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Köthe-Toeplitz duals of some *n*-normed valued difference sequence spaces

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Abstract: First, a few basic results are presented that are relevant to some *n*-normed real linear valued difference sequence spaces. Then Köthe-Toeplitz duals of some such spaces are computed.

Keywords: *n*-norm, difference sequences, Köthe-Toeplitz dual

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INTRODUCTION

A sequence space is a linear space whose elements are sequences chosen from another linear space. The analysis of sequence spaces primarily deals with questions related to the distance between sequences, Cauchy sequences and limits of a sequence of 'points' in the class of all sequences, where 'points' refer to a set of real or complex numbers. The theory of sequence spaces deals with different classes of sequences including those defined by the difference operator and over n (≥ 2)-normed spaces as base space.

Throughout, w, λ_∞, ℓ_1, c and c_0 denote the spaces of all, bounded, absolutely summable, convergent and null sequences $x = (x_k)$ with complex terms respectively.

The zero element of a normed linear space is denoted by θ. A complete normed linear space is called a Banach space. It is well known that λ_∞ is a Banach space under the norm

$$\|x\| = \sup_k |x_k|,$$

which is called the sup-norm or uniform norm. The spaces c and c_0 are complete subspaces of λ_∞.

A BK-space, $(Z, \|.\|)$, introduced by K. Zeller [1], is a Banach space of complex sequences $x = (x_k)$, in which the co-ordinate maps are continuous; that is, $|x_k^n - x_k| \to 0$, whenever $\|x^n - x\| \to 0$ as $n \to \infty$, where $x^n = (x_k^n)$, for all $n \in N$ and $x = (x_k)$.

Let $(X, \|\,.\,\|)$ be a normed linear space and λ be a scalar valued sequence space; then the vector valued sequence space or X-valued sequence space $\lambda(X)$ is defined as

$$\lambda(X) = \{(x_k) : x_k \in X \text{ for all } k \in N \text{ and } \|x_k\| \in \lambda\}$$

Clearly $\lambda(X)$ is a linear space under coordinate-wise addition and scalar multiplication over the field of scalars of *X*. If *X* is a Banach space, then the vector valued sequence spaces $c_0(X)$, $\lambda_\infty(X)$ are also Banach spaces with the norm defined by

$$\|x\| = \sup_k \|x_k\|, \text{ where } x = (x_k)$$

Investigations of spaces are often combined with those of their duals. For the duality theory, the study of sequence spaces is more useful when we consider them equipped with linear topologies. In such cases, however, it is rather cumbersome to obtain their topological duals. Even if we are successful in finding these duals, we would like to deal with only those duals whose members can be represented as sequences. Indeed such situations do not present much difficulty in the analysis. Köthe and Toeplitz [2] first recognised the problem and to overcome the situation, they introduced the notion of α-dual (known also as Köthe-Toeplitz dual), which turns out to be the same as the topological dual in quite many familiar and useful examples of sequence spaces endowed with their natural linear matrices. More details about the notion of algebraic duals can be found in the literature [3, 4].

Let *X* be a normed space. Then the set of all bounded linear functionals on *X* constitutes a normed space with norm defined by

$$\|f\| = \sup_{\substack{x\in X\\ x\neq 0}} \frac{|f(x)|}{\|x\|} = \sup_{\substack{x\in X\\ \|x\|=1}} |f(x)|,$$

which is called the dual space of *X* and is denoted by $X^{/}$.

Let *E* and *F* be two sequence spaces. Then the *F* dual of *E* is defined as

$$E^{F} = \{(x_k)\in w : (x_k y_k)\in F \text{ for all } (y_k)\in E\}$$

For $F = \ell_1$, the dual is termed as α-dual (Köthe-Toeplitz dual) of *E* and denoted by E^{α}. If $X \subset Y$, then $Y^{\alpha} \subset X^{\alpha}$.

Gähler [5-10] introduced and studied the concept of 2-normed spaces extensively at the initial stage. Then White [11], Diminnie et al. [12], Gähler et al. [13], Siddiqi [14], etc. also contributed in popularising the theory. Later on the concept was generalised to *n*(>2)-normed space and some initial work on *n*-normed structures were carried out by Misiak [15, 16]. Since then, Kim and Cho [17], Malčeski [18], Gunawan [19, 20], Gunawan and Mashadi [21, 22], Dutta [23-27], Acikgöz et al. [28], Dutta and Reddy [29], and many others have studied this concept and obtained various results and links with other theories.

The notion of difference sequence space was introduced by Kizmaz [30], who studied the difference sequence spaces $\lambda_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalised by Et and Colak [31] by introducing the spaces $1_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$, where *m* is a fixed non-negative integer.

Let *m* be non-negative integers. Then for *Z*, a given sequence space, we have

$$Z(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in Z\}$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation:

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$$\Delta^m x_k = \sum_{\upsilon=0}^{m} (-1)^{\upsilon} \binom{m}{\upsilon} x_{k+\upsilon}$$

Taking m = 1 and Z = λ_∞, c and c_0, we get the spaces $\lambda_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. For some other notions of difference sequences, one may refer to Dutta [32].

PRELIMINARIES

The definitions of n-norm and several associated preliminary notions are described below in order to make the relevant theories of this paper comprehensible. Let $n \in N$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|.,. . .,.\|$ on X^n satisfying the following four conditions, viz.

(N1) $\|x_1, x_2, . . . , x_n\| = 0$ if and only if $x_1, x_2, . . ., x_n$ are linearly dependent,

(N2) $\|x_1, x_2, . . . , x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1, x_2, . . . , x_n\| = |\alpha| \, \|x_1, x_2, . . . , x_n\|$, for any $\alpha \in R$, and

(N4) $\|x+x', x_2, . . . , x_n\| \leq \|x, x_2, . . . , x_n\| + \|x', x_2, . . . , x_n\|$,

is called an n-norm on X and the pair $(X, \|.,. . .,.\|)$ is called an n-normed space.

A trivial example of an n-normed space is $X = R^n$, equipped with the following Euclidean n-norm:

$$\|x_1,x_2, . . . ,x_n\|_E = \text{abs}\left(\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} \right), \text{ where } x_i=(x_{i1},\ldots,x_{in}) \in R^n \text{ for each } i = 1, 2,. . .,n$$

Gunawan [19] showed how we can actually define an n-inner product and accordingly an n-norm on any inner product space, provided that the dimension is sufficiently large, as follows.

Let $n \in N$ and $(X, \langle .,. \rangle)$ be a real inner product space of dimension $d \geq n$. Define the following function $\langle .,...,. | .,. \rangle$ on $X \times ... \times X$ (n+1 factors) by

$$\langle x_1,...,x_{n-1} \,|\, y,z \rangle = \begin{vmatrix} \langle x_1,x_1 \rangle & \cdots & \langle x_1,x_{n-1} \rangle & \langle x_1,z \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle x_{n-1},x_1 \rangle & \cdots & \langle x_{n-1},x_{n-1} \rangle & \langle x_{n-1},z \rangle \\ \langle y,x_1 \rangle & \cdots & \langle y,x_{n-1} \rangle & \langle y,z \rangle \end{vmatrix}$$

Then one may check that this function satisfies the following five properties:

(I1) $\langle x_1,...,x_{n-1} \,|\, x_n,x_n \rangle \geq 0$; $\langle x_1,...,x_{n-1} \,|\, x_n,x_n \rangle = 0$ if and only if $x_1,\ldots,x_n$ are linearly dependent;

(I2) $\langle x_1,...,x_{n-1} \,|\, x_n,x_n \rangle = \langle x_{i_1},...,x_{i_{n-1}} \,|\, x_{i_n},x_{i_n} \rangle$ for every permutation $(i_1,\ldots,i_n)$ of $(1,\ldots,n)$;

(I3) $\langle x_1,...,x_{n-1} \,|\, y,z \rangle = \langle x_1,...,x_{n-1} \,|\, z,y \rangle$;

(I4) $\langle x_1,...,x_{n-1} \,|\, y,\alpha z \rangle = \alpha \langle x_1,...,x_{n-1} \,|\, y,z \rangle$;

(I5) $\langle x_1,...,x_{n-1} \,|\, y,z+z' \rangle = \langle x_1,...,x_{n-1} \,|\, y,z \rangle + \langle x_1,...,x_{n-1} \,|\, y,z' \rangle$.

Accordingly, we can define $\|.,...,.\|$ on $X \times ... \times X$ (n factors) by

$$\|x_1,...,x_n\| = \langle x_1,...,x_{n-1} \,|\, x_n,x_n \rangle^{1/2};$$

that is,

$$\|x_1,...,x_n\| = \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix}^{\frac{1}{2}}$$

For $n = 1$, we know that $\|.\|$ is a norm, while for $n = 2$, $\|.,.\|$ defines a 2-norm. Note further that for $n = 1$, $\|x_1\|$ gives the length of x_1, while for $n = 2$, $\|x_1, x_2\|$ represents the area of the parallelogram spanned by x_1 and x_2. For $n = 3$ and $X = R^3$, one may observe that $\|x_1, x_2, x_3\|$ is nothing but the volume of the parallelepiped spanned by x_1, x_2 and x_3; that is,

$$\|x_1, x_2, x_3\| = \left|x_1.(x_2 \times x_3)\right|$$

Thus, in general, $\|x_1,...,x_n\|$ can be interpreted as the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in X. Further, it satisfies the four properties (N1)-(N4) of n-norm.

Let $n \in N$ and X be a real vector space of dimension $d \geq n$. A function $\langle .,...,. | .,. \rangle$ on $X \times ... \times X$ (n+1 factors) satisfying the five properties (I1)-(I5) listed above is called an n-inner product on X, and the pair $(X, \langle .,...,. | .,. \rangle)$ is called an n-inner product space. Meanwhile, a function $\|.,...,.\|$ on $X \times ... \times X$ (n factors) satisfying the four properties (N1)-(N4) listed above is called an n-norm on X, and the pair $(X, \|.,...,.\|)$ is called an n-normed space.

The definition of n-inner product given by Gunawan [19] is slightly simpler than that by Misiak [15, 16]. To see that it is equivalent to Misiak's, one only needs to verify that

$$\langle x_1,...,x_{n-1} | y,z \rangle = \langle x_{i_1},...,x_{i_{n-1}} | y,z \rangle$$

for every permutation $(i_1,\ldots,i_{n-1})$ of $(1,\ldots,n-1)$, but this will follow easily from property (I2) and the polarisation identity:

$$\langle x_1,...,x_{n-1} | y,z \rangle = \frac{1}{4}\left[\langle x_1,...,x_{n-1} | y+z, y+z \rangle - \langle x_1,...,x_{n-1} | y-z, y-z \rangle\right]$$

Moreover, the first three properties of the n-norm are easy to prove. To prove the fourth property or the triangle inequality, Cauchy-Schwartz inequality is needed. Gunawan [19] has given the Cauchy-Schwartz inequality for n-inner product spaces as follows.

Proposition 1 (Theorem 3.1[19]) **.** *Let* $(X, \langle .,...,. | .,. \rangle)$ *be an n-inner product space. Then we have*

$$\langle x_1,...,x_{n-1} | y,z \rangle^2 \leq \langle x_1,...,x_{n-1} | y,y \rangle \langle x_1,...,x_{n-1} | z,z \rangle$$

and the equality holds if and only if $x_1, \ldots, x_{n-1}, y, z$ *are linearly dependent.*

Proposition 2 (Corollary 3.2 [19])**.** *On an n-inner product space* $(X, \langle .,...,. | .,. \rangle)$, *the following function,*

$$\|x_1,...,x_n\| = \langle x_1,...,x_{n-1} | x_n,x_n \rangle^{1/2},$$

defines an n-norm. In particular, the triangle inequality,

$$\|x_1,...,x_{n-1}, y+z\| \leq \|x_1,...,x_{n-1}, y\| + \|x_1,...,x_{n-1}, z\| ,$$

holds for all $x_1, \ldots, x_{n-1}, y, z \in X$.

Proposition 3 (Corollary 3.3 [19])**.** *Let* $(X, \langle .,...,. | .,. \rangle)$ *be an n-inner product space. If* $x_1, \ldots, x_{n-1}$, *y*, *z are linearly dependent on X, then*

$$\|x_1,...,x_{n-1},y+z\|=\|x_1,...,x_{n-1},y\|+\|x_1,...,x_{n-1},z\|$$

or

$$\|x_1,...,x_{n-1},y-z\|=\|x_1,...,x_{n-1},y\|+\|x_1,...,x_{n-1},z\|$$

Conversely, if one of the above two equalities holds, then $x_1, \ldots, x_{n-1}, y, z$ *are linearly dependent on* X.

Gunawan and Mashadi [22] showed that if $(X, \|.,. \ldots,.\|)$ is an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ is a linearly independent set in X, then the following function $\|.,. \ldots,.\|_\infty$ on X^{n-1} defined by

$$\|x_1,x_2, \ldots ,x_{n-1}\|_\infty = \max \{\|x_1,x_2, \ldots ,x_{n-1},a_i\| : i = 1, 2, \ldots,n\}$$

defines an $(n$-1) norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$ and this is called the derived $(n$-1)-norm.

A sequence (x_k) in an n-normed space $(X, \|., \ldots,.\|)$ is said to converge to some $L \in X$ in the n-norm if

$$\lim_{k\to\infty}\|x_k - L,u_2,...,u_n\| = 0, \text{ for every } u_2, \ldots, u_n \in X$$

A sequence (x_k) in an n-normed space $(X, \|., \ldots,.\|)$ is said to be Cauchy with respect to the n-norm if

$$\lim_{k,l\to\infty}\|x_k - x_l,u_2,...,u_n\| = 0, \text{ for every } u_2, \ldots, u_n \in X$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Now I state the following results as Lemmas. For details one may refer to Gunawan and Mashadi [22].

Lemma 1. *Every n-normed space is an* $(n$-$r)$*-normed space for all* $r = 1, 2, \ldots\ldots, n$-1. *In particular, every n-normed space is a normed space.*

Lemma 2. *A standard n-normed space is complete if and only if it is complete with respect to the usual norm* $\|\,.\,\|_S = \langle .,. \rangle^{\frac{1}{2}}$.

Lemma 3. *On a standard n-normed space X, the derived* $(n$-1)*-norm* $\|.,\ldots\ldots,.\|_\infty$*, defined with respect to orthonormal set* $\{e_1, e_2,\ldots\ldots,e_n\}$*, is equivalent to the standard* $(n$-1)*-norm* $\|.,\ldots\ldots,.\|_S$. *Precisely, we have*

$$\|x_1,\ldots,x_{n-1}\|_\infty \leq \|x_1,\ldots,x_{n-1}\|_S \leq \sqrt{n}\, \|x_1,\ldots,x_{n-1}\|_\infty$$

for all $x_1, \ldots, x_{n-1}$*, where* $\|x_1,\ldots,x_{n-1}\|_\infty = \max\{\|x_1,\ldots,x_{n-1}, e_i\|_S : i = 1,\ldots,n\}$.

Definition 1. An n-BK-space $\left(X, \|.,...,.\|\right)$ is an n-Banach space of real sequences $x = (x_k)$, in which the co-ordinate maps are continuous.

The well-known spaces c_0, c and λ_∞ can be extended to n-normed space valued difference sequences using mth order difference operator as follows.

Let $(X, \|.,. \ldots, .\|)$ be an n-normed real linear space and $w(n$-$X)$ denote X-valued sequence space. Let m be a non-negative integer; then we have the following sequence spaces:

$$c_0(\Delta^m, \|.,...,.\|) = \{(x_k) \in w(n\text{-}X) : \lim_{\substack{k\to\infty,\\ z_1,...,z_{n-1}\in X}} \left\|\Delta^m x_k, z_1,..., z_{n-1}\right\| = 0 \},$$

$$c(\Delta^m, \|.,. \ldots,.\|) = \{(x_k) \in w(n\text{-}X): \lim_{\substack{k\to\infty,\\ z_1,...z_{n-1}\in X}} \left\|\Delta^m x_k - L, z_1,..., z_{n-1}\right\| = 0, \text{ for some } L \text{ in } X\},$$

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$\ell_\infty(\Delta^m, \|,.,..,.\|) = \{(x_k) \in w(n\text{-}X) : \sup_{\substack{k\geq 1,\\ z_1,...z_{n-1}\in X}} \|\Delta^m x_k, z_1,..., z_{n-1}\| < \infty\}$.

MAIN RESULTS

Throughout this section, Y is assumed to be any one of the spaces $c_0(\Delta^m, \|,.,...,.\|)$, $c(\Delta^m, \|,.,...,.\|)$ and $\ell_\infty(\Delta^m, \|,.,...,.\|)$. First, a few basic descriptions of these spaces with certain immediate results are given. Then the algebraic dual of some of these spaces are computed.

Theorem 1. *The spaces Y are linear over the field of reals.*

Proof. The proof is omitted as it is just a routine verification.

Let us define the function $\|.,.,...,.\|_Y$ on $Y \times ... \times Y$ (n-factors) by

$\|x^1, x^2,..., x^n\|_Y = 0$, if $x^1, x^2,..., x^n$ are linearly dependent and

$\|x^1, x^2,..., x^n\|_Y = \sum_{k=1}^{m} \|x_k^1, z_1,..., z_{n-1}\| + \sup_{k\geq 1} \|\Delta^m x_k^1, z_1,..., z_{n-1}\|$, for every $z_1,..., z_{n-1} \in X$,

if $x^1, x^2,..., x^n$ are linearly independent.

Theorem 2. *The function* $\|.,.,...,.\|_Y$ *is an n-norm on Y.*

Proof: In view of Dutta [24, 25], the proof is easy so it is omitted.

Note 1. We call $\|.,.,...,.\|_Y$ a non-standard n-norm [29].

For any linearly independent set $\{a_1, a_2,..., a_n\}$, we can define

$\|x_k^1, z_1, z_2,..., z_{n-r-1}\|_\infty = \max\{\|x_k^1, z_1, z_2,..., z_{n-r-1}, a_{i_1}, a_{i_2},..., a_{i_r}\|\}$, $\{i_1, i_2,..., i_r\} \subseteq \{1,2,...,n\}$

Then $\|.,...,.\|_\infty$ is an (n-r)-norm on X for each $k \geq 1$ and for all r = 1, 2,…, n-1.

We can define another function $\|.,.,...,.\|_Y$ on $Y \times ... \times Y$ ((n-r)-factors) by

$\|x^1, x^2,..., x^{n-r}\|_Y^{n-r} = 0$ if $x^1, x^2,..., x^{n-r}$ are linearly dependent and

$\|x^1, x^2,..., x^{n-r}\|_Y^{n-r} = \sum_{k=1}^{m} \|x_k^1, z_1,..., z_{n-r-1}\|_\infty + \sup_{k\geq 1} \|\Delta^m x_k^1, z_1,..., z_{n-r-1}\|_\infty$, for every

$z_1,..., z_{n-r-1} \in X$, if $x^1, x^2,..., x^{n-r}$ are linearly independent.

Theorem 3. *The function* $\|.,.,...,.\|_Y$ *is a* (n-r)-*norm on Y for all r* = 1, 2,...,n-1.

Proof. The proof is similar to that of Theorem 2.

Note 2: We call the (n-r)-norm $\|.,.,...,.\|_Y^{n-r}$ on the spaces Y for all r = 1,2,...,n-1 a derived norm.

Corollary 1. *The spaces Y are normed spaces.*

Theorem 4. *If the base space X is an n-Banach space, then the spaces Y are n-Banach spaces under the n-norm* $\|.,.,...,.\|_Y$.

Proof. The proof is easy in view of Dutta [24, 25].

The following corollary is due to Lemma 2.

Corollary 2. *If X is a Banach space under the standard n-norm, then the spaces Y are n-Banach space under the n-norm* $\|.,.,\ldots,.\|_Y$.

In view of the above results, we have the following immediate consequence.

Corollary 3. *The spaces Y are n-BK spaces under the n-norm* $\|.,.,\ldots,.\|_Y$ *, provided that the base space is an n-Banach space or Banach space.*

In order to compute the Köthe-Toeplitz dual, we need the following definitions and results [27]. An *n*-functional is a real valued mapping with domain $A_1 \times \ldots \times A_n$, where $A_1,\ldots,A_n$ are linear manifolds of a linear *n*-normed space.

Let *F* be an *n*-functional with domain $A_1 \times \ldots \times A_n$. *F* is called a linear *n*-functional whenever for all ${}^1a_1, {}^1a_2, \ldots, {}^1a_n \in A_1$, ${}^2a_1, {}^2a_2, \ldots, {}^2a_n \in A_2, \ldots, {}^na_1, {}^na_2, \ldots, {}^na_n \in A_n$ and all $\alpha_1, \ldots, \alpha_n \in R$, we have

i) $F\left({}^1a_1 + {}^1a_2 + \ldots + {}^1a_n, {}^2a_1 + {}^2a_2 + \ldots + {}^2a_n, \ldots, {}^na_1 + {}^na_2 + \ldots + {}^na_n\right) = \sum_{1 \le i_1, i_2, \ldots, i_n \le n} F\left({}^1a_{i_1}, {}^2a_{i_2}, \ldots, {}^na_{i_n}\right)$ and

ii) $F\left(\alpha_1 a_1, \ldots, \alpha_n a_n\right) = \alpha_1 \ldots \alpha_n F\left(a_1, \ldots, a_n\right)$

Let *F* be an *n*-functional with domain *D*(*F*). *F* is called bounded if there is a real constant $K \ge 0$ such that $\left|F\left(a_1,\ldots,a_n\right)\right| \le K\left\|a_1,\ldots,a_n\right\|$ for all $\left(a_1,\ldots,a_n\right) \in D\left(F\right)$. If *F* is bounded, we define the norm of *F*, $\|F\|$, by

$$\|F\| = \text{glb}\left\{K : \left|F\left(a_1,\ldots,a_n\right)\right| \le K\left\|a_1,\ldots,a_n\right\| \text{for all} \left(a_1,\ldots,a_n\right) \in D\left(F\right)\right\}$$

If *F* is not bounded, we define $\|F\| = +\infty$.

For the following two results, one may refer to White [11] and proofs can similarly be obtained.

Theorem 5. *A linear n-functional F is continuous if and only if it is bounded.*

Theorem 6. *Let* B^* *be the set of bounded linear n-functionals with domain* $B_1 \times \ldots \times B_n$. *Then* B^* *is an n-Banach space up to linear dependence.*

As far as is known, the theory of continuous duals for *n*-normed spaces has not been developed well enough. For any *n*-normed space *E*, its continuous dual will be denoted by E^*. Dutta [27] defined the Köthe-Toeplitz dual of sequence spaces, with base space an *n*-normed space, as follows.

Let *E* be an *n*-normed linear space, normed by $\|.,\ldots,.\|_E$. Then the Köthe-Toeplitz dual of the sequence space $Z\left(E\right)$ whose base space is *E* is defined as

$$\left[Z\left(E\right)\right]^\alpha = \{\left(y_k\right) : y_k \in E^*, k \in N \text{ and} \left(\left\|x_k, u_2, \ldots, u_n\right\|_E \left\|y_k, v_2, \ldots, v_n\right\|_{E^*}\right) \in \ell_1,$$
$$\text{for every } v_2, \ldots, v_n \in E^*, u_2, \ldots, u_n \in E, \left(x_k\right) \in Z\left(E\right)\}$$

It is easy to check that $\phi \subset X^\alpha$. If $X \subset Y$, then $Y^\alpha \subset X^\alpha$.

Lemma 4. *For every non-zero* $u_2, \ldots, u_n \in X$, *we have:*

(i) $x \in \ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)$ *implies* $\sup_k k^{-1}\left\|\Delta^{m-1}x_k, u_2, \ldots, u_n\right\| < \infty$

(ii) $x \in \ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)$ *implies* $\sup_k k^{-m}\left\|x_k, u_2, \ldots, u_n\right\| < \infty$

Proof. (i) Let $x \in \ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)$; then

$$\sup_{k\geq 1}\left\|\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\| < \infty, \text{ for every non-zero } z_1, \ldots, z_{n-1} \in X$$

Then there exists a $U > 0$ such that for every non-zero $z_1, \ldots, z_{n-1} \in X$,

$$\left\|\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\| < U \quad \text{for all } k \in N$$

Now for every non-zero $z_1, \ldots, z_{n-1} \in X$,

$$\left\|\Delta^{m-1}x_1 - \Delta^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\| \leq \frac{1}{k}\sum_{l=1}^{k}\left\|\Delta^{m-1}x_l - \Delta^{m-1}x_{l+1}, z_1, \ldots, z_{n-1}\right\| < U$$

Then from the above inequality, we have the desired result using the inequality:

$$\frac{1}{k}\left\|\Delta^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\| \leq \frac{1}{k}\left(\left\|\Delta^{m-1}x_1, z_1, \ldots, z_{n-1}\right\| + \left\|\Delta^{m-1}x_1 - \Delta^{m-1}x_{k+1}, z_1, \ldots, z_{n-1}\right\|\right)$$

(ii) The proof follows from part (i).

Let us set

$$D = \left\{a = (a_k) : \sum_{k=1}^{\infty} k^m \left\|a_k, z_2, \ldots, z_n\right\|_{X^*} < \infty, \text{for every } z_2, \ldots, z_n \in X^*\right\}.$$

Theorem 7. *The Köthe-Toeplitz dual of the spaces* $\ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)$ *and* $c\left(\Delta^m, \|.,\ldots,.\|\right)$ *is D, i.e.*

$$\left[\ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)\right]^\alpha = \left[c\left(\Delta^m, \|.,\ldots,.\|\right)\right]^\alpha = D$$

Proof. Let $a \in D$; then $\sum_{k=1}^{\infty} k^m \left\|a_k, z_2, \ldots, z_n\right\|_{X^*} < \infty$ for every $z_2, \ldots, z_n \in X^*$. Now for any $x \in \ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)$, we have $\sup_k k^{-m}\left\|x_k, u_2, \ldots, u_n\right\|_X < \infty$ for every $u_2, \ldots, u_n \in X$. Then we have

$$\sum_{k=1}^{\infty}\left\|a_k, z_2, \ldots, z_n\right\|_{X^*}\left\|x_k, u_2, \ldots, u_n\right\|_X \leq \sup_k k^{-m}\left\|x_k, u_2, \ldots, u_n\right\|_X \sum_{k=1}^{\infty} k^m \left\|a_k, z_2, \ldots, z_n\right\|_{X^*} < \infty$$

Hence $\quad a \in \left[\ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)\right]^\alpha$

$$D \subseteq \left[\ell_\infty\left(\Delta^m, \|.,\ldots,.\|\right)\right]^\alpha \qquad \text{------------------ (1)}$$

Again, we know that

$$\left[\ell_\infty\left(\Delta^m \|.,\ldots,.\|\right)\right]^\alpha \subseteq \left[c\left(\Delta^m \|.,\ldots,.\|\right)\right]^\alpha \subseteq \left[c_0\left(\Delta^m \|.,\ldots,.\|\right)\right]^\alpha \qquad \text{----------- (2)}$$

Conversely, suppose that $a \in \left[c\left(\Delta^m \|.,\ldots,.\|\right)\right]^\alpha$. Then $\sum_{k=1}^{\infty}\left\|a_k, z_2, \ldots, z_n\right\|_{X^*}\left\|x_k, u_2, \ldots, u_n\right\|_X < \infty$ for each $x \in c\left(\Delta^m \|.,\ldots,.\|\right)$. So we take

$$x_k = k^m, \ k \geq 1$$

and choose $u_2, \ldots, u_n \in X$ such that

$$\left\|k^m, u_2, \ldots, u_n\right\|_X = k^m \left\|1, u_2, \ldots, u_n\right\|_X = k^m, \ k \geq 1.$$

Then

$$\sum_{k=1}^{\infty} k^m \left\|a_k, z_2, \ldots, z_n\right\|_{X^*} =$$

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$$\sum_{k=1}^{\infty}\left\|k^m, u_2, \ldots, u_n\right\|_X \left\|a_k, z_2, \ldots, z_n\right\|_{X^*} = \sum_{k=1}^{\infty}\left\|a_k, z_2, \ldots, z_n\right\|_{X^*} \left\|x_k, u_2, \ldots, u_n\right\|_X < \infty$$

This implies that $a \in D$. Thus,

$$\left[c\left(\Delta^m \|.,\ldots,.\|\right)\right]^{\alpha} \subseteq D \qquad \text{- (3)}$$

Combining (3) with (1) and (2) , it follows:

$$\left[\ell_{\infty}\left(\Delta^m, \|.,\ldots,.\|\right)\right]^{\alpha} = \left[c\left(\Delta^m, \|.,\ldots,.\|\right)\right]^{\alpha} = D$$

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