CHAPTER I

INTRODUCTION

In general, many problems depend not only on a current time, but also on some of *past states* or *time-delays*. It is adequate to model problems related to the history by *delay differential equation* (DDE). Recently, dynamics of delay equations attract many authors, and the study of stability and dynamical properties of DDEs becomes an interested area in mathematical research according to many previous works (see for examples [7, 11, 23, 26]).

This chapter provides fundamental backgrounds and general statements of the thesis. The selected population models with time delay will be presented. We also demonstrate the research objectives and limitation of the work including research methodology which will be used in this thesis. Finally, the advantages of this work will be presented in the last section of this chapter.

1.1 Background and General Statement of the Problem

Delay differential equation (DDE) is a class of differential equations (DEs) in which the time derivative at the current time depends on the solution or its derivative at previous time [15]. The general first-order DDEs can be written as

$$y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)),$$

where τ is a delay. In practice some problems may be acceptable to describe their dynamics by ordinary differential equations (ODEs), where the system's future phenomena depends only on its current-state and ignores the time delay. However, for some problems and more realistic models, the time-delay cannot be ignored [1-4, 15].

In this study we consider a class of population models, which are the DDE of the form

$$y'(t) = f(y(t), y(t-\tau)). \qquad (1\text{-}1)$$

Note that (1-1) is an *autonomous delay differential equation.* Many population models can be described by (1-1) when related rates of the number of population depend only on the population size and the time delay τ [7].

The main aim of this work is to investigate sufficient conditions for asymptotic stability of equilibria. The addition problem is to analyse an existence of a Hopf bifurcation. There are *two main problems*, which we need to investigate: the general population models with and without stage-structure.

Without a stage-structure, we consider the delay population models with the form:

$$N'(t) = -\gamma N(t) + \beta f(N(t-\tau))N(t-\tau), \qquad (1\text{-}2)$$

where $N(t)$ denotes the total population size, $f(N)$ is a birth-rate function, $\beta > 0$ represents a birth-rate constant, $\gamma > 0$ refers to the death-rate constant, and τ is the positive time-delay [27]. In general, many populations have more than one life-stage from their birth to death. In mathematical modeling, many authors use the exponential term $e^{-\delta\tau}$ to present a stage-structure of population [7]. Term of stage-structure can be occurrence by transform partial differential equations (PDEs) into DDEs. Note that more details on the age-structure model can be found in the appendix.

In the second problem, we add the stage-structure $e^{-\delta\tau}$ to (1-2), and the problem becomes

$$N'(t) = -\gamma N(t) + \beta e^{-\delta\tau} f(N(t-\tau))N(t-\tau), \qquad (1\text{-}3)$$

where δ is a positive constant, $N(t)$, $f(N)$, β, γ and τ are defined to be positive.

There are many birth-rate functions, which can be applied to (1-2) and (1-3). For examples,

(I) $f(N) = be^{-aN}; \quad a > 0 \text{ and } b > \gamma,$

(II) $f(N) = \dfrac{b\theta^n}{\theta^n + N^n}; \quad b, \theta, n > 0 \text{ and } \beta > \gamma.$

Nisbet and Gurney [18] used the function f defined in (I) to model the Nicholson's data for the laboratory experiment of blowfly populations. Replacing f from (I) into (1-2), we have

$$N'(t) = -\gamma N(t) + \beta e^{-aN(t-\tau)} N(t-\tau). \quad (1\text{-}4)$$

Equation (1-4) is called *the Nicholson's blowflies equation.* In addition, it has been used to describe also the Chagas disease by Velasco-Hernandez [25]. For case (II), Mackey and Glass [16] proposed the model using the function provided in case (II) to describe a physiological control system for the number of red-blood cells. The equation obtained in this case is

$$N'(t) = -rN(t) + \beta \frac{\theta^n}{\theta^n + N^n(t-\tau)} N(t-\tau), \quad (1\text{-}5)$$

which is usually called *the Mackey-Glass equation.*

In addition, the analysis of (1-2) has been widely presented in many previous papers. For examples, Wei [26] studied the stability of the positive equilibrium and the occurrence of a Hopf bifurcation by using the theory of normal form and centre manifold. He investigated the condition in which a global Hopf bifurcation is established. The delay equation that he considered is

$$x'(t) = -\gamma x(t) + \beta f(x(t-\tau)),$$

where both γ and β are positive and the delay $\tau \geq 0$. In addition, he applied the results obtained to the Mackey-Glass equation. Later, Su *et al.* [23] provided the existence of positive steady state bifurcation for the Nicholson's blowflies equation with a finite delay. They also investigated the occurrence of a Hopf bifurcation by analyzing the distribution of the eigenvalues.

From our knowledge, there are many previous papers dealing with models of single species population growth with various stages of the form (1-3) (see, for examples, [2, 12, 24]). However, our work for the age-structure delay models is inspired by the paper of Cooke *et al.* [7]. In their work, they studied dynamics of (1-3) and applied their results to the model with stage-structure describing the survivor of the red-blood cell production:

$$N'(t) = -\gamma N(t) + \beta e^{-\delta\tau} \frac{\theta^n}{\theta^n + N^n(t-\tau)} N(t-\tau), \tag{1-6}$$

where γ, β and δ are positive parameters and τ is a positive constant delay. It is not difficult to see that (1-6) is a class of (1-3), where $f(x) = \beta e^{-\delta\tau}\, \theta^n / (\theta^n + x^n)$.

Recently, Fan *et al.* [11] have studied stability and analysed the Hopf bifurcation of (1-6) with delay-dependent parameters. They studied dynamical behaviour for the stability and Hopf bifurcation by using the perturbation approach and Floquet technique. They also showed the numerical simulations of the Mackey-Glass equation. Here, with linearisation method, we analyse the general case in (1-2) and apply the obtained results comparing with the previous results of (1-4)-(1-5). Also, we will apply our results of (1-3) to other birth-rate functions, such as (1-6). This is to explore a new idea and fulfill some gaps from previous works.

1.2 Research Objectives

The main objectives of this thesis can be expressed as follows.

- To study qualitative behaviour of the selected DDEs, and compare them with the case of ODEs.
- To investigate the sufficient conditions for the stability of each equilibrium of (1-2) and (1-3), and also apply the analytical results to some related population problems.
- To determine the existence of a Hopf bifurcation of (1-2) and (1-3), and illustrate oscillatory of the solutions near the Hopf bifurcation points.

1.3 Scope of the Research

In this work, we study mainly on the asymptotic stability of the delayed population models (1-2) and the model with stage-structure (1-3). Throughout this work, the delay τ is assumed to be a non-negative constant. The primary aim is to determine sufficient conditions in which the equilibria of the selected models are stable. The main technique used to analyse the selected problems is the linearisation method about an equilibrium, which transforms a nonlinear equation to be a linearised equation. Thus our results are limited to be represented only for local behaviours. In addition, by considering τ as the bifurcation parameter, we need to explore conditions for an existence of a Hopf bifurcation of (1-2) and (1-3). Our obtained results from the general cases will also be applied to the well-known delayed population models, such as the Mackey-Glass equation and the Nicholson's blowflies equation.

1.4 Methodology

The methods which are used in this work to obtain the analytical results can be divided into two main steps. The first step is to analyse stability properties of the population models (1-2) and (1-3). Next we need to investigate an existence of a Hopf bifurcation of (1-2) and (1-3). We summarise details on each step for the research methodology as follows.

Step 1: Analyse stability properties of the population models (1-2) and (1-3).

- Find the equilibria (1-2) and (1-3).
- Apply the linearisation method to nonlinear equations (1-2) and (1-3).
- From the linearised equations, investigate the conditions in which the equilibria are asymptotically stable.
- Determine the existence of a Hopf bifurcation for the selected population models.
- Apply the obtained results to some well-known population models with the form (1-2) and (1-3), and illustrate the graphical results.

Step 2: Determine an existence of a Hopf bifurcation of (1-2) and (1-3).

- Study basic properties for bifurcations of delay differential equations, especially the Hopf bifurcation.

- By considering the delay τ as the bifurcation parameter, determine each bifurcation-point whether it is a Hopf bifurcation point.
- Find the sufficient conditions for an existence of a Hopf bifurcation of (1-2) and (1-3).
- Illustrate graphical results of the selected problems by using mathematical software and compare them with the analytical results of the models (1-2) and (1-3).

In addition, the analytical results for (1-2) and (1-3) will be applied to some well-known population models, such as the Mackey-Glass equation and the Nicholson's blowflies equation. This is to determine whether our obtained results are suited well with the known results from the previous works.

1.5 Utilization of the Research

A primary advantage of this work is that we can apply some mathematical techniques to analyse stability of some classes of DDEs. This is to investigate the conditions for asymptotic stability and the existence of Hopf bifurcations for the selected population models with a time delay. Since the results is obtained by the general forms of DDEs, we can apply them to some delayed problems of the forms (1-2) and (1-3). In addition, the technique used in this work can be a basic idea and fundamental methods for ones who need to study further on the area of dynamical analysis of DDEs.