CHAPTER 2 LITERATURE REVIEWS AND MATHEMATICAL BACKGROUNDS

This chapter is about reviews in relevant publications and mathematical concepts that will be used in this study.

2.1 Literature Reviews

The nonlinear partial differential equation arises in various field of sciences. Recently, there are many analytical methods for solving these problems were developed [1–3] such as inverse scattering method, Hirota's bilinear technique, Bäcklund transformation, Painlevé analysis and including the hyperbolic tangent (tanh) method.

The tanh method are related to the theory of soliton. The solitary wave (soliton) was originated from observation of water wave by Scottish naval engineer named John Scott Russell in 1834 [10]. For many years later, both Boussineq [11] and Lord Rayleigh [12] deduced the equation of motion (fluid dynamics) for an inviscid incompressible fluid and also show that the wave profile $u(x,t)$ was given by

$$u(x,t) = a\,\mathrm{sech}^2\,(\mu(x-ct)), \tag{2.1}$$

where $\mu^{-2} = 4h^2(h+a)/3a$, the speed of wave $c = \sqrt{g(h+a)}$ (as Russell' experiment), a is the amplitude of the wave, h the depth of water and g the acceleration of gravity. However, these authors did not write the equation of motion in simple form for $u(x,t)$ which admits (2.1) as a solution. Korteweg and de Vries [13] was completed this step, the equation is given by

$$\frac{\partial u}{\partial t} = \frac{3}{2}\left(\frac{g}{h}\right)^{1/2}\left(\frac{2}{3}\varepsilon\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + \frac{1}{3}\sigma\frac{\partial^3 u}{\partial x^3}\right), \tag{2.2}$$

where $\sigma = \frac{1}{3}h^3 - Th/g\rho$, T the surface tension, ρ is the density of fluid and ε is an arbitrary parameter. This is important history of soliton theory and nonlinear sciences [14].

Next, the tanh method was developed in several years ago. Huibin and Kelin [15] introduced a finite power series of tanh function as a solution of KdV equation in 1990. For a few years later, Malfliet [16] introduced the tanh function as a new variable. In 1996, Parkes [17] introduced Mathematica package ATFM to deal with the tedious algebra, that arises from using the tanh method. This method is very useful and straight forward to find the analytic solution of nonlinear partial differential equations.

The generalized Burgers–Huxley equation was first investigated by Satsuma [18]. However, the analytical solution did not reveal until 1990, Wang et al. [5] used the relevant nonlinear transformations. Since then many researchs found the analytic solution such as Ismail [19] used Adomian decomposition method, Deng [6] applied the first integral method and Gao et al. [7] applied the exp-function method.

2.2 Mathematical Backgrounds

This section provides fundamental theory of partial differential equations, hyperbolic tangent technique and finite-difference method that will be used in later chapters.

2.2.1 Fundamental Theorems

Theorem 2.1 (*Mean Value Theorem, one variable*) If $f(x)$ is a continuous function on the closed interval $[a,b]$ and f is differentiable on the open interval (a,b), then there exists some c in (a,b) such that

$$f(b)-f(a)=f'(c)(b-a). \tag{2.3}$$

Theorem 2.2 (*Mean Value Theorem, two variables*) If $f(x,y)$ is differentiable at each point of the line segment connecting the points (x_1,y_1) and (x_2,y_2), then there exists a point (x_0,y_0) on this line segment such that

$$f(x_2,y_2)-f(x_1,y_1)=\frac{\partial f}{\partial x}(x_0,y_0)(x_2-x_1)+\frac{\partial f}{\partial y}(x_0,y_0)(y_2-y_1). \tag{2.4}$$

Theorem 2.3 (*Taylor's Theorem, one variable*) Suppose that $f=f(x)$ has continuous derivatives up to order $n+1$ on the open interval (a,b) containing x_0. Then, for each $x\in[a,b]$, there exists a point ξ which lies between x_0 and x such that

$$f(x)=P_n(x)+R_n(x), \tag{2.5}$$

where

$$\begin{aligned} P_n(x) &= f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \end{aligned} \tag{2.6}$$

and

$$R_n(x)=\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \tag{2.7}$$

The function $P_n(x)$ is called the ***nth* Taylor polynomial** of $f(x)$ centered at $x=x_0$ where the values $\frac{f^{(k)}(x_0)}{k!}$ are the **Taylor's coefficients** and $R_n(x)$ is called the **remainder term**. The **Taylor's series** of $f(x)$ centered at x_0 (if limit of $P_n(x)$ exists) is obtained by taking limit of $P_n(x)$ as $n\to\infty$ and the remainder $R_n(x)$ converges to zero as $n\to\infty$.

Theorem 2.4 (*Taylor's Theorem, two variables*) Suppose that the function $f(x,y)$ has continuous $(n+1)th$ partial derivatives on $D = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$. If $(x_0,y_0) \in D$, for each $(x,y) \in D$, there exists ξ between x and x_0 and η between y and y_0 with

$$f(x) = P_n(x) + R_n(x) \tag{2.8}$$

where

$$\begin{aligned} P_n(x) &= f(x_0,y_0) + \frac{1}{1!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial x}\right) f(x_0,y_0) \\ &\quad + \frac{1}{2!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial x}\right)^2 f(x_0,y_0) + \dots \\ &\quad + \frac{1}{n!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial x}\right)^n f(x_0,y_0) \\ &= \sum_{k=0}^{n} \frac{1}{n!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial x}\right)^k f(x_0,y_0) \end{aligned} \tag{2.9}$$

and

$$R_n(x,y) = \frac{1}{(n+1)!}\left((x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial x}\right)^{n+1} f(\xi,\eta). \tag{2.10}$$

The function $P_n(x)$ is called the ***nth* Taylor polynomial in two variables** of $f(x,y)$ centered at (x_0,y_0) and $R_n(x)$ is the remainder term associated with $P_n(x)$.

2.2.2 Partial Differential Equations

The differential equations are equations which contain the unknown functions of one or more independent variables and their derivatives as well. Basically, these equations can be classified as ordinary or partial differential equation. If the unknowns are functions of only one independent variable then the equations are called **ordinary differential equations** (ODE for short). On the other hand, the equations are called **partial differential equations** (PDE for short) if the unknowns are functions of several independent variables.

Suppose that $u = u(x_1,\dots,x_n)$ is a unknown function of n independent variables $x_1,\dots,x_n$. The partial differential equations are commonly written as

$$F(x_1,\dots,x_n,u_{x_1},\dots,u_{x_n},u_{x_1x_1},\dots,u_{x_ix_j,\dots}) = 0, \tag{2.11}$$

where u_{x_i} denotes the partial derivative $\partial u/\partial x_i$. The equation, may be, associated to the initial conditions and/or boundary conditions. The solution of partial differential equation is a function u such that satisfies the partial differential equation. The order of a partial differential equation is the order of the highest derivative appearing in the equation [20].

It is more convenient to consider the partial different equation in the differential operator form. The differential operator L is given by

$$L[u] := F(u_{x_1}, \ldots, u_{x_n}, u_{x_1x_1}, \ldots, u_{x_ix_j,\ldots}). \tag{2.12}$$

The partial differential equation can be rewritten as

$$L[u] = f. \tag{2.13}$$

The partial differential equation is said to be linear if and only the operator L is linear that means satisfy the following properties

1. $L[u+v] = L[u] + L[v]$
2. $L[\alpha u] = \alpha L[u]$.

In the other hand, the partial differential equation is nonlinear. Particularly, a linear differential equation is called homogeneous equation when $L[u] = 0$ means $f = 0$. Otherwise, It is called a nonhomogeneous equation.

The linear operator play an important role in mathematics. If the function u_i satisfies the linear differential equation $L[u_i] = f_i$ for $1 \leq i \leq n$, then the linear combination $z = \sum_{i=1}^{n} \alpha_i u_i$, where α_i are constants, satisfies the equation $L[z] = \sum_{i=1}^{n} f_i$. This property is called superposition principle. It allows the construction of complex solutions through the combinations of simple solutions.

The partial differential equation, perhaps, associated with additional conditions in the form of initial conditions or boundary conditions. These are known as the initial value problems (IVP) and the boundary value problem. The initial conditions are also known as Cauchy conditions which the values of the unknown function u and its derivatives at initial point. The boundary condition, while, are the values on the boundary of the domain under consideration. The three most importance kinds of boundary conditions are:

- Dirichlet conditions: the values of u on the boundary are given at each point of the boundary.
- Neumann conditions: the values of derivative of u on the boundary are given at each point of the boundary.
- Robin conditions: the combination of Dirichlet conditions and Neumann conditions.

2.2.3 Hyperbolic Tangent Method

This subject describes the hyperbolic tangent (tanh) method as represented by Malfliet [1]. The concept of method is the hypothesis that the solution of nonlinear partial differential equation can be expressed in the form of tanh function.

Assume nonlinear partial differential equations

$$F(x,t,u,u_t,u_x,u_{xx},\cdots)=0, \tag{2.14}$$

where u is the function of x and t. The partial derivatives u_t, u_x, and u_{xx} will be used to present $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$, respectively.

The tanh method for solving the travelling wave solution of nonlinear partial differential equations contains five steps.

Step 1 Reduction of (2.14) to nonlinear ordinary differential equation. At this step, the traveling waves can be described by a function of the following

$$u(x,t)=U(x-ct)=U(\xi). \tag{2.15}$$

For $t=0$, $u(x,0)=U(x)$ which is the initial function. The function $u(x,t)$ propagates without change of shape at speed $|c|$ in the positive or negative x-direction when $c\geq 0$ or $c\leq 0$, respectively.

Using (2.15), equation (2.14) is reduced to the nonlinear ordinary differential equation

$$G(U,U',U'',U''',\cdots)=0. \tag{2.16}$$

Step 2 Hypothesis that the solution of (2.16) can be formed in the form of the finite series on hyperbolic tangent.

The exact solutions of (2.16) can be formed in the form

$$U(\xi)=S(Y)=\sum_{n=0}^{N} a_i Y^i, \tag{2.17}$$

where $\xi=x-ct$ and $Y=\tanh(\mu\xi)$, N is an positive integer, the coefficients a_i and parameters μ (denotes wave number, which $\mu>0$), c (denotes velocity of wave) are unknown values that can be found after substitution (2.17) into (2.16). The function $Y=\tanh(\mu\xi)$ leads to the following change of derivatives

$$\begin{aligned}\frac{d}{d\xi}&=\mu(1-Y^2)\frac{d}{dY},\\ \frac{d^2}{d\xi^2}&=\mu^2(1-Y^2)\frac{d}{dY}\left((1-Y^2)\frac{d}{dY}\right).\end{aligned} \tag{2.18}$$

Step 3 Finding positive integer N in the finite series of solution with the hyperbolic tangent.

Substituting (2.17) into (2.16) and equating the maximal power of the hyperbolic tangent to zero. The parameter N can be determined by balancing the highest power of Y in the highest derivative term with the highest power of Y in nonlinear term.

Step 4 Determination of coefficient a_i and parameters μ, c.

With determined N, Substituting expression (2.17) into (2.16) and equating expressions of different power of $\tanh^i(\mu\xi)$ to zero, yields a system of algebraic equations involving parameters a_i $(i = 0, 1, \ldots, N)$, μ and c.

Step 5 Presenting the solutions for (2.16)

The exact solutions of nonlinear differential equation (2.16) are obtained by substituting N, μ, c and a_i $(i = 0, 1, \ldots, N)$, respectively.

Next, the tanh method is applied to obtain the analytic solutions of Burgers and Huxley equations.

1. Burgers Equation

The Burgers equation [21] is a second order of nonlinear partial differential equation given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.19}$$

where u is velocity, x is spatial coordinate, t is time and ν is viscosity coefficient in unit of square length per unit time. The equation is used in fluid dynamics as a simplified model for turbulence, behavior of boundary layer, mass transport and gas dynamics [21–23].

Next, The tanh method, as above representation, is employed to find the travelling wave solution of Burgurs equation.

Step 1 Using $\xi = x - ct$ and $u(x,t) = U(x - ct) = U(\xi)$

$$-c\frac{dU}{d\xi} + U\frac{dU}{d\xi} - \nu\frac{d^2U}{d\xi^2} = 0. \tag{2.20}$$

Step 2 Suppose the solution is expressed in finite series on tanh function

$$U(\xi) = S(Y) = \sum_{n=0}^{N} a_i Y^i, \tag{2.21}$$

where $Y = \tanh(\mu\xi)$. Substituting (2.21) into (2.20) yields

$$-c\mu\left(1-Y^2\right)\frac{dS(Y)}{dY}+\mu S(Y)\left(1-Y^2\right)\frac{dS(Y)}{dY} -\nu\mu^2\left(1-Y^2\right)\frac{d}{dY}\left\{\left(1-Y^2\right)\frac{dS(Y)}{dY}\right\}=0. \quad (2.22)$$

Step 3 From (2.22), the hightest power of Y are Y^{N+2} and Y^{2N+1} in the second term and the last term, respectively. The integer N is obtained by balancing these values gives

$$N+2=2N+1,$$

so that

$$N=1.$$

With $N=1$,

$$\begin{aligned} S(Y) &= \sum_{i=0}^{n} a_i Y^i \\ &= a_0+a_1Y. \end{aligned} \quad (2.23)$$

Step 4 Substituting (2.23) into (2.22) yields

$$(-c\mu a_1+\mu a_0a_1)+\left(\mu a_1^2+2\nu\mu^2a_1\right)Y+(c\mu a_1-\mu a_0a_1)Y^2 +\left(-\mu a_1^2-2\nu\mu^2a_1\right)Y^3=0. \quad (2.24)$$

Equating the coefficients of Y^i to zero give

$$-c\mu a_1+\mu a_0a_1 = 0, \quad (2.25)$$
$$\mu a_1^2+2\nu\mu^2a_1 = 0, \quad (2.26)$$
$$c\mu a_1-\mu a_0a_1 = 0, \quad (2.27)$$
$$-\mu a_1^2-2\nu\mu^2a_1 = 0. \quad (2.28)$$

Solving the system (2.25)-(2.25) gives $a_0=c$ and $a_1=-2\nu\mu$ where c and μ are parameters.

Step 5 Substituting $a_0=c$ and $a_1=-2\nu\mu$ into (2.23). The solution of Burgers equation is given by

$$u(x,t)=c-2\nu\mu\tanh\left(\mu(x-ct)\right). \quad (2.29)$$

2. Huxley Equation

The Huxley equation [24] is a nonlinear partial differential equation of second order given by

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=u(1-u)(u-\gamma), \quad (2.30)$$

where $\gamma\in(0,1)$. The equation is an evolution equation that describes the electrical characteristics of excitable cells such as neurons and cardiac myocytes (heart muscles).

To apply the tanh method, by the above represented algorithm , the results are shown in the following

Step 1 Using $\xi = x - ct$ and $u(x,t) = U(x-ct) = U(\xi)$

$$-c\frac{\mathrm{d}U}{\mathrm{d}\xi} - \frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} = U(1-U)(U-\gamma) \tag{2.31}$$

and rewrite in the form

$$-c\frac{\mathrm{d}U}{\mathrm{d}\xi} - \frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} + \gamma U - (1+\gamma)U^2 + U^3 = 0. \tag{2.32}$$

Step 2 Suppose the solution is expressed in finite series on tanh function

$$U(\xi) = S(Y) = \sum_{n=0}^{N} a_i Y^i, \tag{2.33}$$

where $Y = \tanh(\mu\xi)$, then equation (2.32) becomes

$$-c\mu\left(1-Y^2\right)\frac{dS(Y)}{dY} - \mu^2\left(1-Y^2\right)\frac{d}{dY}\left\{\left(1-Y^2\right)\frac{dS(Y)}{dY}\right\}$$
$$+\gamma S(Y) - (1+\gamma)S^2(Y) + S^3(Y) = 0. \tag{2.34}$$

Step 3 To determine the parameter N, consider the highest power of Y in the second term and the last term which are Y^{N+2} and Y^3, respectively. Balancing these values yields

$$N+2 = 3N$$

and

$$N = 1.$$

Replacing $N = 1$ into (2.34) then gives

$$S(Y) = a_0 + a_1 Y. \tag{2.35}$$

Step 4 Substituting equation(2.35) into (2.34) gives

$$\left(-c\mu a_1 + \gamma a_0 - a_0^2 - \gamma a_0^2 + a_0^4\right) + \left(2\mu^2 a_1 + \gamma a_1 - 2a_0 a_1 - 2\gamma a_0 a_1 + 3a_0^2 a_1\right)Y$$
$$+\left(c\mu a_1 - a_1^2 - \gamma a_1^2 3a_0 a_1^2\right)Y^2 + \left(-2\mu^2 a_1 + a_1^3\right)Y^3 = 0. \tag{2.36}$$

Equating the coefficients of Y^i to zero yield

$$\begin{aligned}
-c\mu a_1 + \gamma a_0 - a_0^2 - \gamma a_0^2 + a_0^4 &= 0, && (2.37)\\
2\mu^2 a_1 + \gamma a_1 - 2a_0 a_1 - 2\gamma a_0 a_1 + 3a_0^2 a_1 &= 0, && (2.38)\\
c\mu a_1 - a_1^2 - \gamma a_1^2 3a_0 a_1^2 &= 0, && (2.39)\\
-2\mu^2 a_1 + a_1^3 &= 0. && (2.40)
\end{aligned}$$

Solving the system of equation gives

$$\left.\begin{aligned}
a_0 &= \frac{1}{2},\\
a_1 &= \pm\frac{1}{2},\\
\mu &= \frac{1}{2\sqrt{2}},\\
c &= \pm\frac{(2\gamma-1)}{\sqrt{2}},
\end{aligned}\right\} \qquad (2.41)$$

$$\left.\begin{aligned}
a_0 &= \frac{\gamma}{2},\\
a_1 &= \pm\frac{\gamma}{2},\\
\mu &= \frac{\gamma}{2\sqrt{2}},\\
c &= \pm\frac{(\gamma-2)}{\sqrt{2}}.
\end{aligned}\right\} \qquad (2.42)$$

Step 5 The solutions of Huxley equation are

$$u(x,t) = \frac{1}{2} \pm \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}(x \pm \frac{(2\gamma-1)}{\sqrt{2}})\right), \qquad (2.43)$$

and

$$u(x,t) = \frac{\gamma}{2} \pm \frac{\gamma}{2}\tanh\left(\frac{1}{2\sqrt{2}}(x \pm \frac{(\gamma-2)}{\sqrt{2}})\right). \qquad (2.44)$$

2.2.4 Finite-Difference Method

The finite-difference method is a well-known numerical method for solving differential equations. The concept of method is to approximate the derivatives in the equation using the finite-difference formulas. Applying Taylor's theorem 2.3, if x and x_0 are replaced by $x+h$ and x, respectively. The Taylor series of $u(x)$ is given by

$$u(x+h) = u(x) + \frac{h}{1!}u'(x) + \frac{h^2}{2!}u''(x) + \ldots \qquad (2.45)$$

and if replacing simultaneously x and x_0 by $x-h$ and x yields

$$u(x-h) = u(x) - \frac{h}{1!}u'(x) + \frac{h^2}{2!}u''(x) - \ldots, \qquad (2.46)$$

where $h > 0$ is called an increment in x. Taking only the first two terms of the right hand sides of (2.45) and (2.46) gives

$$u(x+h) = u(x) + hu'(x) + O(h^2) \tag{2.47}$$

and

$$u(x-h) = u(x) - hu'(x) + O(h^2), \tag{2.48}$$

where the expression $O(h^2)$ indicates the error $R_2(x)$ has principal part proportional to h^2. It follows that

$$\frac{du}{dx} = \frac{u(x+h) - u(x)}{h} + O(h) \tag{2.49}$$

and

$$\frac{du}{dx} = \frac{u(x) - u(x-h)}{h} + O(h). \tag{2.50}$$

These are called **first-order forward-difference** approximant to $\frac{du}{dx}$ and **first-order backward-difference** approximant to $\frac{du}{dx}$, respectively.

Taking terms up to and including h^3 on the right-hand sides of (2.45) and (2.46), and subtracting gives

$$u(x+h) - u(x-h) = 2hu'(x) + O(h^3), \tag{2.51}$$

yields

$$\frac{du}{dx} = \frac{u(x+h) - u(x-h)}{2h} + O(h^2), \tag{2.52}$$

which is a **second-order central-difference** approximant to $\frac{du}{dx}$.

Taking terms up to and including h^4 on the right-hand sides of (2.45) and (2.46), and adding gives

$$u(x+h) + u(x-h) = 2u(x) + h^2u''(x) + O(h^4), \tag{2.53}$$

leads to

$$\frac{d^2u}{dx^2} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2). \tag{2.54}$$

This is called a **second-order central-difference** approximant to $\frac{d^2u}{dx^2}$.

Suppose next that $u = u(x,t)$ is a function of two variables x, t and x is given an increment $h > 0$ but t is fixed. The Taylor series of $u(x+h,t)$ is given by

$$u(x+h,t) = u(x,t) + \frac{h}{1!}u'(x,t) + \frac{h^2}{2!}u''(x,t) + \ldots, \tag{2.55}$$

and the Taylor series of $u(x-h,t)$ becomes

$$u(x-h,t) = u(x,t) - \frac{h}{1!}u'(x,t) + \frac{h^2}{2!}u''(x,t) - \ldots. \tag{2.56}$$

The first and second derivative approximation of function u can be derived as follows:

$$\frac{\partial u}{\partial x} = \frac{u(x+h,t)-u(x,t)}{h}+O(h), \quad (2.57)$$

$$\frac{\partial u}{\partial x} = \frac{u(x,t)-u(x-h,t)}{h}+O(h), \quad (2.58)$$

$$\frac{\partial u}{\partial x} = \frac{u(x+h,t)-u(x-h,t)}{2h}+O(h^2), \quad (2.59)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2}+O(h^2). \quad (2.60)$$

In order to obtain the mixed second derivative $\partial^2 u/\partial x\partial y$, The Taylor's series in which both x, y increments is required. There are four such forms and are given by

$$u(x+h,y+k) = u(x,y)+\frac{1}{1!}\left(h\frac{\partial u(x,y)}{\partial x}+k\frac{\partial u(x,y)}{\partial y}\right)+ \frac{1}{2!}\left(h^2\frac{\partial^2 u(x,y)}{\partial x^2}+2hk\frac{\partial^2 u(x,y)}{\partial x\partial y}+k^2\frac{\partial^2 u(x,y)}{\partial y^2}\right)+\ldots, \quad (2.61)$$

$$u(x+h,y-k) = u(x,y)+\frac{1}{1!}\left(h\frac{\partial u(x,y)}{\partial x}-k\frac{\partial u(x,y)}{\partial y}\right)+ \frac{1}{2!}\left(h^2\frac{\partial^2 u(x,y)}{\partial x^2}-2hk\frac{\partial^2 u(x,y)}{\partial x\partial y}+k^2\frac{\partial^2 u(x,y)}{\partial y^2}\right)+\ldots, \quad (2.62)$$

$$u(x-h,y+k) = u(x,y)+\frac{1}{1!}\left(-h\frac{\partial u(x,y)}{\partial x}+k\frac{\partial u(x,y)}{\partial y}\right)+ \frac{1}{2!}\left(h^2\frac{\partial^2 u(x,y)}{\partial x^2}-2hk\frac{\partial^2 u(x,y)}{\partial x\partial y}+k^2\frac{\partial^2 u(x,y)}{\partial y^2}\right)+\ldots, \quad (2.63)$$

$$u(x-h,y-k) = u(x,y)+\frac{1}{1!}\left(-h\frac{\partial u(x,y)}{\partial x}+k\frac{\partial u(x,y)}{\partial y}\right)+ \frac{1}{2!}\left(h^2\frac{\partial^2 u(x,y)}{\partial x^2}+2hk\frac{\partial^2 u(x,y)}{\partial x\partial y}+k^2\frac{\partial^2 u(x,y)}{\partial y^2}\right)+\ldots, \quad (2.64)$$

Adding (2.61)-(2.64) yields

$$u(x+h,y+k)-u(x+h,y-k)-u(x-h,y+k)+u(x-h,y-k) = 4hk\frac{\partial^2 u}{\partial x\partial y}+O(h+k)^4. \quad (2.65)$$

After rearranging (2.65) leads to

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{u(x+h,y+k)-u(x+h,y-k)-u(x-h,y+k)+u(x-h,y-k)}{4hk} +O\left(\frac{(h+k)^4}{hk}\right). \quad (2.66)$$

For special case $k = h$, (2.66) becomes

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{u(x+h,y+k) - u(x+h,y-k) - u(x-h,y+k) + u(x-h,y-k)}{4h^2} + O\left(\frac{(h+k)^4}{hk}\right). \tag{2.67}$$

These approximations (2.66) and (2.67) are both second order approximants to $\partial^2 u/\partial x \partial y$ [25].