CHAPTER V

CONCLUSION AND RECOMMENDATION

In this thesis, we presented dynamical properties of two classes of delayed population models, which are the population models with and without age-structure. The analysis included stability analysis and Hopf bifurcation analysis. In this chapter, we give readers the conclusions and some recommendations provided for further study in the field of DDEs or related areas.

5.1 Conclusion

We studied the population models with the form of delay differential equations. We started studying fundamental theorems of ODEs to be the basic concepts for study further on dynamics of DDEs. The main aim of the thesis was to analyse the delayed population models with a constant delay. Two types of population models, the models with and without stage-structure, were analysed. The majority method used for the analysis was the linearisation method about the equilibria to transform a nonlinear DDE to be a linear DDE. Apart from that, we also investigated the sufficient conditions for the stability of the equilibrium using the theorem of the stability for linear DDEs. Then we obtained the sufficient conditions for the existence of a Hopf bifurcation of the selected population models. Also, the oscillation of the solutions near the Hopf bifurcation points was examined. Finally, we applied the analytical results and the obtained theorems to two well-known population models, which are the Mackey-Glass equation and the Nicholson's blowflies equation. To support our obtained results, the numerical solutions were illustrated using Matlab® software.

In Chapter II, we presented fundamental concepts of ODEs and DDEs. First, we introduced some theorems of ODEs. For the case of DDE, we demonstrated that the method of steps can transform the DDE to a system of ODEs with infinite equations. In addition, the theorem of the existence and uniqueness for the solution of DDEs was presented using similar ideas from those of ODEs. Some theorems were stated for the dynamical analysis of DDEs. Also, the idea for analysis of the existence of the Hopf bifurcation was provided.

Chapter III provides the analysis of the population model without stage-structure of the form:

$$N'(t) = -\gamma N(t) + \beta f(N(t-\tau))N(t-\tau).$$

In the analysis part, we determined sufficient conditions for the asymptotic stability of the model's equilibrium. From Chapter III, we had the linearised equation as

$$y'(t) = -\gamma y(t) + \eta y(t-\tau),$$

where $\eta = \beta\big(f(\tilde{N}) + \tilde{N}f'(\tilde{N})\big)$, and $\tilde{N}$ is the equilibrium. Then the stability properties were provided in Theorem 3-1 as follows. The zero equilibrium $\tilde{N}_0$ is asymptotically stable when $\gamma > \eta$, and it is unstable when $\gamma < \eta$. On the contrary, we showed that the positive equilibrium $\tilde{N}_+$ exists only if $\gamma < \eta$, and it is asymptotically stable for $\tau \in [0, \tau_0)$, otherwise it is unstable when $\gamma > \eta$. For $\tau > \tau_0$, we obtained the sufficient conditions for the existence of a bifurcation point. The Hopf bifurcation points can be figured out by the following conditions:

$$\tau_k = \begin{cases} \dfrac{1}{\sqrt{\eta^2 - \gamma^2}}\left[\arccos\left(\dfrac{\gamma}{\eta}\right) + 2k\pi\right]; & \eta < 0, \\[2ex] \dfrac{1}{\sqrt{\eta^2 - \gamma^2}}\left[2\pi - \arccos\left(\dfrac{\gamma}{\eta}\right) + 2k\pi\right]; & \eta > 0, \end{cases}$$

and the additional condition of the parameters is $\gamma > \eta$. The theorem for the existence of the Hopf bifurcation was stated in Theorem 3-2.

In the numerical simulation, we applied the analytical results to the Mackey-Glass equation and the Nicholson's blowflies equation. We also illustrated to dynamics of the solutions about the bifurcation parameters τ_0, τ_1 and τ_2. The results showed the behaviours of the solutions before and after the Hopf bifurcation points.

In Chapter IV, the main aim was to analyse the population model with stage-structure.

$$N'(t) = -\gamma N(t) + \beta e^{-\delta\tau} f(N(t-\tau))N(t-\tau).$$

We obtained the stability properties and the conditions for the occurrence of a Hopf bifurcation. The linearised equation of the selected model in this case is

$$y'(t) = -\gamma y(t) + a(\tau)y(t-\tau),$$

where $a(\tau) = \beta e^{-\delta\tau}\left(f(\tilde{N}) + \tilde{N}f'(\tilde{N})\right)$, and $\tilde{N}$ be an equilibrium of the population problem. The sufficient conditions for the asymptotic stability are depended on γ and $a(\tau)$. We shown in Theorem 4-1 that if $\gamma > |a(\tau)|$, then $\tilde{N}$ is asymptotically stable. On the other hand, if $\gamma < a(\tau)$ and $a(\tau) < 0$, then $\tilde{N}$ is unstable. Moreover, if $-\gamma > a(\tau)$, then $\tilde{N}$ is asymptotically stable for $\tau \in [0, \tau_0)$, and it is unstable for $\tau > \tau_0$. In this case we found that the Hopf bifurcation can be occurred when τ is sufficiently large. The bifurcation values are

$$\tau_k = \frac{1}{\omega_k}[\theta + 2k\pi],$$

where $k = 0,1,2,\ldots$, and $\omega_k = \sqrt{a(\tau_k)^2 - \gamma^2}$, which defined in Chapter IV. In this thesis, we considered only the smallest values of bifurcation parameter τ, i.e. τ_0. Then, we applied the obtained results to the Mackey-Glass equation and the Nicholson's blowflies equation. To calculate the bifurcation point τ_0, it is difficult to solve explicitly. Thus the graphical and numerical methods are necessary. We also found that there is an additional condition of parameters for the occurrence of a Hopf bifurcation, which shown in Theorem 4-2, i.e. $a^2(\tau) > \dfrac{\gamma(\delta+\gamma)}{(1-\delta\tau)}$.

To sum up, after we have studied dynamics of DDEs, we know how to investigate the dynamical behaviour for the stability and the existence of a Hopf bifurcation. We found that the problem without stage-structure is simpler than the case of age-structure population models. This is because the equilibrium of the model with age-class contains the bifurcation parameter, or the time-delay τ. Moreover, as shown in the examples in Chapter IV, the value of the bifurcation point τ_0 is difficult to determine. Thus we need some advantages of numerical methods to find it. As the results, we can conclude that age-structure models have more complicated in their dynamics comparing with models without an age-class.

5.2 Suggestions for Further Work

In this thesis, we studied the population models with a constant delay, and they are the retard type of DDEs. For further study, one can apply the methods and techniques used in this thesis to more complicated problems, such as a neutral DDE, or non-constant delay.

In addition, as this thesis was primary focused on the (linear) stability and Hopf bifurcation of the selected DDEs, there are numerous research directions that have not been fully explored, for examples some advance on bifurcation analysis, direction of the Hopf bifurcation and chaos analysis. Interested readers can find in the references, which state that there are many undetermined problems.

Finally, in our work, the behaviour of (local) Hopf bifurcation was investigated using the linearisation method, then the results could only conclude for local behaviour. However, for an advanced analysis, a global analysis might be interested, and our results can be improved for some advanced techniques.