CHAPTER II

THEORETICAL PRELIMINARY

In this chapter, we describe the fundamental concepts of ordinary differential equations (ODEs) and delay differential equations (DDEs). Firstly, we determine the existence and uniqueness theorem of the solutions and stability properties of ODEs. Next, we introduce population models obtained by DDEs. We also show some analytical methods for solutions of simple linear DDEs and present the existence and uniqueness theorem. Some useful theorems for stabilities of equilibria are described. We also compare some similarities and differences between ODEs and DDEs. In addition, the idea to investigate Hopf bifurcations for both ODEs and DDEs are presented, together with some useful theorems to show an existence of a Hopf bifurcation for first-order ODEs and DDEs.

2.1 Ordinary Differential Equations

Many problems, such as epidemics, population and economics problems, can be modeled by ODEs. In general, the ODE is an equation which depends on an independent variable, a function of the independent variable, and derivatives of the function or state variable [9].

2.1.1 Existence and uniqueness theorems of the solutions

Most differential equations will have solutions and that solutions of initial-value problems should be unique. Real life, however, is not idyllic. Thus it is desirable to know in advance of trying to solve an initial-value problem whether a solution exits and, when it does, whether it is the only solution of the problem [21].

In this part, we introduce the theorem for the existence and uniqueness of solution of ODEs.

Firstly, consider the initial value problem:

$$\begin{aligned} y'(t) &= f(t, y(t)); \qquad t > t_0, \\ y(t_0) &= y_0. \end{aligned} \tag{2-1}$$

where $y(t)$ is the state variable at current time t. The following theorem represents the condition for the existence and uniqueness of a solution of the initial value problem (2-1).

Theorem 2-1 (Local existence and uniqueness theorem, [9])

Let $\mathcal{R}$ be a rectangular region in the ty-plane, defined by $a \le t \le b$, $c \le y \le d$, that contains the point (t_0, y_0) in its interior. If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous on $\mathcal{R}$, then there exists an interval $\mathcal{I}$ centered at t_0 and a unique function $y(t)$ defined on $\mathcal{I}$ satisfying the initial-value problem (2-1).

Theorem 2-1 shows that the existence and uniqueness of a solution of the initial value problem (2-1) depends on the continuities of $f(t,y)$ and $\frac{\partial f}{\partial y}$, which are relatively easy to examine. The geometry representation of Theorem 2-1 is illustrated in Figure 2-1

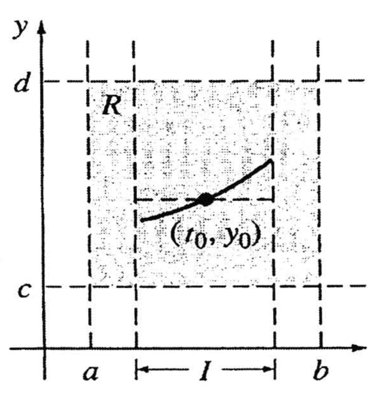

FIGURE 2-1 Rectangular region $\mathcal{R}$ (adapted from [30])

2.1.2 Stability properties

We focus in this part for stability properties of the equilibria of (2-1). Here we state definitions of some types of stability of the equilibria. In the simplest case, the first-order autonomous equation is

$$\begin{aligned} y'(t) &= f(y(t)); \qquad t > t_0, \\ y(t_0) &= y_0. \end{aligned} \tag{2-2}$$

The main objective in this part is to state definitions of stability types of the equilibrium (2-2). Here we state definitions for some types of stability.

Definition 2-2 (Stability of an equilibrium, [19])
Let $\tilde{y}$ be an equilibrium and u be a solution of (2-2).

(1) The equilibrium $\tilde{y}$ is called *stable* if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|u(t_0) - \tilde{y}| < \delta$ for some t_0, provided that $|u(t) - \tilde{y}| < \varepsilon$ for all $t \geq t_0$. If the equilibrium $\tilde{y}$ is not stable, then it is *unstable*.

(2) The equilibrium $\tilde{y}$ is called *asymptotically stable* if it is stable and $\delta > 0$ can be chosen so that $|u_0(t) - \tilde{y}| < \delta$ and $\lim_{t\to\infty} u(t) = \tilde{y}$.

Note that in this thesis, we focus on long term behaviour of solutions, thus we concern mainly on the asymptotic stability properties.

2.2 Delay Differential Equations

More realistic models, in general, should include some of the past states; that is ideally. A real system should be modeled by differential equations with time delays. Indeed, according to recent papers, the use of DDEs in modeling of population dynamics is currently very active. Many previous results show that dynamics of DDEs are more complicated than in cases of ODEs. In this part we demonstrate fundamental terminologies of DDEs, and some well-known methods to find analytical solutions of DDEs.

Consider a simple first-order DDE with a delay:

$$y'(t) \;=\; f(t, y(t), y(t-\tau), y'(t-\tau)). \tag{2-3}$$

In general the DDEs models of the form (2-3) can be classified by types of the time delay τ as follows.

1) If τ is a constant, then τ is called a *discrete delay.*
2) If τ is a function of t, i.e. $\tau \;=\; \tau(t)$, then τ is called a *variable delay* or a *time-dependent delay.*
3) If τ is a function of both t and y, i.e. $\tau \;=\; \tau(t, y(t))$, then τ is called a *state- dependent delay.*

Some examples on different types of the delay τ in DDEs are as follows.

A discrete delay:	$y'(t) = -y(t-1)$,
A time-dependent delay:	$y'(t) = \lambda y(2t+1)$,
A state-dependent delay:	$y'(t) = \beta y(t - \sin y(t))$.

In addition, the general DDEs can be classified also by an existence of delay terms in the state-variable, namely $y(t-\tau)$ and $y'(t-\tau)$, as follows.

1) If DEE has only the term of $y(t-\tau)$ and there is no term of $y'(t-\tau)$, then it is called *retarded delay differential equation* (RDDE).
2) If DEE has term $y'(t-\tau)$, which is the highest derivative of (2-3), then it is called *neutral delay differential equation* (NDDE).

Examples of RDDE and NDDE are as follows.

RDDE: $$N'(t) = -rN(t) + \frac{\beta\theta^n N(t-\tau)}{\theta^n + N^n(t-\tau)},$$

NDDE: $$x'(t) = \gamma x(t-1) + \beta x'(t-1).$$

In general, a first-order RDDE is the equation with the form:

$$y'(t) = f(t, y(t), y(t-\tau)), \tag{2-4}$$

where the term $y'(t-\tau)$ is omitted. On the other hand, an NDDE is the equation which includes a term of $y'(t-\tau)$. Thus, general equation of the NDDE type is

$$y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)). \tag{2-5}$$

Note that, throughout this work, all problems are DDEs with retarded argument, i.e. they are of the form (2-4). Next, we will provide some analytical methods to investigate solutions of DDEs.

2.2.1 Analytical methods for solutions of linear DDEs

In general, it is difficult to determine solutions of a DDE. However, some simple delay problems can be solved for analytical or exact solutions by the Laplace transform and the method of steps. In this part, we show readers how those methods can be used to solve analytically for simple linear DDEs.

The method of Laplace transform

Laplace transform is a widely technique for solving differential equations with initial conditions. In this part, we will find analytical solutions of a simple linear DDE using the Laplace transformation method.

First, consider the following DDE:

$$x'(t) = ax(t-\tau); \qquad t > 0, \tag{2-6}$$

with the initial function

$$x(t) = \varphi(t); \qquad -\tau \le t \le 0. \tag{2-7}$$

Like with many cases of ODEs, the method of Laplace transform can be applied to evaluate solutions of DDEs with the same technique. Firstly, we state an example to solution of DDEs by the method of Laplace transform. By taking the Laplace transform, $\mathcal{L}\{\cdot\}$, to both sides of (2-6). It yields that

$$\mathcal{L}\{x'(t)\} = \mathcal{L}\{ax(t-\tau)\}.$$

Hence

$$s\mathcal{L}\{x(t)\} - x(0) = ae^{-\tau s}\left(\int_{-\tau}^{0} e^{-st}x(t)dt + \int_{0}^{\infty} x(t)e^{-st}dt\right)$$

$$= ae^{-\tau s}\left(\int_{-\tau}^{0} e^{-st}\varphi(t)dt + \mathcal{L}\{x(t)\}\right).$$

Let $\mathcal{L}\{x(t)\} = X(s)$ and $x(0) = \varphi(0) = x_0$, then we have

$$sX(s) - x_0 = ae^{-\tau s}\left(\int_{-\tau}^{0} e^{-st}x(t)dt + X(s)\right). \tag{2-8}$$

In the simple case, let the initial function (2-7) be $\varphi(t) = 1$. It is not difficult to show that

$$\int_{-\tau}^{0} e^{-st}\varphi(t)dt = -\frac{1}{s} + \frac{e^{\tau s}}{s}.$$

Then, (2-8) becomes

$$sX(s) - 1 = -\frac{ae^{-\tau s}}{s} + \frac{a}{s} + ae^{-\tau s}X(s).$$

After rearrange above equation, we can see that

$$X(s) = \frac{s - a(e^{-\tau s} - 1)}{s(s - ae^{-\tau s})}. \tag{2-9}$$

The solution $x(t)$ can be determined by taking the inverse Laplace transform to (2-9), then

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{s - a(e^{-\tau s} - 1)}{s(s - ae^{-\tau s})}\right\}.$$

According to the definition of inverse Laplace transform [11], we have

$$x(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\left\{\frac{s - a(e^{-\tau s} - 1)}{s(s - ae^{-\tau s})}\right\}ds; \quad t > 0. \tag{2-10}$$

Note that to find the solution $x(t)$ from (2-10), it is difficult to obtain an explicit solution. However, the solution can be determined using residues at poles technique from complex analysis [5], or by a numerical computation [3].

We can see that, in the general solution of (2-10), the method of Laplace transform cannot solve for the solutions of (2-6) by itself, even though the problem looks so simple. The analytical solution of (2-6) can be determined by applying other techniques, such as residues at poles theorem, or some theorems provided by complex analysis, or by a numerical computation. As the results, the method of Laplace transform will always be used only in simple linear differential equations but, it is not good for our research, which the problems are nonlinear.

The method of steps

One of the most popular methods to determine solutions of a DDE is *the method of steps*. This method is used to solve a DDE on the interval $\left[n\tau,(n+1)\tau\right]$, where $n = 0, 1, 2, \ldots$, which based on the previous solutions on $\left[(n-1)\tau,n\tau\right]$.

Consider the DDE with the initial function:

$$\begin{aligned} y'(t) &= f(t,y(t),y(t-\tau)); && t>0, \\ y(t) &= \varphi(t); && -\tau \le t \le 0. \end{aligned} \tag{2-11}$$

To find the solution on the interval $\left[0,\tau\right]$, we replace $y(t-\tau)$ with the initial function $\varphi(t-\tau)$, and then solve instead for the solution of ODE:

$$y_1'(t) = f(t,y_1(t),\varphi(t-\tau)),$$

with the initial condition $y_1(0) = \varphi(0) = \varphi_0$. Next, we use the obtained solution $y_1(t)$ to find the solution $y_2(t)$ on $\left[\tau,2\tau\right]$, i.e. the function $y_2(t)$ is the solution of

$$y_2'(t) = f(t,y_2(t),y_1(t-\tau)).$$

Suppose that the solution of (2-11) is continuous, we set the initial condition $y_2(\tau) = y_1(\tau)$. Solutions on the successive intervals can be determined by the same technique. Since the solutions are solved step by step, it is named *"the method of steps"*. The processes of finding solutions of DDEs can be illustrated in Figure 2-2.

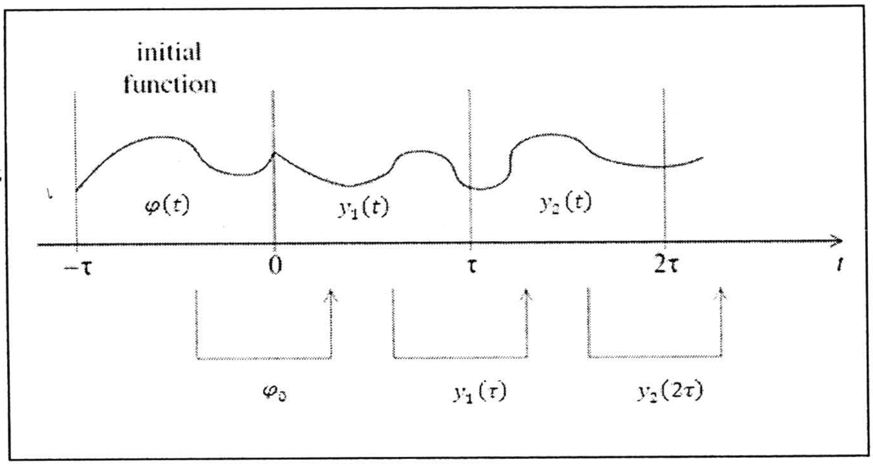

FIGURE 2-2 Processes to obtain solutions of a DDE by the method of steps (adapted from [20])

Figure 2-2 shows the processes to solve a DDE by the method of steps. To be clearer, we give readers an example of the method of steps to find solutions of a DDE in the following example.

Example 2-1 Consider the DDE:

$$y'(t) = -y(t-1); \quad t > 0, \tag{2-12}$$

$$y(t) = \varphi(t) = 1; \quad t \le 0. \tag{2-13}$$

Firstly, to find the solution of (2-12) - (2-13) on $[0,1]$, we need to solve $y_1(t)$ from

$$y_1'(t) = -\varphi(t-1) = -1, \tag{2-14}$$

with the initial condition $y_1(0) = \varphi(0) = 1$. Note that, to obtain (2-14), we replace $y(t-1)$ from (2-12) with the initial function $\varphi(t-\tau)$ in (2-13). The solution of (2-14) with the initial condition $y_1(0) = 1$ is

$$y_1(t) = 1 - t,$$

where $0 < t < 1$.

Next, to find the solution of (2-12) on the interval $[1,2]$, we replace $y(t)$ with the previous solution $y_1(t-1)$ and then solve for $y_2(t)$ from

$$y_2'(t) = -y_1(t-1) = t - 2, \qquad \text{(2-15)}$$

with the initial condition $y_2(1) = 0$. It is not difficult to show that the solution of (2-15) with $y_2(1) = 0$ is

$$y_2(t) = \frac{t^2}{2} - 2t + \frac{3}{2},$$

where $1 < t < 2$.

Similarly, for the solution of (2-12) on the next interval $[2,3]$, we need to solve for $y_3(t)$ from

$$y_3'(t) = -y_2(t-1) = \frac{-t^2}{2} + 3t - 4, \qquad \text{(2-16)}$$

with the initial condition $y_3(2) = -\frac{1}{2}$. The solution of (2-16) with the given condition is

$$y_3(t) = -\frac{t^3}{6} + \frac{3t^2}{2} - 4t + \frac{17}{6},$$

where $2 < t < 3$.

Using the same technique, we can solve for the solutions on the successive intervals, namely $y_4(t), y_5(t), \ldots$. Figure 2-3 illustrates the solutions of (2-12) with the initial function (2-13), which obtained by the method of steps. ■

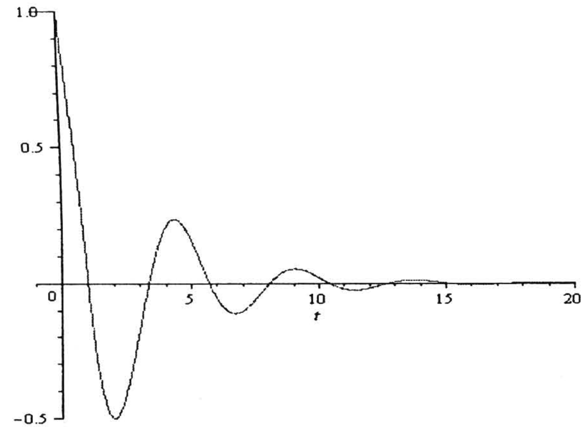

FIGURE 2-3 The solution of (2-12) - (2-13) obtained by the method of steps.

To sum up, the method of steps can be used to find the solution of (2-12) - (2-13), and it can be written as the system of differential equations:

$$\begin{aligned} y_1'(t) &= f(t, y_1(t), \varphi(t-1)); & 0 \le t \le 1, \\ y_2'(t) &= f(t, y_2(t), y_1(t-1)); & 1 \le t \le 2, \\ &\vdots & \vdots \\ y_n'(t) &= f(t, y_n(t), y_{n-1}(t-1)); & n-1 \le t \le n, \\ &\vdots & \vdots \end{aligned} \tag{2-17}$$

with the initial conditions $y_1(0) = \varphi(0)$ and $y_{n+1}(n) = y_n(n)$, where $n = 1,2,3,\ldots$. One can see that the system of differential equations (2-17) can be written in the matrix form as

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(t-\tau)), \tag{2-18}$$

where $\mathbf{y}, \mathbf{f} \in \mathbb{R}^{\infty}$ an infinitely dimensional vectors. As the results, a single DDE can be transformed to a system of ODEs with infinite equations. Hence, dynamics of the solutions of DDEs can be examined by an infinitely dimensional ODEs.

2.2.2 Existence and uniqueness theorem of the solutions of DDEs

Consider the DDE:

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)); \qquad & t > t_0, \\ y(t) &= \varphi(t); & t_0 - \tau \le t \le t_0. \end{aligned} \qquad (2\text{-}19)$$

Let $y(t; t_0, \varphi)$ be a *solution* of (2-19), where t_0 and φ are defined, respectively, as the initial point and initial function $y(t) = \varphi(t)$, where $t \in [t_0 - \tau, t_0]$. Thus $y(t; t_0, \varphi)$ must satisfy the equation and condition in (2-19). For convenience, we define $y(t) \equiv y(t; t_0, \varphi)$ as the solution of (2-19).

Now we will state the theorem for an existence and uniqueness of the solution of DDEs. In many monographs, the theorem is stated in term of functional. define $y_t \in C$ as a function of both $y(t)$ and $y(t-\tau)$, where C is the space of continuous functions, hence y_t is a functional and $y_t = g(y(t), y(t-\tau))$. Thus equation (2-19) can be rewritten in term of the functional as

$$\begin{aligned} y'(t) &= f(t, y_t); \qquad & t > t_0, \\ y(t) &= \varphi(t); & t_0 - \tau \le t \le t_0. \end{aligned} \qquad (2\text{-}20)$$

Throughout this study, we assume that f is continuous for all arguments.

Since a DDE can be transformed as infinite equations of ODEs, the existence and uniqueness theorem of infinite-dimension ODEs is enough to explain the case of the DDEs. We now state Theorem 2-3 for the conditions in which a DDE with initial condition (2-20) exists a unique solution.

Theorem 2-3 (The existence and uniqueness theorem; [15])

Suppose that A is an open subset in $\mathbb{R} \times C$ and $f : A \to \mathbb{R}$ is continuous on A. If $(t_0, \varphi) \in A$, and f is Lipschitz in the second argument, i.e.,

$$\left| f(t, y_1) - f(t, y_2) \right| \le k \left| y_1 - y_2 \right|,$$

then there is a unique solution of (2-20) *satisfying $y(t) = \varphi(t)$, where $t \in [t_0 - \tau, t_0]$ and k is a Lipschitz constant.*

2.2.3 Stability properties of DDEs

In this part, we are interested in classes and some useful theorems for the stability of the equilibrium of autonomous DDEs of the form

$$y'(t) = f(y(t), y(t-\tau)). \tag{2-21}$$

Firstly, we state definitions of some stability types of the equilibrium.

Definition 2-4 (Stability of the solution, [15])

Suppose that $\tilde{y}$ is an equilibrium and u is the solution of (2-21).

(1) The equilibrium $\tilde{y}$ is called *stable* if, for any $t_0 \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all initial value u_0 with $|u_0(t) - \tilde{y}| < \delta$, provided that $|u(t) - \tilde{y}| < \varepsilon$ for all $t \geq t_0$. If $\tilde{y}$ is not stable, then it is *unstable*.

(2) The equilibrium $\tilde{y}$ is called *asymptotically stable* if it is stable and $\delta > 0$ can be chosen so that $|u_0(t) - \tilde{y}| < \delta$ and $\lim_{t \to \infty} u(t) = \tilde{y}$.

(3) If the number δ in the definition of stable is independent of t_0, then the equilibrium $\tilde{y}$ is (locally) uniformly stable.

In Definition 2-4, we provide different types of stabilities. In general the equilibrium $\tilde{y}$ is stable whenever it is bounded when the initial point t_0 is given. Otherwise it is unstable. If the equilibrium $\tilde{y}$ is stable and the solution tends to the equilibrium itself as $t \to \infty$, then it is asymptotically stable. In addition, if the equilibrium is stable without any conditions of the initial data t_0, then the equilibrium $\tilde{y}$ is called (locally) uniformly stable.

To study (local) stability of a linear DDE, it is easily to study from its characteristic equation. Consider the following linear DDE:

$$y'(t) = -\gamma y(t) + \beta y(t-\tau); \qquad t > t_0. \tag{2-22}$$

Like with the case of an ODE, suppose that the solution $y(t)$ of (2-22) is the exponential form, i.e. $y(t) = e^{\lambda t}$. Substituting $y(t)$ into (2-22), we have

$$\lambda e^{\lambda t} = -\gamma e^{\lambda t} + \beta e^{\lambda(t-\tau)}.$$

Multiply both sides by $e^{-\lambda t}$, then it yields

$$\lambda = -\gamma + \beta e^{-\lambda\tau}. \tag{2-23}$$

Equation (2-23) is called the *characteristic equation* of (2-22), which is different from a case of ODE. The characteristic equation of DDE has a term of $e^{-\lambda\tau}$ which cannot be found in ODE.

In the case of ODEs, stability properties can be determined by roots of the characteristic equation λ, the so called *eigenvalues*. If all eigenvalues have negative real-parts, then the trivial solution, $y(t) \equiv 0$, is asymptotically stable. If at least one of the eigenvalues has a positive real part, then the trivial solution is unstable. These properties are similar to the case of DDEs. Here we state the theorem for stability of linear DDE.

Theorem 2-5 [15]

Assume that $y(t)$ *is a non-zero solution of* (2-22), *and* λ *is the root of* (2-23).

If $\text{Re}(\lambda) < 0$ *for all* λ, *then* $y(t) \to 0$ *as* $t \to \infty$, *i.e. the trivial solution* $y = 0$ *is asymptotically stable.*

Note that, If all real parts of eigenvalues are less than zero, then $y(t)$ tends to zero as $t \to \infty$. From Theorem 2-5, the trivial solution of (2-22), $y \equiv 0$, is asymptotically stable.

Next, we are interested on the behaviour of a linear DDE as $t \to \infty$. We state the theorem for a sufficient condition of the equilibrium of (2-22) to be asymptotically stable as follow.

Theorem 2-6 [15]

Let γ, $\beta \in \mathbb{R}$ *and* $y(t)$ *be a solution of* (2-22), *where* $\tau > 0$. *If* $|\beta| < \gamma$, *then* $\lim_{t\to\infty} y(t) = 0$.

Proof. From (2-22), the characteristic equation is

$$\lambda = -\gamma + \beta e^{-\lambda\tau}. \quad (2\text{-}24)$$

Let $\lambda = u + \mathrm{i}\,v$, where $u, v \in \mathbb{R}$. Hence (2-24) becomes

$$\begin{aligned} u + \mathrm{i}\,v &= -\gamma + \beta e^{-\tau(u+\mathrm{i}\,v)} \\ &= -\gamma + \beta e^{-u\tau}(\cos v\tau - \mathrm{i}\sin v\tau). \end{aligned}$$

We can see that

$$\mathrm{Re}(\lambda) = u = -\gamma + \beta e^{-u\tau} \cos v\tau.$$

Assume that $\mathrm{Re}(\lambda) = u < 0$, then

$$\gamma > \beta e^{-u\tau} \cos v\tau > \beta \cos v\tau.$$

Since $|\cos v\tau| < 1$. Hence, it is not difficult to show that $-\gamma < \beta < \gamma$, or $|\beta| < \gamma$. As the results, we can conclude that if the condition $\gamma > |\beta|$ is satisfied, then $\mathrm{Re}(\lambda) = u < 0$ for all λ. As in Theorem 2-5, it is adequate to conclude that $y(t) \to 0$ as $t \to \infty$. This completes the proof. ■

Next, we show the conditions for the stability properties of linear delay differential equation in the following lemma.

Lemma 2-7 (Crauste [8], pp. 270-272)

Consider the linear delay differential equation:

$$x'(t) = Ax(t) + Bx(t-\tau), \tag{2-25}$$

where $A, B \in \mathbb{R}$. *Here,* (2-25) *has only unique equilibrium, namely* $\tilde{x} = 0$. *The stability properties of the equilibrium* $\tilde{x}$ *are as follows.*

1. *If* $B \geq 0$, *then* $\tilde{x} = 0$ *is asymptotically stable for all* $\tau \geq 0$ *when* $A + B < 0$, *and unstable for all* $\tau \geq 0$ *when* $A + B > 0$.
2. *If* $B < 0$, *then* $\tilde{x} = 0$ *is asymptotically stable for all* $\tau \geq 0$ *when* $A < B$, *and unstable for all* $\tau \geq 0$ *when* $A + B > 0$.
3. *If* $B < 0$ *and* $-B > |A|$, *then* $\tilde{x} = 0$ *is asymptotically stable for* $\tau \in [0, \tau_0)$, *and it is unstable for* $\tau > \tau_0$.

Note that we can apply Lemma 2-7 to the linearised DDE (2-22). By comparison (2-22) with (2-25), it follows that $A = -\gamma$ and $B = \beta$. According to conditions (1) and (2) in Lemma 2-7, we find that if $\gamma > \beta$, then the equilibrium is asymptotically stable. From condition (3), we can conclude that there exists a value of τ_0 so that the equilibrium is asymptotically stable if $\tau \in [0, \tau_0)$, and it is unstable when $\tau > \tau_0$.

2.3 Similarities and Differences between ODEs and DDEs

In this section, we present some similarities and differences between ODEs and DDEs. Consider the first-order initial value problems (IVPs):

$$\text{ODE:} \quad \begin{aligned} y'(t) &= f(t, y(t)); & t > t_0, \\ y(t_0) &= y_0, \end{aligned} \tag{2-26}$$

and

$$\text{DDE:} \quad \begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)); & t > t_0, \\ y(t) &= \varphi(t); & t_0 - \tau \leq t \leq t_0. \end{aligned} \tag{2-27}$$

It can be easily seen that if $\tau = 0$, the DDE becomes an ODE.

The most obvious difference between ODE and DDE is the initial conditions. In the initial value problems (2-26), the initial condition of ODE is a point of the initial time t_0, whereas the initial data of DDE in (2-27) is a function of *history* or an interval of time before the initial point t_0, i.e. $\tau \leq t_0$ or $t_0 - \tau \leq t \leq t_0$, where $\tau > 0$.

In term of dynamical properties, one can see that even a simple DDE may have more complicate behaviour compare with an ODE. To show this statement, consider the following examples:

$$y'(t) = -y(t - \frac{\pi}{2}), \tag{2-28}$$

and

$$y'(t) = -y(t). \tag{2-29}$$

Equations (2-28) and (2-29) are examples of the first-order linear DDE and ODE, respectively. Solutions of (2-28) can be oscillated, for examples $y(t) = \cos t$ and $y(t) = \sin t$ are both periodic solutions of (2-28). On the contrary, solutions of (2-29) can only be trivial (constant) solution or monotone solutions.

Next, we show the simplest example. Consider the ODE:

$$y'(t) = \gamma y(t). \tag{2-30}$$

It is not difficult to show that the solution of (2-30) with the initial condition $y(0) = y_0$ is

$$y(t) = y_0 e^{\gamma t}.$$

Next consider the DDE:

$$y'(t) = \gamma y(t - \tau), \tag{2-31}$$

with the initial condition $y(t) = \varphi(t)$. Solutions of (2-31) are more complicated than solutions of (2-30). Also its dynamics are even richer.

Finally, we also show some similarities and differences between the characteristic equations of ODEs and DDEs in the following example.

Example 2-2 Consider the simplest ODE:

$$y'(t) = \gamma y(t). \tag{2-32}$$

Suppose that the solution of (2-32) is the exponential form. Let $y(t) = e^{\lambda t}$, then $y'(t) = \lambda e^{\lambda t}$. Hence (2-32) becomes

$$\lambda e^{\lambda t} = \gamma e^{\lambda t}.$$

The characteristic equation of (2-32) is

$$\lambda = \gamma.$$

Next, consider the DDE:

$$y'(t) = \gamma y(t) + \beta y(t - \tau). \tag{2-33}$$

Use the same idea as in the ODE. Put $y(t) = e^{\lambda t}$ and $y'(t) = \lambda e^{\lambda t}$ into (2-33), we have

$$\lambda e^{\lambda t} = \gamma e^{\lambda t} + \beta e^{\lambda(t-\tau)}.$$

Hence, the characteristic equation of (2-33) is $\lambda = \gamma + \beta e^{-\lambda\tau}$. ■

From Example 2-2, we may conclude that similarity between ODE and DDE is that the solutions can be defined by the exponential form. The difference between ODE and DDE is that the characteristic equation of a DDE has term(s) of $e^{-\lambda\tau}$, which cannot be found in ODEs. Moreover, the solutions and dynamics of DDEs are more complicated.

2.4 Hopf Bifurcations of the First-Order Differential Equations

2.4.1 Hopf bifurcations of the first-order ODEs

In this section, we present a basic idea of bifurcations for ODEs. Consider the first-order system of differential equations:

$$\mathbf{y}'(t) = \mathbf{f}(\mathbf{y}, \mu); \quad \mathbf{y}(t_0) = \underset{\sim}{\xi}_0, \tag{2-34}$$

where $\mathbf{y} \in \mathbb{R}^p$ is the state variable, and $\mu \in \mathbb{R}$ is the parameter. Suppose that $\tilde{\mathbf{y}}$ is the equilibrium of (2-34), then we have the linearised equation

$$\mathbf{x}'(t) = \mathrm{D}\mathbf{f}(\tilde{\mathbf{y}}, \mu)\mathbf{x}(t); \tag{2-35}$$

where $\mathrm{D}\mathbf{f}(\tilde{\mathbf{y}}, \mu)$ is the first derivative with respect to $\mathbf{y}$ of $\mathbf{f}$ at $\tilde{\mathbf{y}}$, obtained from the Taylor's series.

A bifurcation, in general, is a change of the topological type of the system as its parameters pass through a *bifurcation* (*critical*) *value*. In (2-34), regarding to the parameter μ as the bifurcation parameter, we need to find the bifurcation values or bifurcation points. First we provide readers the definition and a theorem dealing with a Hopf bifurcation of (2-34) and (2-35).

Suppose that $\lambda_{1,2} = \alpha \pm i\omega_0$ be the eigenvalues of (2-35), where $\alpha \in \mathbb{R}$ and $\omega_0 > 0$. Hence, we classify the type of bifurcation and we state definition of Fold and Hopf bifurcations as follows.

Definition 2-8 (Fold and Hopf bifurcations: [28], pp. 80)

1) The bifurcation associated with the appearance of an eigenvalue $\lambda_1 = 0$ is called a fold (or tangent) bifurcation.

2) The bifurcation corresponding to the presence of eigenvalues with a pair of purely imaginary roots, i.e. $\lambda_{1,2} = \pm i\omega_0$, where $\omega_0 > 0$, is called a Hopf (or Andronov-Hopf) bifurcation.

Remark 2-1:

1. The fold bifurcation has a lot of other names, including *limit point, turning point*, and *saddle-node bifurcation*.

2. In (2-34), with $\mathbf{y} \in \mathbb{R}^p$, the fold bifurcation is possible if $p \geq 1$, but for the Hopf bifurcation we need $p \geq 2$ (see [28] for more details).

Next, we state the Hopf bifurcation theorem of ODEs as follows.

Theorem 2-9 (Hopf Bifurcation Theorem; [22])

Consider equation (2-34) *in dimension* $p = 2$. *Assume that*

(a) $\mathbf{f} \in C^r(\mathbb{R}^p \times \mathbb{R}, \mathbb{R}^p)$ *for* $r \geq 2$ *and that* $\mathbf{f}(\tilde{\mathbf{y}}, \mu) = \mathbf{0}$ *for all* $\mu \in \mathbb{R}$.

(b) $\mathrm{D}\mathbf{f}(\tilde{\mathbf{y}}, \mu)$ *has a pair of complex conjugate eigenvalues* $\lambda_{\pm}(\mu)$ *with satisfy* $\lambda_{\pm}(\mu_0) = \pm i\omega$ *for some* $\omega \in \mathbb{R} \setminus \{0\}$.

(c) $\dfrac{d}{d\mu}(\mathrm{Re}(\lambda(\mu))) \neq 0$ *for* $\mu = \mu_0$.

Then, μ_0 *is a Hopf bifurcation point from* $\tilde{\mathbf{y}}$.

Proof See [22] and references therein.

Remark 2-2: In dimension $p > 2$, condition (b) is still necessary for a bifurcation of ODEs. However, the additional technicalities are far more complicated than (c). For more advanced theorems, we recommend [22] and [28] as good resources for dynamical systems and bifurcations.

To be an introductory, we study the existence of a Hopf bifurcation point for a system of two equations ($p = 2$). In this case, the mathematical technicalities required to determine a Hopf bifurcation are fewer than in the case of general p-dimensional cases. Next, we show an example in which a Hopf bifurcation of a linear system of ODEs can be obtained.

Example 2-3 Find bifurcation points of

$$\begin{aligned} y_1'(t) &= \mu y_1 - \mu y_2, \\ y_2'(t) &= (2 + \mu) y_1 + (\mu - 3) y_2, \end{aligned} \tag{2-36}$$

and determine whether they are a Hopf bifurcation point or not.

Solution: Since (2-36) is a homogeneous linear system, for all $\mu \in \mathbb{R}$, there exists only one equilibrium, namely $\tilde{\mathbf{y}} = (0, 0)$. Next, we will investigate all values of bifurcations of $\tilde{\mathbf{y}}$. According to (2-36), the Jacobian matrix at $\tilde{\mathbf{y}}$ is

$$J(\tilde{y}) = J(0,0) = \begin{pmatrix} \mu & -\mu \\ 2+\mu & \mu-3 \end{pmatrix}. \tag{2-37}$$

Let λ be an eigenvalue of (2-37), then the determinant of $J(\tilde{y}) - \lambda I$, where I is the identity matrix, must be zero, i.e.

$$\begin{vmatrix} \mu-\lambda & -\mu \\ 2+\mu & \mu-3-\lambda \end{vmatrix} = 0,$$

which leads to the polynomial function $\Phi(\lambda)$:

$$\Phi(\lambda) = \lambda^2 + (3-2\mu)\lambda + (\mu^2 - \mu) = 0.$$

The eigenvalues of (2-37) are

$$\lambda_{\pm} = \frac{(2\mu-3) \pm \sqrt{-4\mu^2 - 8\mu + 9}}{2}. \tag{2-38}$$

Next, we will find values of μ that alter stability properties of (2-37) near the equilibrium. From (2-38), the system is stable if and only if $\mathrm{Re}(\lambda) < 0$. In a two-dimensional system, the negative real parts of all eigenvalues must satisfy the conditions

$$\lambda_+ + \lambda_- < 0 \quad \text{and} \quad \lambda_+\lambda_- > 0.$$

For $\lambda_+ + \lambda_- < 0$, we have

$$\mu < \frac{3}{2}. \tag{2-39}$$

In addition, the condition $\lambda_+\lambda_- > 0$ gives that

$$\mu < 0 \quad \text{and} \quad \mu > \frac{1}{2}. \tag{2-40}$$

Conditions (2-39) and (2-40) are necessary and sufficient conditions for the equilibrium $\tilde{y} = (0,0)$ is stable. Figure 2-4 shows intervals of stability.

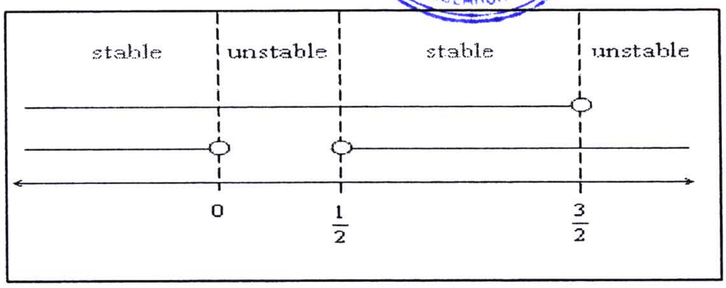

FIGURE 2-4 Bifurcation points of (2-36).

From Figure 2-4, the values of μ in which $\mu = 0, \mu = 1/2$ and $\mu = 3/2$ are bifurcation points. Now we will test whether they are a Hopf bifurcation point or not.

From Theorem 2-9(b), the parameter μ is a Hopf bifurcation point provided that $\lambda_{\pm}(\mu)$ has a purely imaginary part $(\mathrm{Re}(\lambda) = 0)$. From (2-38), substituting values of μ we can see that

$$\lambda_{\pm}(0) = 0, -3; \quad \lambda_{\pm}\left(\frac{1}{2}\right) = 0, -2 \quad \text{and} \quad \lambda_{\pm}\left(\frac{3}{2}\right) = \pm 2\sqrt{3}i.$$

Both $\mu = 0$ and $\mu = 1/2$ are not a bifurcation of the Hopf type, because they are not purely imaginary numbers. In contrast to the case that $\mu = 3/2$, all eigenvalues satisfy the condition (b) in Theorem 2-9. However, to show that $\mu = 3/2$ is a Hopf bifurcation point, we also need to show that it satisfies Theorem 2-9(c), i.e. $\frac{d}{d\mu}(\mathrm{Re}(\lambda(\mu))) \neq 0$. From (2-38),

$$\frac{d}{d\mu}\lambda(\mu) = 1 \pm \frac{-2(\mu+1)}{\sqrt{-4\mu^2 - 8\mu + 9}}. \tag{2-41}$$

Substitute $\mu = \frac{3}{2}$ into (2-41), then it yields

$$\left.\frac{d}{d\mu}\lambda(\mu)\right|_{\mu=\frac{3}{2}} = 1 \mp \frac{5\sqrt{3}}{6}\mathrm{i}.$$

Hence,

$$\frac{d}{d\mu}(\mathrm{Re}(\lambda(3/2)) = \mathrm{Re}(\lambda'(3/2)) = 1 \neq 0.$$

As the results, from Theorem 2-9 (a)-(c), $\mu = 3/2$ is a Hopf bifurcation point of differential system (2-37). ■

Note that both $\mu = 0$ and $\mu = 1/2$ are not Hopf bifurcation points. But they are the other type of bifurcations, namely the *fold bifurcation* as shown in Definition 2-8. Here, we are interested only in the bifurcations of the Hopf type. For interested readers, more details of bifurcations, we refer to [22] and [28] for the good resources on stability and bifurcation analysis.

2.4.2 Hopf bifurcations of the first-order DDEs

In this part, we study the stability of equilibrium and the existence of a (local) Hopf bifurcation of a DDE. Consider the linear delay differential equation:

$$y'(t) = -\gamma y(t) + \beta y(t-\tau), \qquad (2\text{-}42)$$

where γ and β are positive parameters, τ is a positive delay. The characteristic equation of (2-42) is

$$\lambda = -\gamma + \beta e^{-\lambda\tau}. \qquad (2\text{-}43)$$

Next, we state the theorem for the occurrence of Hopf bifurcation of DDEs.

Theorem 2-10 (Hopf Bifurcation Theorem [8], pp. 267-268)
Let λ *be an eigenvalue of equation* (2-42).

(1) *Suppose that* (2-43) *has a simple nonzero purely imaginary eigenvalue* λ_0, *and that all other eigenvalues are not integer multiples of* λ_0.

(2) *Suppose that the branch of eigenvalues* $\lambda(\tau_0)$, *which satisfies* $\lambda(0) = \lambda_0$, *is such that* $\mathrm{Re}(\lambda'(\tau_0)) \neq 0$.

Then, for τ *close to* τ_0, (2-43) *has nontrivial periodic solutions, with period closes to* $2\pi/\mathrm{Im}(\lambda_0)$.

From Theorem 2-10, the point $\tau = \tau_0$ is a Hopf bifurcation point of (2-42) provided that the characteristic equation (2-43) has a simple nonzero purely imaginary root for $\tau = \tau_0$; and the derivative $\mathrm{Re}(\lambda'(\tau_0))$ must not be zero. In our work, we will apply Theorem 2-10 to analyse the bifurcation of the selected delayed population models.