CHAPTER 3 ANALYTIC SOLUTION AND NUMERICAL METHOD FOR GENERALIZED BURGERS–HUXLEY EQUATION

This chapter is the main result of this thesis. The analytic solution will be obtained and numerical methods will be developed based on finite-different method.

3.1 Generalized Burgers–Huxley Equation

The generalized Burgers–Huxley equation

$$\frac{\partial u}{\partial t}+\alpha u^{\delta}\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^{\delta})(u^{\delta}-\gamma), \tag{3.1}$$

where α, β, γ and δ are parameters such that $\alpha,\beta \geq 0$, $\gamma \in (0,1)$ and δ is a positive integer. This model is a nonlinear partial differential equation which describes relation between diffusion, convection and reaction process. For $\alpha = 1$ and $\beta = 0$ then (3.1) reduces to Burgers equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=0, \tag{3.2}$$

and for $\alpha = 0$, $\beta = 1$ and $\delta = 1$ then (3.1) reduces to Huxley equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=u(1-u)(u-\gamma),\ \gamma \neq 0. \tag{3.3}$$

Using the wave variable $\xi = x - ct$, (3.1) is transformed to an ordinary differential equation

$$\begin{aligned}-c\frac{dU}{d\xi}+\alpha U^{\delta}\frac{dU}{d\xi}-\frac{d^2U}{d\xi^2} &= \beta U(1-U^{\delta})(U^{\delta}-\gamma)\\ &= -\beta\gamma U+\beta U^{\delta+1}+\beta\gamma U^{\delta+1}-\beta U^{2\delta+1}\\ &= -\beta\gamma U+\beta(\gamma+1)U^{\delta+1}-\beta U^{2\delta+1}. \end{aligned} \tag{3.4}$$

Rearranging (3.4) yields

$$-c\frac{dU}{d\xi}+\alpha U^{\delta}\frac{dU}{d\xi}-\frac{d^2U}{d\xi^2}+\beta\gamma U-\beta(\gamma+1)U^{\delta+1}+\beta U^{2\delta+1}=0. \tag{3.5}$$

By changing the variable U to V by given $V = U^{\delta}$ gives

$$U = V^{\frac{1}{\delta}}. \tag{3.6}$$

This leads to the first and second derivative of U with respect to ξ

$$\frac{dU}{d\xi} = \frac{1}{\delta} V^{\frac{1}{\delta}-1} \frac{dV}{d\xi} \tag{3.7}$$

and

$$\frac{d^2U}{d\xi^2} = \frac{1}{\delta} V^{\frac{1}{\delta}-1} \frac{d^2V}{d\xi^2} + \frac{1}{\delta}\left(\frac{1}{\delta}-1\right) V^{\frac{1}{\delta}-2}\left(\frac{dV}{d\xi}\right)^2. \tag{3.8}$$

Substituting (3.6)-(3.8) into (3.5) yields

$$\begin{aligned} & -c\left(\frac{1}{\delta} V^{\frac{1}{\delta}-1} \frac{dV}{d\xi}\right) + \alpha V\left(\frac{1}{\delta} V^{\frac{1}{\delta}-1} \frac{dV}{d\xi}\right) \\ & -\left(\frac{1}{\delta} V^{\frac{1}{\delta}-1} \frac{d^2V}{d\xi^2} + \frac{1}{\delta}\left(\frac{1}{\delta}-1\right) V^{\frac{1}{\delta}-2}\left(\frac{dV}{d\xi}\right)^2\right) \\ & +\beta\gamma V^{\frac{1}{\delta}} - \beta(\gamma+1)V^{1+\frac{1}{\delta}} + \beta V^{2+\frac{1}{\delta}} = 0. \end{aligned} \tag{3.9}$$

Next, rearranging by multiplying (3.9) with $\delta^2 V^{2-\frac{1}{\delta}}$, (because of δ is a positive integer) and rearranging gives

$$\begin{aligned} & -c\delta V\frac{dV}{d\xi} + \alpha\delta V^2\frac{dV}{d\xi} - \delta V\frac{d^2V}{d\xi^2} - (1-\delta)\left(\frac{dV}{d\xi}\right)^2 \\ & +\beta\gamma\delta^2 V^2 - \beta(\gamma+1)\delta^2 V^3 + \beta\delta^2 V^4 = 0. \end{aligned} \tag{3.10}$$

The generalized Burgers–Huxley equation (3.1) is now then reduced to an ordinary differential equation.

Next, it is assumed that the solution of (3.10) can be expressed in the form

$$V(\xi) = S(Y) = \sum_{k=0}^{N} a_k Y^k, \tag{3.11}$$

where $Y = \tanh(\mu\xi)$. Substituting (3.11) into (3.10) leads to

$$\begin{aligned} & -c\delta S(Y)\left(\mu(1-Y^2)\frac{dS(Y)}{dY}\right) + \alpha\delta S(Y)^2\left(\mu(1-Y^2)\frac{dS(Y)}{dY}\right) \\ & -\delta S(Y)\left(\mu(1-Y^2)\frac{d}{dY}\left(\mu(1-Y^2)\frac{dS(Y)}{dY}\right)\right) - (1-\delta)\left(\mu(1-Y^2)\frac{dS(Y)}{dY}\right)^2 \\ & +\beta\gamma\delta^2 S(Y)^2 - \beta(\gamma+1)\delta^2 S(Y)^3 + \beta\delta^2 S(Y)^4 = 0. \end{aligned} \tag{3.12}$$

In order to determine value of N, as 3.11, the highest power of Y appear as Y^{2N+2} in the third term and Y^4 in the last term. Balancing these values then lead to

$$2N + 2 = 4N,$$

so that

$$N = 1.$$

With $N=1$, then (3.11) remains

$$
\begin{aligned}
S(Y) &= \sum_{k=0}^{1} a_k Y^k \\
&= a_0 + a_1 Y. \qquad (3.13)
\end{aligned}
$$

Then the unknown a_0, a_1, μ and c will be determined by substituting (3.13) into (3.12) and equating the coefficients of Y^i $(i=0,1,...,4)$, yields

$$
\begin{aligned}
-cn\mu a_0a_1 + \beta\gamma n^2a_0^2 - \mu^2a_1^2 + n\mu^2a_1^2 + \alpha n\mu a_0^2a_1 \\
-\beta n^2a_0^3 - \beta\gamma n^2a_0^3 + \beta n^2a_0^4 &= 0, \qquad (3.14) \\
2n\mu^2a_0a_1 + 2\beta\gamma n^2a_0a_1 - cn\mu a_1^2 - 3\beta n^2a_0^2a_1 - 3\beta\gamma n^2a_0a_1^2 \\
+2\alpha n\mu a_0a_1^2 + 4\beta n^2a_0^3a_1 &= 0, \qquad (3.15) \\
cn\mu a_0a_1 + 2\mu^2a_1^2 + \beta\gamma n^2a_1^2 - \alpha n\mu a_0^2a_1 - 3\beta n^2a_0a_1^2 \\
-3\beta\gamma n^2a_0a_1^2 + \alpha n\mu a_1^3 + 6\beta n^2a_0^2a_1^2 &= 0, \qquad (3.16) \\
-2n\mu^2a_0a_1 + cn\mu a_1^2 - 2\alpha n\mu a_0a_1^2 - \beta n^2a_1^3 \\
-\beta\gamma n^2a_1^3 + 4\beta n^2a_0a_1^3 &= 0, \qquad (3.17) \\
-\mu^2a_1^2 - n\mu^2a_1^2 - \alpha n\mu a_1^3 + \beta n^2a_1^4 &= 0. \qquad (3.18)
\end{aligned}
$$

Solving the system of algebraic equations (3.14)-(3.18) is explored as follows. Firstly, reducing the system (3.14)-(3.18) to two equations by adding (3.14), (3.16) and (3.18) gives

$$
\begin{aligned}
\beta\gamma\delta^2a_0^2 + \beta\gamma\delta^2a_1^2 - 3\beta\delta^2a_0a_1^2 - 3\beta\gamma\delta^2a_0a_1^2 - \beta\delta^2a_0^3 \\
-\beta\gamma\delta^2a_0^3 + 6\beta\delta^2a_0^2a_1^2 + \beta\delta^2a_0^4 + \beta\delta^2a_1^4 &= 0. \qquad (3.19)
\end{aligned}
$$

By adding (3.15) and (3.14), it yields

$$
\begin{aligned}
2\beta\gamma\delta^2a_0a_1 - 3\beta\delta^2a_0^2a_1 - 3\beta\gamma\delta^2a_0^2a_1 - \beta\delta^2a_1^3 \\
-\beta\gamma\delta^2a_1^3 + 4\beta\delta^2a_0^3a_1 + 4\beta\delta^2a_0a_1^3 &= 0. \qquad (3.20)
\end{aligned}
$$

Solving (3.19)-(3.20) for a_0, a_1 give the following four cases:

Case 1: $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}$,

Case 2: $a_0 = \frac{1}{2}$, $a_1 = -\frac{1}{2}$,

Case 3: $a_0 = \frac{\gamma}{2}$, $a_1 = \frac{\gamma}{2}$,

Case 4: $a_0 = \frac{\gamma}{2}$, $a_1 = -\frac{\gamma}{2}$.

In each cases, substituting a_0, a_1 into (3.14)-(3.18) yields

Case 1: $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}$,

$$-\frac{1}{4}c\mu\delta+\frac{1}{8}\mu\alpha\delta-\frac{1}{4}\mu^2+\frac{1}{4}\mu^2\delta-\frac{1}{16}\beta\delta^2+\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.21)$$

$$-\frac{1}{4}c\mu\delta+\frac{1}{4}\mu\alpha\delta+\frac{1}{2}\mu^2\delta-\frac{1}{8}\beta\delta^2+\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.22)$$

$$\frac{1}{4}c\mu\delta+\frac{1}{2}\mu^2-\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.23)$$

$$\frac{1}{4}c\mu\delta-\frac{1}{4}\mu\alpha\delta-\frac{1}{2}\mu^2\delta+\frac{1}{8}\beta\delta^2-\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.24)$$

$$-\frac{1}{8}\mu\alpha\delta-\frac{1}{4}\mu^2-\frac{1}{4}\mu^2\delta+\frac{1}{16}\beta\delta^2 = 0. \quad (3.25)$$

Case 2: $a_0 = \frac{1}{2}$, $a_1 = -\frac{1}{2}$,

$$\frac{1}{4}c\mu\delta-\frac{1}{8}\mu\alpha\delta+\frac{1}{4}\mu^2-\frac{1}{4}\mu^2\delta-\frac{1}{16}\beta\delta^2+\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.26)$$

$$-\frac{1}{4}c\mu\delta+\frac{1}{4}\mu\alpha\delta-\frac{1}{2}\mu^2\delta+\frac{1}{8}\beta\delta^2-\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.27)$$

$$-\frac{1}{4}c\mu\delta-\frac{1}{2}\mu^2-\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.28)$$

$$\frac{1}{4}c\mu\delta-\frac{1}{4}\mu\alpha\delta+\frac{1}{2}\mu^2\delta-\frac{1}{8}\beta\delta^2+\frac{1}{8}\beta\gamma\delta^2 = 0, \quad (3.29)$$

$$\frac{1}{8}\mu\alpha\delta+\frac{1}{4}\mu^2+\frac{1}{4}\mu^2\delta+\frac{1}{16}\beta\delta^2 = 0. \quad (3.30)$$

Case 3: $a_0 = \frac{\gamma}{2}$, $a_1 = \frac{\gamma}{2}$,

$$-\frac{1}{4}c\mu\gamma^2\delta+\frac{1}{8}\mu\alpha\gamma^3\delta-\frac{1}{4}\mu^2\gamma^2+\frac{1}{4}\mu^2\gamma^2\delta-\frac{1}{16}\beta\gamma^4\delta^2+\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.31)$$

$$-\frac{1}{4}c\mu\gamma^2\delta+\frac{1}{4}\mu\alpha\gamma^3\delta+\frac{1}{2}\mu^2\gamma^2\delta-\frac{1}{8}\beta\gamma^4\delta^2+\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.32)$$

$$\frac{1}{4}c\mu\gamma^2\delta+\frac{1}{2}\mu^2\gamma^2-\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.33)$$

$$\frac{1}{4}c\mu\gamma^2\delta-\frac{1}{4}\mu\alpha\gamma^3\delta-\frac{1}{2}\mu^2\gamma^2\delta+\frac{1}{8}\beta\gamma^4\delta^2-\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.34)$$

$$-\frac{1}{8}\mu\alpha\gamma^3\delta-\frac{1}{4}\mu^2\gamma^2-\frac{1}{4}\mu^2\gamma^2\delta+\frac{1}{16}\beta\gamma^4\delta^2 = 0. \quad (3.35)$$

Case 4: $a_0 = \frac{\gamma}{2}$, $a_1 = -\frac{\gamma}{2}$,

$$\frac{1}{4}c\mu\gamma^2\delta-\frac{1}{8}\mu\alpha\gamma^3\delta+\frac{1}{4}\mu^2\gamma^2-\frac{1}{4}\mu^2\gamma^2\delta-\frac{1}{16}\beta\gamma^4\delta^2+\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.36)$$

$$-\frac{1}{4}c\mu\gamma^2\delta+\frac{1}{4}\mu\alpha\gamma^3\delta+\frac{1}{2}\mu^2\gamma^2\delta+\frac{1}{8}\beta\gamma^4\delta^2-\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.37)$$

$$-\frac{1}{4}c\mu\gamma^2\delta-\frac{1}{2}\mu^2\gamma^2-\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.38)$$

$$\frac{1}{4}c\mu\gamma^2\delta-\frac{1}{4}\mu\alpha\gamma^3\delta-\frac{1}{2}\mu^2\gamma^2\delta-\frac{1}{8}\beta\gamma^4\delta^2+\frac{1}{8}\beta\gamma^3\delta^2 = 0, \quad (3.39)$$

$$\frac{1}{8}\mu\alpha\gamma^3\delta+\frac{1}{4}\mu^2\gamma^2+\frac{1}{4}\mu^2\gamma^2\delta+\frac{1}{16}\beta\gamma^4\delta^2 = 0. \quad (3.40)$$

Solving each cases of four systems: (3.21)-(3.25), (3.26)-(3.30), (3.31)-(3.35) and (3.36)-(3.40) , for μ and c, lead to the four possible solutions of (3.14)-(3.18) as follows.

Case 1:

$$\left.\begin{aligned} a_0 &= \frac{1}{2}, \\ a_1 &= \frac{1}{2}, \\ \mu &= \frac{n\left(\sqrt{\alpha^2+4\beta(\delta+1)}-\alpha\right)}{4(\delta+1)}, \\ c &= \frac{\alpha-\sqrt{\alpha^2+4\beta(\delta+1)}+\left(\alpha+\sqrt{\alpha^2+4\beta(\delta+1)}\right)(\delta+1)\gamma}{2(\delta+1)}, \end{aligned}\right\} \tag{3.41}$$

Case 2:

$$\left.\begin{aligned} a_0 &= \frac{1}{2}, \\ a_1 &= -\frac{1}{2}, \\ \mu &= \frac{\delta\left(\sqrt{\alpha^2+4\beta(\delta+1)}+\alpha\right)}{4(\delta+1)}, \\ c &= \frac{\alpha+\sqrt{\alpha^2+4\beta(\delta+1)}+\left(\alpha-\sqrt{\alpha^2+4\beta(\delta+1)}\right)(\delta+1)\gamma}{2(\delta+1)}, \end{aligned}\right\} \tag{3.42}$$

Case 3:

$$\left.\begin{aligned} a_0 &= \frac{\gamma}{2}, \\ a_1 &= \frac{\gamma}{2}, \\ \mu &= \frac{\delta\gamma\left(\sqrt{\alpha^2+4\beta(\delta+1)}-\alpha\right)}{4(\delta+1)}, \\ c &= \frac{\left(\alpha-\sqrt{\alpha^2+4\beta(\delta+1)}\right)\gamma+\left(\alpha+\sqrt{\alpha^2+4\beta(\delta+1)}\right)(\delta+1)}{2(\delta+1)}, \end{aligned}\right\} \tag{3.43}$$

Case 4:

$$\left.\begin{aligned} a_0 &= \frac{\gamma}{2}, \\ a_1 &= -\frac{\gamma}{2}, \\ \mu &= \frac{\delta\gamma\left(\sqrt{\alpha^2+4\beta(\delta+1)}+\alpha\right)}{4(\delta+1)}, \\ c &= \frac{\left(\alpha+\sqrt{\alpha^2+4\beta(\delta+1)}\right)\gamma+\left(\alpha-\sqrt{\alpha^2+4\beta(\delta+1)}\right)(\delta+1)}{2(\delta+1)}. \end{aligned}\right\} \tag{3.44}$$

Finally, substituting (3.41)-(3.44) into (3.13) and $U^\delta = V$ give the four solutions of (3.1) in the closed form

$$u_1(x,t) = \left[\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{\delta(\rho-\alpha)}{4(\delta+1)}\left[x-\frac{\alpha-\rho+(\alpha+\rho)(\delta+1)\gamma}{2(\delta+1)}t\right]\right)\right]^{\frac{1}{\delta}}, \tag{3.45}$$

$$u_2(x,t) = \left[\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\delta(\rho+\alpha)}{4(\delta+1)}\left[x-\frac{\alpha+\rho+(\alpha-\rho)(\delta+1)\gamma}{2(\delta+1)}t\right]\right)\right]^{\frac{1}{\delta}}, \tag{3.46}$$

$$u_3(x,t) = \left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left(\frac{\delta\gamma(\rho-\alpha)}{4(\delta+1)}\left[x-\frac{(\alpha-\rho)\gamma+(\alpha+\rho)(\delta+1)}{2(\delta+1)}t\right]\right)\right]^{\frac{1}{\delta}}, \tag{3.47}$$

$$u_4(x,t) = \left[\frac{\gamma}{2}-\frac{\gamma}{2}\tanh\left(\frac{\delta\gamma(\rho+\alpha)}{4(\delta+1)}\left[x-\frac{(\alpha+\rho)\gamma+(\alpha-\rho)(\delta+1)}{2(\delta+1)}t\right]\right)\right]^{\frac{1}{\delta}}, \tag{3.48}$$

where $\rho = \sqrt{\alpha^2+4\beta(\delta+1)}$.

3.2 Numerical Method

The generalized Burgers–Huxley equation has the general form

$$\frac{\partial u}{\partial t}+\alpha u^\delta\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2} = \beta u(1-u^\delta)(u^\delta-\gamma), \tag{3.49}$$

where $u = u(x,t)$, the parameters α, β, γ and δ are constants such that $\alpha, \beta \geq 0$, $\gamma \in (0,1)$ and δ is a positive integer, respectively. Equation (3.49) is defined in the region $\Omega \times [0 < t < T]$, where $\Omega = \{x \mid 0 < x < L\}$ and T is a positive integer.

The initial condition consists of

$$u(x,0) = \phi(x), \qquad 0 \le x \le L \tag{3.50}$$

and boundary condition consist of

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad t \ge 0. \tag{3.51}$$

3.2.1 Discretization and Notation

The interval $0 \le x \le L$ is divided into M subintervals each of width h so that $Mh = L$ and the time interval $t \ge 0$ is discretized in step of length ℓ. The open region $\Omega = [0 < x < L] \times [t > 0]$ and its boundary $\partial\Omega$ consisting of the lines $x = 0$, $x = L$ and $t = 0$ have thus been covered by a rectangular mesh having coordinates of the form (x_m, t_n) where $x_m = mh$ $(m = 0, 1, 2, \ldots, M)$ and $t_n = nl$ $(n = 0, 1, 2, \ldots)$.

The solutions of (3.49)-(3.51) at the typical mesh point (x_m, t_n) is, of course, $u(x_m, t_n)$ which may be denoted by u_m^n. The solution of an approximating difference scheme at the same point will be denoted by U_m^n, while the numerical value of U_m^n actually obtained (for instance, subject to computer round-off errors) will be denoted by U_m^n. Collectively, the value $\mathbf{U}^n$ will be written in vector form as

$$\mathbf{U}^n = \left[U_0^n, U_1^n, \ldots, U_M^n\right]^T, \quad n = 0, 1, 2, \ldots,$$

and T denotes transpose. Clearly, there are $M+1$ values to be determined at each time step.

3.2.2 Finite-Difference Method

Two numerical methods, based on finite-difference method, are developed by approximating the time derivative in (3.49) by the first-order forward-difference replacement

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+\ell) - u(x,t)}{\ell}, \tag{3.52}$$

and the space derivatives by the second-order approximation

$$\frac{\partial u}{\partial x} \approx \frac{u(x+h,t) - u(x-h,t)}{2h}, \tag{3.53}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} \approx\ & \frac{\theta}{h^2}\Big\{u(x-h,t+\ell) - 2u(x,t+\ell) + u(x+h,t+\ell)\Big\} \\ & + \frac{(1-\theta)}{h^2}\Big\{u(x-h,t) - 2u(x,t) + u(x+h,t)\Big\}, \end{aligned} \tag{3.54}$$

in which $0 \le \theta \le 1$ is a parameter. When $\theta = 0$, (3.54) is $O(h^2)$ as $h, \ell \to 0$ and is $O(h^2 + \ell)$ as $h, \ell \to 0$ otherwise.

The non-derivative terms in the right-hand side of (3.49) may be replaced in the following two ways

(a) $\beta U_m^n\left\{1-(U_m^n)^\delta\right\}\left\{(U_m^n)^\delta-\gamma\right\}$,

(b) $(1+\gamma)\beta U_m^n(U_m^n)^\delta-\beta U_m^{n+1}\left\{\gamma+(U_m^n)^{2\delta}\right\}$.

These approximation, together with the replacement for the time and space derivatives in (3.52)-(3.54), gives rise to two numerical methods, Method $M_1(\theta)$ and Method $M_2(\theta)$ for the numerical solutions of (3.49) -(3.51). These methods are as follows:

Method $M_1(\theta), 0\le\theta\le 1$:

$$\frac{U_m^{n+1}-U_m^n}{\ell}+\alpha(U_m^n)^\delta\frac{U_{m+1}^n-U_{m-1}^n}{2h}-\frac{1}{h^2}\Big[\theta\left\{U_{m-1}^{n+1}-2U_m^{n+1}+U_{m+1}^{n+1}\right\}$$
$$+(1-\theta)\left\{U_{m-1}^n-2U_m^n+U_{m+1}^n\right\}\Big]=\beta U_m^n\left\{1-(U_m^n)^\delta\right\}\left\{(U_m^n)^\delta-\gamma\right\}, \quad (3.55)$$

Method $M_2(\theta), 0\le\theta\le 1$:

$$\frac{U_m^{n+1}-U_m^n}{\ell}+\alpha(U_m^n)^\delta\frac{U_{m+1}^n-U_{m-1}^n}{2h}-\frac{1}{h^2}\Big[\theta\left\{U_{m-1}^{n+1}-2U_m^{n+1}+U_{m+1}^{n+1}\right\}$$
$$+(1-\theta)\left\{U_{m-1}^n-2U_m^n+U_{m+1}^n\right\}\Big]=\beta U_m^n\left\{(U_m^n)^\delta+\gamma(U_m^n)^\delta\right\}-\beta U_m^{n+1}\left\{\gamma+(U_m^n)^{2\delta}\right\}.$$
$$(3.56)$$

It follows that equations (3.55)-(3.56) can be rewritten, equivalently, as

Method $M_1(\theta), 0\le\theta\le 1$:

$$\begin{aligned}-p\theta U_{m-1}^{n+1}+(1+2p\theta)U_m^{n+1}-p\theta U_{m+1}^{n+1}&\\ =\ &\left\{r(U_m^n)^\delta+p(1-\theta)\right\}U_{m-1}^n\\ &+\left[1-2p(1-\theta)+\ell\beta\left\{1-(U_m^n)^\delta\right\}\left\{(U_m^n)^\delta-\gamma\right\}\right]U_m^n\\ &+\left\{-r(U_m^n)^\delta+p(1-\theta)\right\}U_{m+1}^n, \qquad (3.57)\end{aligned}$$

Method $M_2(\theta), 0\le\theta\le 1$:

$$\begin{aligned}-p\theta U_{m-1}^{n+1}+\left[1+2p\theta+\ell\beta\left\{\gamma+(U_m^n)^{2\delta}\right\}\right]U_m^{n+1}-p\theta U_{m+1}^{n+1}&\\ =\ &\left\{r(U_m^n)^\delta+p(1-\theta)\right\}U_{m-1}^n\\ &+\left[1-2p(1-\theta)+\ell\beta\left\{(U_m^n)^\delta+\gamma(U_m^n)^\delta\right\}\right]U_m^n\\ &+\left\{-r(U_m^n)^\delta+p(1-\theta)\right\}U_{m+1}^n, \qquad (3.58)\end{aligned}$$

where $p=\frac{\ell}{h^2}$, $r=\frac{\alpha\ell}{2h}$, $m=0,1,2,\ldots,M$ and $n=0,1,2,\ldots$.

3.2.3 Local Truncation Errors

To verify the accuracy of the methods $M_1(\theta)$ and $M_2(\theta)$, the local truncation error associated with (3.57) and (3.58) at the point $(x,t) = (x_m, t_n)$ can be obtained from (3.55) and (3.56), respectively, and given by

Method $M_1(\theta)$, $0 \leq \theta \leq 1$:

$$\begin{aligned}
\mathscr{L}_{M_1}[u(x,t):h,\ell] &= \frac{1}{\ell}[u(x,t+\ell)-u(x,t)] \\
&+\frac{1}{2h}\alpha u^{\delta}(x,t)\,[u(x+h,t)-u(x-h,t)] \\
&-\frac{1}{h^2}\theta\,[u(x-h,t+\ell)-2u(x,t+\ell)+u(x+h,t+\ell)] \\
&-\frac{1}{h^2}(1-\theta)\,[u(x-h,t)-2u(x,t)+u(x+h,t)] \\
&-\beta u(x,t)(1-u^{\delta}(x,t))(u^{\delta}(x,t)-\gamma), \qquad (3.59)
\end{aligned}$$

Method $M_2(\theta)$, $0 \leq \theta \leq 1$:

$$\begin{aligned}
\mathscr{L}_{M_2}[u(x,t):h,\ell] &= \frac{1}{\ell}[u(x,t+\ell)-u(x,t)] \\
&+\frac{1}{2h}\alpha u^{\delta}(x,t)\,[u(x+h,t)-u(x-h,t)] \\
&-\frac{1}{h^2}\theta\,[u(x-h,t+\ell)-2u(x,t+\ell)+u(x+h,t+\ell)] \\
&-\frac{1}{h^2}(1-\theta)\,[u(x-h,t)-2u(x,t)+u(x+h,t)] \\
&-\beta u(x,t)(u^{\delta}(x,t)+\gamma u^{\delta}(x,t)) \\
&+\beta u(x,t+\ell)(\gamma+u^{2\delta}(x,t)). \qquad (3.60)
\end{aligned}$$

Expanding $u(x,t+\ell)$, $u(x\pm h,t+\ell)$ and $u(x\pm h,t)$ in (3.59) and (3.60) as Taylor's series about (x,t) lead to

Method $M_1(\theta)$, $0 \leq \theta \leq 1$:

$$\begin{aligned}
\mathscr{L}_{M_1}[u(x,t):h,\ell] &= \frac{1}{6}h^2\left(\alpha u^{\delta}\frac{\partial^3 u}{\partial x^3}+\frac{1}{2}(1-\theta)\frac{\partial^4 u}{\partial x^4}\right) \\
&+\ell\left(\frac{1}{2}\frac{\partial^2 u}{\partial t^2}-\theta\frac{\partial^3 u}{\partial x^2 \partial t}\right)+\ldots, \qquad (3.61)
\end{aligned}$$

Method $M_2(\theta)$, $0 \leq \theta \leq 1$:

$$\begin{aligned}
\mathscr{L}_{M_2}[u(x,t):h,\ell] &= \frac{1}{6}h^2\left(\alpha u^{\delta}\frac{\partial^3 u}{\partial x^3}+\frac{1}{2}(1-\theta)\frac{\partial^4 u}{\partial x^4}\right) \\
&+\ell\left(\beta\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial^2 u}{\partial t^2}-\theta\frac{\partial^3 u}{\partial x^2 \partial t}\right)+\ldots. \qquad (3.62)
\end{aligned}$$

Equations (3.61) and (3.62) verify that the methods $M_1(\theta)$ and $M_2(\theta)$ are $O(h^2+\ell)$ as $h,\ell \to 0$. These mean that the accuracy of the methods $M_1(\theta)$ and $M_2(\theta)$ are first-order in time and second-order in space. The implementations of the methods $M_1(\theta)$ and $M_2(\theta)$ are given in the next section.

3.2.4 Implementation

Let $\mathbf{U}^{n+1} = \left[U_1^{n+1}, U_2^{n+1}, \ldots, U_{M-1}^{n+1}\right]^T$ where T denotes transpose. The modification to the formulae of the two numerical methods and their implication, are as follows.

Method $M_1(\theta)$:

Taking $m=1; m=2,\ldots,M-2; m=M-1$ in (3.57) give

$m=1$,

$$
\begin{aligned}
(1+2p\theta)U_1^{n+1} - p\theta U_2^{n+1} \\
&= \left[p\theta U_0^{n+1} + \left\{r(U_1^n)^\delta + p(1-\theta)\right\}U_0^n\right] \\
&+ \left[1-2p(1-\theta)+\ell\beta\left\{1-(U_1^n)^\delta\right\}\left\{(U_1^n)^\delta-\gamma\right\}\right]U_1^n \\
&+ \left\{-r(U_1^n)^\delta + p(1-\theta)\right\}U_2^n,
\end{aligned}
\tag{3.63}
$$

$m=2,\ldots,M-2$,

$$
\begin{aligned}
-p\theta U_{m-1}^{n+1} + (1+2p\theta)U_m^{n+1} - p\theta U_{m+1}^{n+1} \\
&= \left\{r(U_m^n)^\delta + p(1-\theta)\right\}U_{m-1}^n \\
&+ \left[1-2p(1-\theta)+\ell\beta\left\{1-(U_m^n)^\delta\right\}\left\{(U_m^n)^\delta-\gamma\right\}\right]U_m^n \\
&+ \left\{-r(U_m^n)^\delta + p(1-\theta)\right\}U_{m+1}^n,
\end{aligned}
\tag{3.64}
$$

and $m=M-1$,

$$
\begin{aligned}
-p\theta U_{M-2}^{n+1} + (1+2p\theta)U_{M-1}^{n+1} \\
&= \left\{r(U_{M-1}^n)^\delta + p(1-\theta)\right\}U_{M-2}^n \\
&+ \left[1-2p(1-\theta)+\ell\beta\left\{1-(U_{M-1}^n)^\delta\right\}\left\{(U_{M-1}^n)^\delta-\gamma\right\}\right]U_{M-1}^n \\
&+ \left[p\theta U_M^{n+1} + \left\{-r(U_{M-1}^n)^\delta + p(1-\theta)\right\}U_M^n\right].
\end{aligned}
\tag{3.65}
$$

The solution vector $\mathbf{U}^{n+1}$ may be written in the matrix form as

$$
A\mathbf{U}^{n+1} = B\mathbf{U}^n + C, \tag{3.66}
$$

where A is a constant, tridiagonal matrix of order $M-1$ given by

$$A = \begin{bmatrix} 1+2p\theta & -p\theta & 0 & \dots & 0 \\ -p\theta & 1+2p\theta & -p\theta & & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & -p\theta & 1+2p\theta & -p\theta \\ 0 & \dots & 0 & -p\theta & 1+2p\theta \end{bmatrix}, \tag{3.67}$$

and $C = [p\theta U_0^{n+1} + \Big\{r(U_1^n)^\delta + p(1-\theta)\Big\} U_0^n, 0, \dots, 0, p\theta U_M^{n+1} + \Big\{-r(U_{M-1}^n)^\delta + p(1-\theta)\Big\} U_M^n]^T$ is vector of order $M-1$. The matrix B is also tridiagonal matrix of order $M-1$ and is given by

$$B = \begin{bmatrix} f_1 & g_1 & 0 & \dots & 0 \\ h_2 & f_2 & g_2 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & h_{M-2} & f_{M-2} & g_{M-2} \\ 0 & \dots & 0 & h_{M-1} & f_{M-1} \end{bmatrix}, \tag{3.68}$$

where

$$\begin{aligned} f_i &= 1 - 2p(1-\theta) + \ell\beta\Big\{1 - (U_i^n)^\delta\Big\}\Big\{(U_i^n)^\delta - \gamma\Big\}, & i &= 1, \dots, M-1, \\ g_i &= -r(U_i^n)^\delta + p(1-\theta), & i &= 1, \dots, M-2, \\ h_i &= r(U_i^n)^\delta + p(1-\theta), & i &= 2, \dots, M-1. \end{aligned}$$

The vector solution $\mathbf{U}^{n+1}$ of (3.49) using Method $M_1(\theta)$ is obtained by solving linear algebraic system (3.66), which is of order $M-1$, on a single process.

Method $M_2(\theta)$:

Taking $m = 1; m = 2, \dots, M-2; m = M-1$ in (3.58) give

$m = 1$,

$$\begin{aligned} \Big[1 + 2p\theta + \ell\beta\Big\{\gamma + (U_1^n)^{2\delta}\Big\}\Big] U_1^{n+1} - p\theta U_2^{n+1} \\ = \Big[p\theta U_0^{n+1} + \Big\{r(U_1^n)^\delta + p(1-\theta)\Big\} U_0^n\Big] \\ + \Big[1 - 2p(1-\theta) + \ell\beta\Big\{(U_1^n)^\delta + \gamma(U_1^n)^\delta\Big\}\Big] U_1^n \\ + \Big\{-r(U_1^n)^\delta + p(1-\theta)\Big\} U_2^n, \end{aligned} \tag{3.69}$$

$m = 2, \ldots, M-2$,

$$-p\theta U_{m-1}^{n+1} + \left[1 + 2p\theta + \ell\beta\left\{\gamma + (U_m^n)^{2\delta}\right\}\right] U_m^{n+1} - p\theta U_{m+1}^{n+1}$$
$$= \left\{r(U_m^n)^\delta + p(1-\theta)\right\} U_{m-1}^n$$
$$+ \left[1 - 2p(1-\theta) + \ell\beta\left\{(U_m^n)^\delta + \gamma(U_m^n)^\delta\right\}\right] U_m^n$$
$$+ \left\{-r(U_m^n)^\delta + p(1-\theta)\right\} U_{m+1}^n, \qquad (3.70)$$

and $m = M-1$,

$$-p\theta U_{M-2}^{n+1} + \left[1 + 2p\theta + \ell\beta\left\{\gamma + (U_{M-1}^n)^{2\delta}\right\}\right] U_{M-1}^{n+1}$$
$$= \left\{r(U_{M-1}^n)^\delta + p(1-\theta)\right\} U_{M-2}^n$$
$$+ \left[1 - 2p(1-\theta) + \ell\beta\left\{(U_{M-1}^n)^\delta + \gamma(U_{M-1}^n)^\delta\right\}\right] U_{M-1}^n$$
$$+ \left[p\theta U_M^{n+1} + \left\{-r(U_{M-1}^n)^\delta + p(1-\theta)\right\} U_M^n\right]. \qquad (3.71)$$

The solution vector $\mathbf{U}^{n+1}$ may be written in the matrix form as

$$A\mathbf{U}^{n+1} = B\mathbf{U}^n + C, \qquad (3.72)$$

where

$$A = \begin{bmatrix} a_1 & -p\theta & 0 & \ldots & 0 \\ -p\theta & a_2 & -p\theta & & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & -p\theta & a_{M-2} & -p\theta \\ 0 & \ldots & 0 & -p\theta & a_{M-1} \end{bmatrix}, \qquad (3.73)$$

with $a_i = 1 + 2p\theta + \ell\beta\{\gamma + (U_i^n)^{2\delta}\}$,

$$B = \begin{bmatrix} f_1 & g_1 & 0 & \ldots & 0 \\ h_2 & f_2 & g_2 & \ldots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & h_{M-2} & f_{M-2} & g_{M-2} \\ 0 & \ldots & 0 & h_{M-1} & f_{M-1} \end{bmatrix}, \qquad (3.74)$$

with

$$
\begin{aligned}
f_i &= 1-2p(1-\theta)+\ell\beta\Big\{(U_i^n)^\delta+\gamma(U_i^n)^\delta\Big\}, && i=1,\ldots,M-1\\
g_i &= -r(U_i^n)^\delta+p(1-\theta), && i=1,\ldots,M-2\\
h_i &= r(U_i^n)^\delta+p(1-\theta), && i=2,\ldots,M-1
\end{aligned}
$$

and

$$
C=\begin{bmatrix}
p\theta U_0^{n+1}+\Big\{r(U_1^n)^\delta+p(1-\theta)\Big\}U_0^n\\
0\\
\vdots\\
0\\
p\theta U_M^{n+1}+\Big\{-r(U_{M-1}^n)^\delta+p(1-\theta)\Big\}U_M^n
\end{bmatrix}. \tag{3.75}
$$

The matrices A and B are both of order $M-1$ and the vector C is a column-vector of order $M-1$.

Further research reveals that the vector solutions $\mathbf{U}^{n+1}$ of (3.49) using Method $M_2(\theta)$ is obtained by solving the linear algebraic system (3.72), which is of order $M-1$, on a single process.