterise meets in  $R(q)$  under  $\subseteq$ .

**Theorem 3.4.4.** *Let  $\alpha, \beta \in R(q)$  and  $E = E(\alpha, \beta)$ . Then  $\gamma \subseteq \alpha, \beta$  for some non-empty  $\gamma \in R(q)$  if and only if*

- (i)  $E \neq \emptyset$ ,
- (ii)  $\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$ , and
- (iii)  $\max(|\text{dom } \alpha \setminus E|, |\text{dom } \beta \setminus E|) \leq q$ .

*Moreover, when this occurs,  $\alpha|E$  (equals  $\beta|E$ ) is the non-empty meet of  $\alpha, \beta$  under  $\subseteq$ .*

**Proof.** Suppose that  $\emptyset \neq \gamma \subseteq \alpha, \beta \in R(q)$ . Since  $R(q) \subseteq PS(q)$ , Theorem 3.4.2 implies that (i) and (ii) hold. Since  $\text{dom } \gamma \subseteq E \subseteq \text{dom } \alpha$ , we have

$$|\text{dom } \alpha \setminus E| \leq |\text{dom } \alpha \setminus \text{dom } \gamma| \leq |X \setminus \text{dom } \gamma| = q.$$

Similarly,  $|\text{dom } \beta \setminus E| \leq q$  and hence (iii) holds. Conversely, suppose the conditions hold. By Theorem 3.4.2 again, (i) and (ii) imply that  $\gamma = \alpha|E = \beta|E$  is a non-empty element of  $PS(q)$  and it is also the meet of  $\alpha, \beta$  in  $PS(q)$  under  $\subseteq$ . Also, since  $\text{dom } \gamma = E \subseteq \text{dom } \alpha$  and  $g(\alpha) = q$ , we have

$$X \setminus \text{dom } \gamma = (X \setminus \text{dom } \alpha) \dot{\cup} (\text{dom } \alpha \setminus E).$$

Then (iii) implies that  $g(\gamma) = q$ , hence  $\gamma \in R(q)$ . ■

From Theorem 3.1.1, we have that  $\leq$  equals  $\subseteq \cap \mathbb{L}$  on  $PS(q)$ , where  $\mathbb{L}$  is the relation defined on  $PS(q)$  by

$$(\alpha, \beta) \in \mathbb{L} \quad \text{if and only if} \quad PS(q)^1\alpha \subseteq PS(q)^1\beta.$$

It is equivalent to say that

$$(\alpha, \beta) \in \mathbb{L} \quad \text{if and only if} \quad \alpha = \lambda\beta \quad \text{for some } \lambda \in PS(q)^1.$$

Hence, by using Theorem 2.3.10, we can simplify the relation  $\mathbb{L}$  by  $(\alpha, \beta) \in \mathbb{L}$  if and only if

$$\alpha = \beta \quad \text{or} \quad X\alpha \subseteq X\beta \quad \text{and} \quad q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q).$$

Note that if  $\alpha \wedge \beta = \emptyset$  in  $PS(q)$  under  $\leq$ , then  $p = q$ . In this case, if  $x \in E = E(\alpha, \beta)$  and  $x\alpha = x\beta = y$  then  $x_y \in PS(q)$  and  $x_y \subseteq \alpha, \beta$ . Also, since

$$|X\alpha \setminus \{y\}| = |\text{dom } \alpha \setminus \{x\}| \quad \text{and} \quad g(\alpha) = |X \setminus \text{dom } \alpha|,$$

we have

$$q = p = \max(g(\alpha), |X\alpha \setminus \{y\}|) \leq \max(g(x_y), q) = p = q.$$

Similarly,

$$q = \max(g(\beta), |X\beta \setminus \{y\}|) \leq \max(g(x_y), q) = q.$$

That is,  $x_y \leq \alpha, \beta$ , so  $x_y \leq \alpha \wedge \beta = \emptyset$ , a contradiction. In other words, if  $\alpha \wedge \beta = \emptyset$  then  $E = \emptyset$  and so  $\alpha|E = \beta|E = \emptyset$ .

As usual, if  $\preceq$  is a partial order on a set  $S$ , we say  $a, b \in S$  are *non-comparable* if  $a \not\preceq b$  and  $b \not\preceq a$ .

**Theorem 3.4.5.** *Suppose that  $\alpha, \beta \in PS(q)$  are non-comparable under  $\leq$  and let  $E = E(\alpha, \beta)$ . Then  $\gamma \leq \alpha, \beta$  for some non-empty  $\gamma \in PS(q)$  if and only if there exists a non-empty  $Y \subseteq E$  such that*

- (i)  $\max(|X\alpha \setminus Y\alpha|, |X\beta \setminus Y\beta|) \leq q$ , and
- (ii)  $q \leq \max(g(\alpha), |X\alpha \setminus Y\alpha|)$  and  $q \leq \max(g(\beta), |X\beta \setminus Y\beta|)$ .

*In this event,  $\gamma = \alpha|Y = \beta|Y$ .*

**Proof.** Suppose that  $\emptyset \neq \gamma \leq \alpha, \beta$  and let  $Y = \text{dom } \gamma$ . Then  $\gamma \subseteq \alpha, \beta$  and so  $x\alpha = x\gamma = x\beta$  for all  $x \in Y$ . That is,  $\emptyset \neq Y \subseteq E$  and  $X\gamma = Y\gamma = Y\alpha = Y\beta$ . Since  $d(\gamma) = q$ , we see that

$$|X\alpha \setminus Y\alpha| \leq |X \setminus Y\alpha| \leq |X \setminus X\gamma| = q$$

and likewise  $|X\beta \setminus Y\beta| \leq q$ , so (i) holds. Also, since  $\leq$  equals  $\subseteq \cap \mathbb{L}$ , we have  $(\gamma, \alpha) \in \mathbb{L}$  and  $(\gamma, \beta) \in \mathbb{L}$  and these imply

$$q \leq \max(g(\alpha), |X\alpha \setminus Y\alpha|) \quad \text{and} \quad q \leq \max(g(\beta), |X\beta \setminus Y\beta|),$$

that is (ii) holds.

Conversely, suppose the conditions hold. We write  $Y = \{y_i\}$  and  $E = Y \dot{\cup} \{e_j\}$  (possibly  $J = \emptyset$ ). We also write

$$\alpha = \begin{pmatrix} y_i & e_j & u_m \\ a_i & a_j & a_m \end{pmatrix}, \quad \beta = \begin{pmatrix} y_i & e_j & v_n \\ a_i & a_j & b_n \end{pmatrix}, \quad \gamma = \begin{pmatrix} y_i \\ a_i \end{pmatrix} \quad (1)$$

Clearly,  $\gamma \subseteq \alpha, \beta$  and thus  $g(\alpha) \leq g(\gamma)$  and  $g(\beta) \leq g(\gamma)$ . By condition (i),

$$|X\alpha \setminus X\gamma| = |J| + |M| = |X\alpha \setminus Y\alpha| \leq q,$$

thus  $d(\gamma) = |J| + |M| + d(\alpha) = q$  and so  $\gamma \in PS(q)$ . These also imply

$$\max(g(\alpha), |X\alpha \setminus X\gamma|) \leq \max(g(\gamma), q).$$

Hence, the above and condition (ii) imply that  $(\gamma, \alpha) \in \mathbb{L}$  and similarly  $(\gamma, \beta) \in \mathbb{L}$ .

Thus, we have shown that  $\gamma \leq \alpha, \beta$  and, we also see that  $\gamma = \alpha|Y = \beta|Y$ .  $\blacksquare$

**Corollary 3.4.6.** *Suppose that  $\alpha, \beta \in PS(q)$  are non-comparable under  $\leq$  and let  $E = E(\alpha, \beta)$ . Then  $\alpha \wedge \beta$  exists in  $PS(q)$  under  $\leq$  and it is non-empty if and only if  $E$  is non-empty and  $\alpha, \beta$  satisfy conditions*

(i)  $\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$ , and

(ii)  $q \leq \max(g(\alpha), |X\alpha \setminus E\alpha|)$  and  $q \leq \max(g(\beta), |X\beta \setminus E\beta|)$ .

*In this case  $\alpha \wedge \beta = \alpha|E = \beta|E$ .*

**Proof.** Suppose that  $\alpha \wedge \beta = \gamma \in PS(q)$  and it is non-empty. Then  $\gamma \leq \alpha, \beta$ . Thus, Theorem 3.4.5 implies that there exists a non-empty  $Y = \text{dom } \gamma \subseteq E$  and  $\alpha$  and  $\beta$  satisfy (i) and (ii) in Theorem 3.4.5. So we can write  $\alpha, \beta$  as in (1) in Theorem 3.4.5. If  $g(\gamma) < q$ , then Theorem 3.3.3 implies that  $\gamma$  is maximal under  $\leq$  and so  $\gamma = \alpha = \beta$ , contradicting the supposition. Hence  $g(\gamma) \geq q$ . If there exists  $e_0 \in E \setminus Y$  for some  $0 \in J$ , we can define  $\gamma' \in PS(q)$  by

$$\gamma' = \begin{pmatrix} y_i & e_0 \\ a_i & a_0 \end{pmatrix}.$$

Then  $\gamma \subseteq \gamma' \subseteq \alpha, \beta$  and  $|X\gamma' \setminus X\gamma| = 1$ , and we see that

$$g(\gamma) = |J| + |M| + g(\alpha) \quad (\text{and this implies } |J| + |M| \geq q \text{ or } g(\alpha) \geq q),$$

$$g(\gamma') = |J \setminus \{0\}| + |M| + g(\alpha).$$

Thus, if  $|J|+|M| \geq \aleph_0$  then  $g(\gamma) = g(\gamma') \geq g(\alpha)$ ; and if  $|J|+|M| < \aleph_0$  then  $\gamma \leq \alpha$  implies  $q \leq \max(g(\alpha), |J| + |M|)$ , so  $g(\alpha) \geq q$  and hence  $g(\gamma) = g(\gamma') = g(\alpha)$ . Since  $q \leq g(\gamma)$ , we have  $g(\gamma') = g(\gamma) \geq q$  in both cases. Therefore,

$$q \leq g(\gamma') = \max(g(\gamma'), 1) \leq \max(g(\gamma), q),$$

that is,  $(\gamma, \gamma') \in \mathbb{L}$ . Next, since  $\gamma \leq \alpha$ , we have  $(\gamma, \alpha) \in \mathbb{L}$  and so

$$q \leq \max(g(\alpha), |J| + |M|).$$

This implies

$$q \leq \max(g(\alpha), |J \setminus \{0\}| + |M|).$$

We also recall that  $|X\alpha \setminus X\gamma'| \leq |X \setminus X\gamma'| = q$  and  $g(\alpha) \leq g(\gamma')$  (since  $\gamma' \subseteq \alpha$ ).

Then we have

$$q \leq \max(g(\alpha), |J \setminus \{0\}| + |M|) = \max(g(\alpha), |X\alpha \setminus X\gamma'|) \leq \max(g(\gamma'), q),$$

that is,  $(\gamma', \alpha) \in \mathbb{L}$  and likewise we can show  $(\gamma', \beta) \in \mathbb{L}$ . In other words, we have shown that  $\gamma < \gamma' \leq \alpha, \beta$ , and this contradicts to  $\gamma = \alpha \wedge \beta$ . Hence, it follows that  $Y = E$ , that is,  $\alpha$  and  $\beta$  satisfy (i) and (ii).

Conversely, suppose  $E$  is non-empty and  $\alpha$  and  $\beta$  satisfy (i) and (ii). Then, by Theorem 3.4.5,  $\gamma \leq \alpha, \beta$  where  $\gamma = \alpha|E = \beta|E \in PS(q)$ . Moreover, if  $\gamma \leq \gamma' \leq \alpha, \beta$  for some  $\gamma' \in PS(q)$  then,  $\gamma \subseteq \gamma' \subseteq \alpha, \beta$  and thus  $x\gamma' = x\alpha = x\beta$  for all  $x \in \text{dom } \gamma'$ , so  $E = \text{dom } \gamma \subseteq \text{dom } \gamma' \subseteq E$ , and it follows that  $\gamma = \gamma'$ . That is,  $\gamma = \alpha \wedge \beta$ . ■

In effect, by Theorem 3.3.3, the next result determines when two elements of  $PS(q)$ , which are maximal under  $\leq$ , possess a meet under  $\leq$ .

**Corollary 3.4.7.** *Suppose that  $\alpha, \beta \in PS(q)$  are non-comparable under  $\leq$  and let  $E = E(\alpha, \beta)$ . If  $g(\alpha) < q$  and  $g(\beta) < q$ , then  $\alpha \wedge \beta$  exists in  $PS(q)$  under  $\leq$  if and only if  $|X\alpha \setminus E\alpha| = q = |X\beta \setminus E\beta|$ .*

**Proof.** Suppose that  $g(\alpha), g(\beta) < q$ . If  $\alpha \wedge \beta$  exists under  $\leq$ , then Theorem 3.4.5 (ii) implies that  $q \leq |X\alpha \setminus E\alpha|$  which is at most  $q$  by Theorem 3.4.5 (i). Thus  $|X\alpha \setminus E\alpha| = q$  and likewise  $g(\beta) < q$  implies  $|X\beta \setminus E\beta| = q$ .

Conversely, if  $|X\alpha \setminus E\alpha| = q = |X\beta \setminus E\beta|$  then both (i) and (ii) in Theorem 3.4.5 hold for  $E = E(\alpha, \beta)$ , so  $\alpha \wedge \beta$  exists. ■

**Example 3.4.8.** Suppose that  $X = M \dot{\cup} N \dot{\cup} \{b, c\}$ , where  $|M| = p$ ,  $|N| = q$  and

$$\alpha = \begin{pmatrix} M \cup N & b \\ M & b \end{pmatrix}, \quad \beta = \begin{pmatrix} M \cup N & c \\ M & c \end{pmatrix}$$

where  $E = E(\alpha, \beta) = M \cup N$ . Then  $d(\alpha) = q = d(\beta)$ , so  $\alpha, \beta \in PS(q)$  and  $\alpha|E = \beta|E \in PS(q)$ . But,  $|X\alpha \setminus E\alpha| = 1 = |X\beta \setminus E\beta|$  and  $g(\alpha) = 1 = g(\beta)$ , so  $E$  satisfies condition (i) in Theorem 3.4.5 but not condition (ii), and hence  $\alpha \wedge \beta$  does not exist in  $(PS(q), \leq)$ . That is, although  $\alpha|E$  may be the greatest lower bound under  $\subseteq$ , that may not be true for  $\leq$  since  $\leq \neq \subseteq$  on  $PS(q)$ .

**Theorem 3.4.9.** *Let  $\alpha, \beta \in I(X)$  under  $\subseteq$ . Then  $\alpha, \beta \subseteq \gamma$  for some  $\gamma \in I(X)$  if and only if*

- (i)  $\text{dom } \alpha \cap \text{dom } \beta \subseteq E(\alpha, \beta)$  and
- (ii)  $(\text{dom } \alpha \setminus \text{dom } \beta)\alpha \cap (\text{dom } \beta \setminus \text{dom } \alpha)\beta = \emptyset$ .

*Moreover, in this case,  $\alpha \vee \beta$  exists and equals  $\alpha \cup \beta$ .*

**Proof.** Suppose that  $\alpha, \beta \subseteq \gamma \in I(X)$ . If  $x \in \text{dom } \alpha \cap \text{dom } \beta$  then  $x\alpha = x\gamma = x\beta$ , and so  $x \in E(\alpha, \beta)$ . On the other hand, if there exist  $y \in \text{dom } \alpha \setminus \text{dom } \beta$  and  $z \in \text{dom } \beta \setminus \text{dom } \alpha$  such that  $y\alpha = z\beta$ , then  $y\gamma = z\gamma$ . Since  $\gamma$  is injective, this implies that  $y = z$ , a contradiction.

Conversely, suppose that the conditions hold and let  $\gamma = \alpha \cup \beta$  (as sets). Then (i) says that  $\gamma$  is a mapping and (ii) says it is injective, so  $\gamma \in I(X)$  and clearly it is an upper bound of  $\{\alpha, \beta\}$ . Moreover, if (i) and (ii) hold, then  $\gamma = \alpha \vee \beta$ , since  $\alpha, \beta \subseteq \lambda \in I(X)$  implies  $\alpha, \beta \subseteq \alpha \cup \beta \subseteq \lambda$  (as sets) where  $\alpha \cup \beta \in I(X)$ . ■

Like before, the result for joins in  $PS(q)$  under  $\subseteq$  involves an extra condition.

**Theorem 3.4.10.** *Let  $\alpha, \beta \in PS(q)$  under  $\subseteq$ . Then  $\alpha, \beta \subseteq \gamma$  for some  $\gamma \in PS(q)$  if and only if the following conditions hold.*

- (i)  $\text{dom } \alpha \cap \text{dom } \beta \subseteq E(\alpha, \beta)$ ,
- (ii)  $(\text{dom } \alpha \setminus \text{dom } \beta)\alpha \cap (\text{dom } \beta \setminus \text{dom } \alpha)\beta = \emptyset$ , and
- (iii)  $|X \setminus (X\alpha \cup X\beta)| = q$ .

Moreover, in this case,  $\alpha \vee \beta$  exists and equals  $\alpha \cup \beta$ .

**Proof.** Suppose that  $\alpha, \beta \subseteq \gamma$  in  $PS(q)$ . Then, conditions (i) and (ii) hold since  $PS(q) \subseteq I(X)$ . Since  $X\alpha \cup X\beta \subseteq X\gamma$ , we also have

$$q = |X \setminus X\gamma| \leq |X \setminus (X\alpha \cup X\beta)| \leq |X \setminus X\alpha| = q.$$

Hence (iii) holds. Conversely, suppose (i), (ii) and (iii) hold and let  $\gamma = \alpha \cup \beta$ . Then (i) and (ii) imply that  $\gamma \in I(X)$ , and (iii) implies that

$$d(\gamma) = |X \setminus X\gamma| = |X \setminus (X\alpha \cup X\beta)| = q,$$

that is,  $\gamma \in PS(q)$ . Since  $\gamma = \alpha \cup \beta$ , it follows that  $\alpha, \beta \subseteq \gamma$ . Finally, as in Theorem 3.4.9, we can show that  $\alpha \vee \beta = \gamma$ . ■

**Theorem 3.4.11.** *Let  $\alpha, \beta \in R(q)$ . Then  $\alpha, \beta \subseteq \gamma$  for some  $\gamma \in R(q)$  if and only if the following conditions hold.*

- (i)  $\text{dom } \alpha \cap \text{dom } \beta \subseteq E(\alpha, \beta)$ ,
- (ii)  $(\text{dom } \alpha \setminus \text{dom } \beta)\alpha \cap (\text{dom } \beta \setminus \text{dom } \alpha)\beta = \emptyset$ ,
- (iii)  $|X \setminus (X\alpha \cup X\beta)| = q$ , and
- (iv)  $|X \setminus (\text{dom } \alpha \cup \text{dom } \beta)| = q$ .

Moreover, when this occurs,  $\alpha \cup \beta$  is the join of  $\alpha, \beta$  under  $\subseteq$ .

**Proof.** Suppose that  $\alpha, \beta \subseteq \gamma$  in  $R(q)$ . Since  $R(q) \subseteq PS(q)$ , Theorem 3.4.10 implies that (i), (ii) and (iii) hold. Since  $\text{dom } \alpha \cup \text{dom } \beta \subseteq \text{dom } \gamma$ , we have

$$q = |X \setminus \text{dom } \gamma| \leq |X \setminus (\text{dom } \alpha \cup \text{dom } \beta)| \leq |X \setminus \text{dom } \alpha| = q.$$

Hence (iv) holds. Conversely, suppose that the conditions hold. By Theorem 3.4.10 again, (i), (ii) and (iii) imply that  $\gamma = \alpha \cup \beta$  is an element of  $PS(q)$  and it is also a join of  $\alpha, \beta$  under  $\subseteq$ . Also, (iv) implies that

$$g(\gamma) = |X \setminus \text{dom } \gamma| = |X \setminus (\text{dom } \alpha \cup \text{dom } \beta)| = q,$$

so  $\gamma \in R(q)$ . ■

To characterize joins in  $PS(q)$  under  $\leq$ , we need two lemmas. In effect, the first provides a description of  $\leq$  in terms of  $\subseteq$  which differs from that in Theorem 3.1.1.

**Lemma 3.4.12.** *Suppose that  $\alpha, \beta \in PS(q)$  and  $\alpha \neq \beta$ . Then  $\alpha < \beta$  if and only if  $\alpha \subsetneq \beta$  and  $g(\alpha) \geq q$ .*

**Proof.** If  $\alpha < \beta$ , then  $\alpha \subsetneq \beta$  and  $(\alpha, \beta) \in \mathbb{L}$ . Therefore,  $\text{dom } \alpha \subsetneq \text{dom } \beta$  and  $X\alpha \subseteq X\beta$ , and hence

$$X \setminus \text{dom } \alpha = (X \setminus \text{dom } \beta) \dot{\cup} (\text{dom } \beta \setminus \text{dom } \alpha), \text{ and} \quad (2)$$

$$X\beta = [(\text{dom } \beta \setminus \text{dom } \alpha)\beta] \dot{\cup} [(\text{dom } \alpha)\beta].$$

Now,  $(\text{dom } \alpha)\beta = (\text{dom } \alpha)\alpha = X\alpha$  (since  $\alpha \subsetneq \beta$ ) and so

$$|X\beta \setminus X\alpha| = |(\text{dom } \beta \setminus \text{dom } \alpha)\beta| = |\text{dom } \beta \setminus \text{dom } \alpha|. \quad (3)$$

By Theorem 2.3.10, we also know that

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q).$$

Hence, if  $\max(g(\beta), |X\beta \setminus X\alpha|) = g(\beta)$ , then  $q \leq g(\beta) \leq g(\alpha)$  by (2); and if  $\max(g(\beta), |X\beta \setminus X\alpha|) = |X\beta \setminus X\alpha|$ , then

$$q \leq |X\beta \setminus X\alpha| = |\text{dom } \beta \setminus \text{dom } \alpha| \leq |X \setminus \text{dom } \alpha| = g(\alpha)$$

by (3). That is, the conditions hold.

Conversely, suppose that the conditions hold. Then (2) and (3) hold (since  $\alpha \subsetneq \beta$ ),  $\max(g(\alpha), q) = g(\alpha) \geq g(\beta)$  and  $|X\beta \setminus X\alpha| \leq |X \setminus X\alpha| = d(\alpha) = q$ . Since  $g(\alpha) \geq q$ , (2) implies that  $|X \setminus \text{dom } \beta| \geq q$  or  $|\text{dom } \beta \setminus \text{dom } \alpha| \geq q$ . By this result together with (3), we deduce that

$$g(\beta) = |X \setminus \text{dom } \beta| \geq q \text{ or } |X\beta \setminus X\alpha| = |\text{dom } \beta \setminus \text{dom } \alpha| \geq q.$$

Consequently,

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q),$$

and so  $(\alpha, \beta) \in \mathbb{L}$ . By Theorem 3.1.1, it follows that  $\alpha < \beta$ . ■

**Lemma 3.4.13.** *Suppose that  $\alpha, \beta \in PS(q)$  are non-comparable under  $\leq$ . Then  $\alpha, \beta \leq \gamma$  for some  $\gamma \in PS(q)$  if and only if*

- (i)  $\alpha, \beta \subseteq \theta$  for some  $\theta \in PS(q)$ , and
- (ii)  $g(\alpha) \geq q$  and  $g(\beta) \geq q$ .

**Proof.** If  $\alpha, \beta \leq \gamma \in PS(q)$ , then  $\alpha, \beta \subseteq \gamma$ , so (i) holds. In addition, if  $g(\alpha) < q$ , then  $\alpha$  is maximal under  $\leq$  (by Theorem 3.3.3). Hence  $\alpha \leq \gamma$  implies  $\alpha = \gamma$  and so  $\beta \leq \alpha$ , contradicting the supposition. Therefore,  $g(\alpha) \geq q$  and likewise  $g(\beta) \geq q$ . That is, (ii) holds.

Conversely, suppose that (i) and (ii) hold. Then (i) and Theorem 3.4.10 imply that  $\pi = \alpha \cup \beta \in PS(q)$  is the join of  $\{\alpha, \beta\}$  under  $\subseteq$ . So, if  $\alpha = \pi$ , then  $\beta \subsetneq \alpha$  (since they are non-comparable). Thus, (ii) and Lemma 3.4.12 imply  $\beta < \alpha$ , which contradicts the supposition. Therefore,  $\alpha \subsetneq \pi$  and  $g(\alpha) \geq q$ , so  $\alpha < \pi$  by Lemma 3.4.12 again. Similarly,  $\beta < \pi$  and so  $\alpha, \beta$  have an upper bound in  $PS(q)$  under  $\leq$ . ■

**Example 3.4.14.** Surprisingly, (i) and (ii) in Lemma 3.4.13 do not ensure that  $\alpha \cup \beta$  equals  $\alpha \vee \beta$  in  $PS(q)$  under  $\leq$ . For example, write  $X = A \dot{\cup} B \dot{\cup} C \dot{\cup} D \dot{\cup} \{a\}$  where  $|A| = p = |X|$  and  $|B| = |C| = |D| = q$ . Let

$$\alpha = \left( \begin{array}{c} A \cup B \\ A \end{array} \right) \cup \text{id}_C, \quad \beta = \left( \begin{array}{c} A \cup B \\ A \end{array} \right) \cup \text{id}_D$$

where  $x\alpha = x\beta$  for all  $x \in A \cup B$ . Then  $\alpha, \beta \in PS(q)$  and they are non-comparable under  $\leq$  (since  $\alpha \not\subseteq \beta$  and  $\beta \not\subseteq \alpha$ ). If  $\theta = \alpha \cup \beta$ , then  $\alpha, \beta \subseteq \theta \in PS(q)$  (since  $d(\theta) = |B| = q$ ), hence  $\alpha$  and  $\beta$  satisfy (i). Also,  $g(\alpha) = |D| = q = |C| = g(\beta)$ , and hence  $\alpha$  and  $\beta$  satisfy (ii). By Lemma 3.4.12,  $\alpha, \beta < \theta' = \theta \cup \text{id}_{\{a\}} \in PS(q)$ , but  $\theta \not\leq \theta'$  since  $g(\theta) = 1 \not\geq q$ , and thus  $\alpha \cup \beta$  does not equal  $\alpha \vee \beta$ .

**Theorem 3.4.15.** *Suppose that  $\alpha, \beta \in PS(q)$  are non-comparable under  $\leq$ . Then  $\alpha \vee \beta$  exists if and only if*

- (i)  $\alpha, \beta < \theta$  for some  $\theta \in PS(q)$ , and

(ii) either  $X = \text{dom } \alpha \cup \text{dom } \beta$  or  $|X \setminus (\text{dom } \alpha \cup \text{dom } \beta)| \geq q$ .

Moreover, when this occurs,  $\alpha \vee \beta$  equals  $\alpha \cup \beta$ .

**Proof.** Suppose that  $\alpha \vee \beta$  exists under  $\leq$  and write  $\gamma = \alpha \vee \beta$ . Then  $\alpha, \beta < \gamma$  (since  $\alpha$  and  $\beta$  are non-comparable), so (i) holds. Consequently,  $\alpha, \beta \not\subseteq \gamma$  and so Theorem 3.4.10 implies that  $\pi = \alpha \cup \beta \in PS(q)$  is the join of  $\{\alpha, \beta\}$  under  $\subseteq$  and this follows that  $\pi \subseteq \gamma$ . Now, to prove (ii), suppose  $\text{dom } \alpha \cup \text{dom } \beta \not\subseteq X$ . Choose  $a \in X \setminus (\text{dom } \alpha \cup \text{dom } \beta) = X \setminus \text{dom } \pi$  and, for any  $x \in X \setminus X\pi$  (non-empty since  $d(\pi) = q$ ), we let

$$\mu_x = \begin{pmatrix} \text{dom } \pi & a \\ X\pi & x \end{pmatrix}$$

where  $\mu_x|_{\text{dom } \pi} = \pi$ . Then  $\mu_x \in PS(q)$  since  $d(\mu_x) = |X \setminus X\pi| = d(\pi) = q$ . Clearly,  $\alpha \subseteq \mu_x$  and  $\alpha \neq \mu_x$  (since  $a \in \text{dom } \mu_x \setminus \text{dom } \alpha$ ). Therefore, using the fact that  $\alpha < \gamma$ , Lemma 3.4.12 implies that  $g(\alpha) \geq q$  and thus  $\alpha < \mu_x$  by Lemma 3.4.12 again. Similarly,  $\beta < \mu_x$ . It follows that  $\gamma \leq \mu_x$  for all  $x \in X \setminus X\pi$  since  $\gamma = \alpha \vee \beta$  under  $\leq$ . If  $\gamma = \mu_x$  for all  $x \in X \setminus X\pi$ , then  $\mu_x = \mu_y$  for all  $x \neq y$  in  $X \setminus X\pi$ , a contradiction. Hence,  $\gamma < \mu_z$  for some  $z \in X \setminus X\pi$ , and so  $\gamma$  is not maximal. Therefore, by Theorem 3.3.3,

$$q \leq g(\gamma) \leq g(\pi) = |X \setminus (\text{dom } \alpha \cup \text{dom } \beta)|$$

since  $\pi \subseteq \gamma$ , and so we have proved (ii).

Conversely, suppose the conditions hold. Then (i) implies that  $\alpha, \beta \subseteq \theta$  since  $\subseteq$  contains  $\leq$ . Therefore, Lemma 3.4.13 (i) and Theorem 3.4.10 imply (say)  $\pi = \alpha \cup \beta \in PS(q)$  is the join of  $\{\alpha, \beta\}$  under  $\subseteq$  and we claim that it is also the join under  $\leq$ . In addition, (i) and Lemma 3.4.12 imply that  $g(\alpha), g(\beta) \geq q$ . Now, if  $\pi = \alpha$ , then  $\beta \not\subseteq \alpha$  (since they are non-comparable) and so  $\beta < \alpha$  by Lemma 3.4.12, which contradicts the supposition. Thus,  $\alpha \not\subseteq \pi$  and this follows that  $\alpha < \pi$  by Lemma 3.4.12 again. Likewise, we have  $\beta < \pi$ . Finally, if  $\alpha, \beta \leq \mu$  for some  $\mu \in PS(q)$ , then  $\alpha, \beta \subseteq \mu$  and so  $\pi \subseteq \mu$ . Since (ii) holds, if  $X = \text{dom } \alpha \cup \text{dom } \beta$ , then  $X = \text{dom } \pi$  and so  $\pi = \mu$ . Otherwise, if  $|X \setminus (\text{dom } \alpha \cup \text{dom } \beta)| \geq q$ , then

$g(\pi) \geq q$  and so  $\pi < \mu$  by Lemma 3.4.12. In other words,  $\pi$  is the join of  $\alpha$  and  $\beta$  in  $PS(q)$  under  $\leq$ . ■