Chapter 1

Introduction

Let X be a non-empty set. As usual, $P(X)$ denotes the set of all *partial transformations* of X: that is, all transformations α whose *domain*, $\operatorname{dom}\alpha$, and *range*, $X\alpha$ (or $\operatorname{ran}\alpha$) are subsets of X. Let $T(X)$ denote the subsemigroup of $P(X)$ consisting of all $\alpha \in P(X)$ with $\operatorname{dom}\alpha = X$, which is called the *full transformation semigroup*. Also, let $I(X)$ denote the *symmetric inverse semigroup* on X: that is, the set of all injective mappings in $P(X)$.

Transformation semigroups play an important role in semigroup theory since it is well known that every semigroup is isomorphic to a subsemigroup of a suitable full transformation semigroup. Moreover, the Wagner-Preston Theorem states that every inverse semigroup is isomorphic to a subsemigroup of a suitable symmetric inverse semigroup. Therefore, in a sense, in order to study semigroups it suffices to consider transformation semigroups.

When X is an infinite set of cardinality p, and q is a cardinal such that $p \geq q \geq \aleph_0$. Write

$$BL(q) = \{\alpha \in T(X) \cap I(X) : d(\alpha) = q\}$$

where the *defect* of α, $d(\alpha) = |X \backslash X\alpha|$. Then $BL(q)$ is called the *Baer-Levi semigroup of type* (p, q).

As shown in [1] vol 2, Section 8.1, the Baer-Levi semigroup is an example of a right simple (it has no proper right ideals), right cancellative semigroup which is not a group, also, it has no idempotents. Moreover, it is known that $BL(q)$ is a model of a right simple, right cancellative semigroup without idempotents since any semigroup S with these properties can be *embedded* in some Baer-Levi semigroups, that is, there is a monomorphism $\varphi : S \to BL(q)$ for some cardinals p and q.

In 1984, Levi and Wood [9] defined the first class of maximal subsemigroups of $BL(q)$ by letting

$$M_A = \{\alpha \in BL(q) : A \not\subseteq X\alpha \text{ or } (A\alpha \subseteq A \text{ or } |X\alpha \backslash A| < q)\}$$

where A is a non-empty subset of X with $|X\backslash A| \geq q$. The authors showed that M_A is a maximal subsemigroup of $BL(q)$, and

$$I_A = \{\alpha \in M_A : A \not\subseteq X\alpha \text{ or } (A\alpha \subsetneq A \text{ or } |X\alpha \backslash A| < q)\}$$

is a prime maximal ideal of M_A. Later, Hotzel [2] studied maximal subsemigroups and maximal left unitary subsemigroups of $BL(q)$. He showed that there are many other maximal subsemigroups of $BL(q)$ and they are very complicated to describe.

A semigroup S of transformations of X is said to be *G_X-normal* if for every $\alpha \in G(X)$, $\alpha S \alpha^{-1} \subseteq S$ where $G(X)$ is the permutation group on a set X. Also, an automorphism φ of S is said to be *inner* if there exists $\gamma \in G(X)$ such that $\varphi(\beta) = \gamma\beta\gamma^{-1}$ for all $\beta \in S$. In 1983, Levi, Schein, Sullivan and Wood [8] showed that every automorphism of $BL(q)$ is inner. Moreover, the Baer-Levi semigroup is an example of a G_X-normal semigroup. In 1992, Levi [7] gave a complete description of injective endomorphisms of a G_X-normal semigroup S of injective transformations with infinite defect smaller than $|X|$. Also, the injective endomorphisms of the Baer-Levi semigroup were characterized.

In 1986, Mitsch [11] defined the *natural partial order* for any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1.$$

In 2003, Marques-Smith and Sullivan [10] studied various properties of the partial order $\leq$ and the *containment order* $\subseteq$ on $P(X)$, where $\subseteq$ is defined by, for $\alpha, \beta \in P(X)$,

$$\alpha \subseteq \beta \text{ if and only if } \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \text{ and } x\alpha = x\beta \text{ for all } x \in \operatorname{dom}\alpha.$$

They determined an upper bound Ω' and the join Ω of $\leq$ and $\subseteq$ (the smallest partial order on $P(X)$ containing $\leq$ and $\subseteq$). They also described the existence of maximal and minimal elements and the compatibility under these partial orders.

In this work we are interested in a related semigroup of $BL(q)$, namely, the *partial Baer-Levi semigroup on* X (as first defined in [13], p. 82) defined by

$$PS(q) = \{\alpha \in I(X) : d(\alpha) = q\}.$$

It is clear that $BL(q)$ is a subsemigroup of $PS(q)$. In 1975, Sullivan showed that, when $p = q$, every automorphism of $PS(q)$ is inner and the set of all automorphisms of $PS(q)$, Aut $PS(q)$ is isomorphic to $G(X)$. Later, in 2004, Pinto and Sullivan [12] showed that this is also true when $p > q$. They also determined the largest regular subsemigroup

$$R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}$$

of $PS(q)$, where $g(\alpha) = |X \setminus \operatorname{dom}\alpha|$ is called the *gap* of α, and studied the subsemigroup

$$S_r = \{\alpha \in PS(q) : g(\alpha) \leq r\}$$

where $\aleph_0 \leq r \leq p$. They also showed that some properties of $PS(q)$ differ from those of $BL(q)$. For example, $PS(q)$ is not right cancellative nor right simple, but, it is left and right reductive. Moreover, they showed that $PS(q)$ is a G_X-normal semigroup. Finally, they determined all ideals and described the Green's relations for $PS(q)$.

Although there are some research describe the algebraic properties and automorphisms of $PS(q)$, but there are still many properties have not been described. Since $BL(q)$ is a subsemigroup of $PS(q)$ and the researches on $BL(q)$ are a lot more than on $PS(q)$, we can modify the arguments and methods that were used for $BL(q)$ to describe some properties of $PS(q)$.

This thesis is divided into six chapters. Chapter 1 is an introduction to the research problems. Chapter 2 deals with some preliminaries and some useful results those will be used in later chapters. Chapter 3 to Chapter 5 are the main results of this research work, and the conclusion is in Chapter 6. In Chapter 3 we describe the existence of maximal and minimal elements, the compatibility, the meet $\alpha \wedge \beta$ and the join $\alpha \vee \beta$ under the partial orders $\leq$, $\subseteq$, Ω and Ω' on

$PS(q)$. We also describe the existence of the meet $\alpha \wedge \beta$ and the join $\alpha \vee \beta$ under each of these partial orders for $\alpha, \beta \in R(q)$ and for $\alpha, \beta \in I(X)$. In Chapter 4 we study the automorphisms of $R(q)$. Moreover, we study the isomorphisms between $R(X, p, q)$ and $R(Y, r, s)$ where $|X| = p \geq q \geq \aleph_0$, $|Y| = r \geq s \geq \aleph_0$ and the notations $R(X, p, q)$ and $R(Y, r, s)$ are written in place of the subsemigroups $R(q)$ on the set X and Y, respectively. In Chapter 5, we present some classes of maximal subsemigroups of $PS(q)$ when $p = q$. We also give necessary and sufficient conditions for subsemigroups of $PS(q)$ to be maximal when $p > q$.