GOODNESS-OF-FIT TESTS FOR TWO-DIMENSIONAL CIRCULAR NORMAL PROBABILITY DISTRIBUTION

Voinov Vassilly
KIMEP University, 2 Abai Av. Almaty 050010, Republic of Kazakhstan
Email: voinovv@kimep.kz

Pya Natalya
KIMEP University, 2 Abai Av. Almaty 050010, Republic of Kazakhstan
Email: N.Y.Pya@bath.ac.uk

Makarov Rashid
KIMEP University, 2 Abai Av. Almaty 050010, Republic of Kazakhstan
Email: rashid.m@list.ru

Voinov Yevgeniy
Institute for Mathematics and Mathematical Modelling of the Ministry of Education and Science of the Republic of Kazakhstan, 125 Pushkin St. Almaty 050010
Email: v.yevgeniy@yahoo.com

ABSTRACT

Based on the idea of McCulloch (1985) we introduced new modified chi-squared test, S_n^2, *for bivariate circular normality. We compare the power of* S_n^2 *with that for other modified chi-squared tests and with the well-known Anderson–Darling* A^2, *Cramer-von Mises* W^2, *and Mardia's (1970) tests. The results reveal two advantages of* S_n^2 *statistic over abovementioned tests: while possessing a comparable power, the* S_n^2 *does not require simulation of critical values to define rejection region even for small samples, and removes requirement for defining the optimal number of grouping intervals.*

INTRODUCTION

Multivariate normality is required for correct statistical inferences in very many cases. It is well known that the joint normality does not follow from the normality of marginal univariate distributions (see, e.g., Kotz et al. (2000)). The literature on tests of fit for the multivariate normal family is not such an extensive as for the assessing the univariate normality, but the last twenty years have seen an increased activity in this regard. Mardia (1970), and Malkovich and Afifi (1973) derived several tests based on a generalizations of the univariate skewness and kurtosis statistics. For alternative generalizations see Balakrishnan et al. (2007), and Balakrishnan and Scarpa (2012). For other pertinent work in this regard, see Shapiro and Wilk (1965), Royston (1983), Srivastava and Hui (1987), Tserenbat (1990), Looney (1995), Henze and Wagner (1997), Henze (2002), Doornik and Hansen (2008).
More elaborate lists of references can be obtained from the review articles of Henze (2002) and Mecklin and Mundfrom (2004).

MODIFIED CHI-SQUARED TESTS

Moore and Stubblebine (1981) developed a test for multivariate normality as follows. Let $\mathbf{X}_1,...,\mathbf{X}_n$ be independent identically distributed p-variate ($p \geq 2$) normal random vectors with the following joint probability density function

$$f\left(\mathbf{x}\,|\,\boldsymbol{\theta}\right)=\left(2\pi\right)^{-p/2}|\,\boldsymbol{\Sigma}\,|^{-1/2}\exp\left[-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^{T}\boldsymbol{\Sigma}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)\right],$$

where $\boldsymbol{\mu}$ is a p -vector of means and $\boldsymbol{\Sigma}$ is a non-singular $p\times p$ matrix. Let a given vector of unknown parameters be

$\boldsymbol{\theta}=\left(\mu_1,...,\mu_p,\sigma_{11},\sigma_{12},\sigma_{22},...,\sigma_{1j},\sigma_{2j},...,\sigma_{jj},...,\sigma_{pp}\right)^T$, where the elements of the matrix $\boldsymbol{\Sigma}$ are arranged column-wise by taking the elements of the upper-triangular sub matrix of $\boldsymbol{\Sigma}$. Given $0=c_0<c_1<...<c_M=\infty$, the M random grouping cells can be defined as (Moore and Stubblebine, 1981)

$$E_{in}\left(\hat{\boldsymbol{\theta}}_n\right)=\left\{\mathbf{X}\text{ in R}^p:\ c_{i-1}\leq\left(\mathbf{X}-\bar{\mathbf{X}}\right)^T\mathbf{S}^{-1}\left(\mathbf{X}-\bar{\mathbf{X}}\right)<c_i\right\},\ i=1,...,M,$$

where $\bar{\mathbf{X}}$ and $\mathbf{S}$ are maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ correspondingly. The estimated probability to fall into each cell is

$$p_{in}\left(\hat{\boldsymbol{\theta}}_n\right) = \int_{E_{in}\left(\hat{\boldsymbol{\theta}}_n\right)} f(\mathbf{x} \mid \hat{\boldsymbol{\theta}}_n) d\mathbf{x}.$$

If c_i is the i/M point of the central chi-squared distribution with p degrees of freedom, χ_p^2, then the cells are equiprobable under the estimated parameter value, $\hat{\boldsymbol{\theta}}_n$, and $p_{in}\left(\hat{\boldsymbol{\theta}}_n\right) = 1/M$, $i = 1,...,M$. Denote $\mathbf{V}_n$ a vector of standardized cell frequencies with the components

$$V_{in}\left(\hat{\boldsymbol{\theta}}_n\right) = \frac{\left(N_{in} - n/M\right)}{\sqrt{n/M}}, \quad i = 1,...,M,$$

where N_{in} is the number of random vectors $\mathbf{X}_1,...,\mathbf{X}_n$ falling in $E_{in}\left(\hat{\boldsymbol{\theta}}_n\right)$. The Fisher information matrix $\mathbf{J}\left(\boldsymbol{\theta}\right)$ for one observation can be evaluated as (Moore and Stubblebine, 1981)

$$\mathbf{J}\left(\boldsymbol{\theta}\right) = \begin{bmatrix} \boldsymbol{\Sigma}^{-1} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{Q}^{-1} \end{bmatrix}.$$

The dimension of $\mathbf{J}\left(\boldsymbol{\theta}\right)$ is $m \times m$, where $m = p + p(p+1)/2$ is the dimension of $\boldsymbol{\theta}$. $\mathbf{Q}$ is the $p(p+1)/2 \times p(p+1)/2$ covariance matrix of $\mathbf{r} = \left(s_{11}, s_{12}, s_{22}, s_{13}, s_{23}, s_{33}, ..., s_{pp}\right)^T$, a vector of the entries of $\sqrt{n}\mathbf{S}$ (arranged column-wise by taking the upper triangular elements).

The elements of **Q** can be written as (Press, 1972)

$$\mathbf{Var}\left(s_{ij}\right) = \sigma_{ij}^2 + \sigma_{ii}\sigma_{jj}, \quad i, j = 1,...,p, \; i \le j,$$

$$\mathbf{Cov}\left(s_{ij}, s_{kl}\right) = \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}, \quad i, j, k, l = 1,...,p, \; i \le j, k \le l,$$

where σ_{ij}, $i, j = 1,..., p,$ are elements of $\boldsymbol{\Sigma}$. For example, in the two-dimensional case the matrix $\mathbf{Q}$ will be (see also McCulloch, 1980)

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Var}(s_{11}) & \mathbf{Cov}(s_{11}, s_{12}) & \mathbf{Cov}(s_{11}, s_{22}) \\ \mathbf{Cov}(s_{12}, s_{11}) & \mathbf{Var}(s_{12}) & \mathbf{Cov}(s_{12}, s_{22}) \\ \mathbf{Cov}(s_{22}, s_{11}) & \mathbf{Cov}(s_{22}, s_{12}) & \mathbf{Var}(s_{22}) \end{pmatrix} = \begin{pmatrix} 2\sigma_{11}^2 & 2\sigma_{11}\sigma_{12} & 2\sigma_{12}^2 \\ 2\sigma_{11}\sigma_{12} & \sigma_{11}\sigma_{22} + \sigma_{12}^2 & 2\sigma_{12}\sigma_{22} \\ 2\sigma_{12}^2 & 2\sigma_{12}\sigma_{22} & 2\sigma_{22}^2 \end{pmatrix}.$$

Following Moore and Stubblebine (1981) for a specified $\boldsymbol{\theta}_0$ define

$$p_i(\boldsymbol{\theta}, \boldsymbol{\theta}_0) = \int_{E_i(\boldsymbol{\theta}_0)} f(\mathbf{x} \,|\, \boldsymbol{\theta}) d\mathbf{x},$$

where

$$E_i(\boldsymbol{\theta}_0) = \left\{ \mathbf{X} \text{ in } \mathrm{R}^p : \; c_{i-1} \le (\mathbf{X} - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1} (\mathbf{X} - \boldsymbol{\mu}_0) < c_i \right\}, \; i = 1, \ldots, M.$$

Define $M \times m$ matrix $\mathbf{B}(\boldsymbol{\theta}, \boldsymbol{\theta}_0)$ with elements

$$B_{ij} = \frac{1}{\sqrt{p_i(\boldsymbol{\theta}, \boldsymbol{\theta}_0)}} \frac{\partial p_i(\boldsymbol{\theta}, \boldsymbol{\theta}_0)}{\partial \theta_j}.$$

From Lemma 1 of Moore and Stubblebine (1981) it follows that for any c_i and $\boldsymbol{\theta}_0$

$$\left. \frac{\partial p_i(\boldsymbol{\theta}, \boldsymbol{\theta}_0)}{\partial \mu_j} \right|_{\boldsymbol{\theta} = \boldsymbol{\theta}_0} = 0, \;\; 1 \le i \le M, \; 1 \le j \le p,$$

$$\left. \frac{\partial p_i(\boldsymbol{\theta}, \boldsymbol{\theta}_0)}{\partial \sigma_{jk}} \right|_{\boldsymbol{\theta} = \boldsymbol{\theta}_0} = d_i \sigma^{jk}, \;\; 1 \le i \le M, \; 1 \le j \le k \le p,$$

where σ^{jk} are the elements of $\boldsymbol{\Sigma}_0^{-1}$ and

$$d_i = \left(c_{i-1}^{p/2} e^{-c_{i-1}/2} - c_i^{p/2} e^{-c_i/2} \right) b_p / 2,$$

$$b_p = \left[p(p-2) \ldots 4 \cdot 2 \right]^{-1} \;\; \text{if } p \text{ is even},$$

$$b_p = (2/\pi)^{1/2} \left[p(p-2) \ldots 5 \cdot 3 \right]^{-1} \;\; \text{if } p \text{ is odd}.$$

The Nikulin-Rao-Robson (NRR) statistic based on the MLEs can be presented as (Nikulin 1973; Rao and Robson, 1974; Voinov and Nikulin, 2011)

$$Y_n^2 = \mathbf{V}_n^T\left(\hat{\boldsymbol{\theta}}_n\right)\mathbf{V}_n\left(\hat{\boldsymbol{\theta}}_n\right) + \mathbf{V}_n^T\left(\hat{\boldsymbol{\theta}}_n\right)\mathbf{B}_n\left[\mathbf{J}_n - \mathbf{B}_n^T\mathbf{B}_n\right]^{-1}\mathbf{B}_n^T\mathbf{V}_n\left(\hat{\boldsymbol{\theta}}_n\right),$$

(1)

where $\mathbf{J}_n = \mathbf{J}(\hat{\boldsymbol{\theta}}_n)$ and $\mathbf{B}_n = \mathbf{B}(\hat{\boldsymbol{\theta}}_n, \hat{\boldsymbol{\theta}}_n)$.

Unfortunately, in this case, the limiting covariance matrix $\mathbf{I} - \mathbf{q}\mathbf{q}^T - \mathbf{B}\mathbf{J}^{-1}\mathbf{B}^T$ of the standardized frequencies $\mathbf{V}_\mathbf{n}$, where $\mathbf{q}$ is a M-vector with its entries as $1/\sqrt{M}$, depends on the unknown parameters and, hence, in accordance with Lemma 9 of Khatri (1968), Y_n^2 in (1) can not be distributed as chi-square. Moreover, the statistic Y_n^2 is not distribution-free and so can not be used for hypothesis testing, in principle.

TESTING FOR BIVARIATE CIRCULAR NORMALITY

Kowalski (1970) possibly was the first who considered “some rough tests for bivariate normality”. Gumbel (1954) considered applications of the circular normal distribution in “economic statistics”, geophysics and medical investigations.

Following Moore and Stubblebine (1981) consider testing for bivariate circular normality. The hypothesized probability density function is

$$f\left(x, y \mid \boldsymbol{\theta}\right) = \left(2\pi\sigma^2\right)^{-1}\exp\left\{-\frac{1}{2\sigma^2}\left[\left(x-\mu_1\right)^2 + \left(y-\mu_2\right)^2\right]\right\},$$

(2)

where $\boldsymbol{\theta} = \left(\mu_1, \mu_2, \sigma^2\right)^T$. Using a random sample $\left(X_1, Y_1\right), ..., \left(X_n, Y_n\right)$, the MLEs of $\mu_1, \mu_2,$ and σ^2, can be obtained as $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$, $\bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j$, and

$$s^2 = \frac{1}{2n}\left\{\sum_{j=1}^{n}(X_j - \bar{X})^2 + \sum_{j=1}^{n}(Y_j - \bar{Y})^2\right\}.$$ If

$c_i = -2\log(1 - i/M),\ i = 1,\ ...,\ M-1,\ c_M = +\infty$, then $\hat{p}_{in} = 1/M$. It can be shown that (Moore and Stubblebine, 1981)

$$\left.\frac{\partial p_{in}}{\partial \mu_1}\right|_{\hat{\theta}_n} = \left.\frac{\partial p_{in}}{\partial \mu_2}\right|_{\hat{\theta}_n} = 0,$$

$$\left.\frac{\partial p_{in}}{\partial \sigma}\right|_{\hat{\theta}_n} = v_i / s,$$

where $\nu_i = 2\left[\left(1-\frac{i}{M}\right)\log\left(1-\frac{i}{M}\right)-\left(1-\frac{i-1}{M}\right)\log\left(1-\frac{i-1}{M}\right)\right]$. From this it follows

that the matrix $\mathbf{B}_n$ is

$$\mathbf{B}_n = \begin{pmatrix} 0 & 0 & \sqrt{M}\nu_1 / s \\ 0 & 0 & \sqrt{M}\nu_2 / s \\ \cdots & \cdots & \cdots \\ 0 & 0 & \sqrt{M}\nu_M / s \end{pmatrix}.$$

(3)

Since the estimate $\mathbf{J}_n$ of the Fisher information matrix for the family (2) is (Moore and Stubblebine, 1981)

$$\mathbf{J}_n = \begin{pmatrix} 1/s^2 & 0 & 0 \\ 0 & 1/s^2 & 0 \\ 0 & 0 & 4/s^2 \end{pmatrix},$$

(4)

then

$$(\mathbf{J}_n - \mathbf{B}_n^T\mathbf{B}_n)^{-1} = \begin{pmatrix} s^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & \dfrac{s^2}{4 - M\sum \nu_i^2} \end{pmatrix}.$$

(5)

Denoting $\mathbf{V}_n = \left(\frac{(N_{1n} - n/M)}{\sqrt{n/M}},\ ...,\ \frac{(N_{Mn} - n/M)}{\sqrt{n/M}}\right)^T = \left(\tilde{N}_1,\ ...,\tilde{N}_M\right)^T$, where

$N_{jn},\ j = 1,...,M,$ is the number of $d_i = \frac{1}{s^2}\left[\left(X_i - \bar{X}\right)^2 + \left(Y_i - \bar{Y}\right)^2\right],\ i = 1,\ ...,n,$ that

fall into the interval $[c_{j-1}, c_j),\ j = 1,\ ..., M$, the statistic (1) is easily derived as

$$Y_n^2 = \sum_{i=1}^{M} \tilde{N}_i^2 + \frac{M}{4 - M\sum_{i=1}^{M} \nu_i^2} \left(\sum_{i=1}^{M} \tilde{N}_i \nu_i \right)^2 .$$

(6)

Note that formula for R_n (denoted as Y_n^2 in the current paper) given by Moore and Stubblebine (1981), p.724, is not correct. The correct formula is

$$R_n = \sum_{i=1}^{M} \tilde{N}_i^2 + \frac{M^2}{n\left(4 - M\sum_{i=1}^{M} \nu_i^2\right)} \left(\sum_{i=1}^{M} N_{in} \nu_i \right)^2 .$$

The second term of Y_n^2 in (6) recovers information lost due to data grouping. Another useful decomposition $Y_n^2 = U_n^2 + S_n^2$ of Y_n^2 has been proposed by McCulloch (1985). In $Y_n^2 = U_n^2 + S_n^2$ the first term U_n^2 is the Dzhaparidze-Nikulin (DN) statistic (Dzhaparidze and Nikulin, 1974)

$$U_n^2 = \mathbf{V}_n^T \left(\hat{\boldsymbol{\theta}}_n\right) \left[\mathbf{I} - \mathbf{B}_n (\mathbf{B}_n^T \mathbf{B}_n)^{-1} \mathbf{B}_n^T \right] \mathbf{V}_n \left(\hat{\boldsymbol{\theta}}_n\right),$$

(7)

and the second term is

$$S_n^2 = Y_n^2 - U_n^2 = \mathbf{V}_n^T \left(\hat{\boldsymbol{\theta}}_n\right) \mathbf{B}_n \left[(\mathbf{J}_n - \mathbf{B}_n^T \mathbf{B}_n)^{-1} + (\mathbf{B}_n^T \mathbf{B}_n)^{-1} \right] \mathbf{B}_n^T \mathbf{V}_n \left(\hat{\boldsymbol{\theta}}_n\right).$$

(8)

McCulloch (1985), theorem 4.2, showed that if rank of $\mathbf{B}_n$ is s, then U_n^2 and S_n^2 are asymptotically independent and distributed in the limit as χ^2_{M-s-1} and χ^2_s respectively. Since the first two columns of the matrix $\mathbf{B}_n$ in our case are columns of zeros and the rest are linearly dependent, the matrix $\mathbf{B}_n$ has rank 1. From the above it follows that $(\mathbf{B}_n^T \mathbf{B}_n)^{-1}$ does not exist, but, using well known facts from the theory of multivariate normal distribution (Moore, 1977, p.132), we may replace $\mathbf{A}^{-1} = (\mathbf{B}_n^T \mathbf{B}_n)^{-1}$ by $\mathbf{A}^{-} = (\mathbf{B}_n^T \mathbf{B}_n)^{-}$, where

$\mathbf{A}^{-}$ is any generalized matrix inverse of $\mathbf{A}$. It can be computed, e.g., by using singular value decomposition. So, for testing two-dimensional circular normality with random cells $E_{in}\left(\hat{\boldsymbol{\theta}}_n\right)$ we may use: the NRR statistic Y_n^2 defined by (1), the DN statistic

$$U_n^2 = \mathbf{V}_n^T\left(\hat{\boldsymbol{\theta}}_n\right)\left[\mathbf{I} - \mathbf{B}_n(\mathbf{B}_n^T\mathbf{B}_n)^{-}\mathbf{B}_n^T\right]\mathbf{V}_n\left(\hat{\boldsymbol{\theta}}_n\right),$$

(9)

where $\mathbf{I}$ is the $M \times M$ identity matrix, and

$$S_n^2 = Y_n^2 - U_n^2 = \mathbf{V}_n^T\left(\hat{\boldsymbol{\theta}}_n\right)\mathbf{B}_n\left[(\mathbf{J}_n - \mathbf{B}_n^T\mathbf{B}_n)^{-1} + (\mathbf{B}_n^T\mathbf{B}_n)^{-}\right]\mathbf{B}_n^T\mathbf{V}_n\left(\hat{\boldsymbol{\theta}}_n\right)$$

(10)

which have asymptotically χ^2_{M-1} , χ^2_{M-2} and χ^2_1 distributions respectively.

From (3) it follows that

$$\mathbf{B}_n^T\mathbf{B}_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix},$$

(11)

where $a = \frac{M}{s^2}\sum_{i=1}^{M} v_i^2$. From the singular value decomposition it follows that

$$\left(\mathbf{B}_n^T\mathbf{B}_n\right)^{-} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/a \end{pmatrix}.$$

(12)

After simple matrix algebra one gets:

$$U_n^2 = \sum_{i=1}^{M} \tilde{N}_i^2 - \frac{\left(\sum_{i=1}^{M} \tilde{N}_i v_i\right)^2}{\sum_{i=1}^{M} v_i^2},$$

(13)

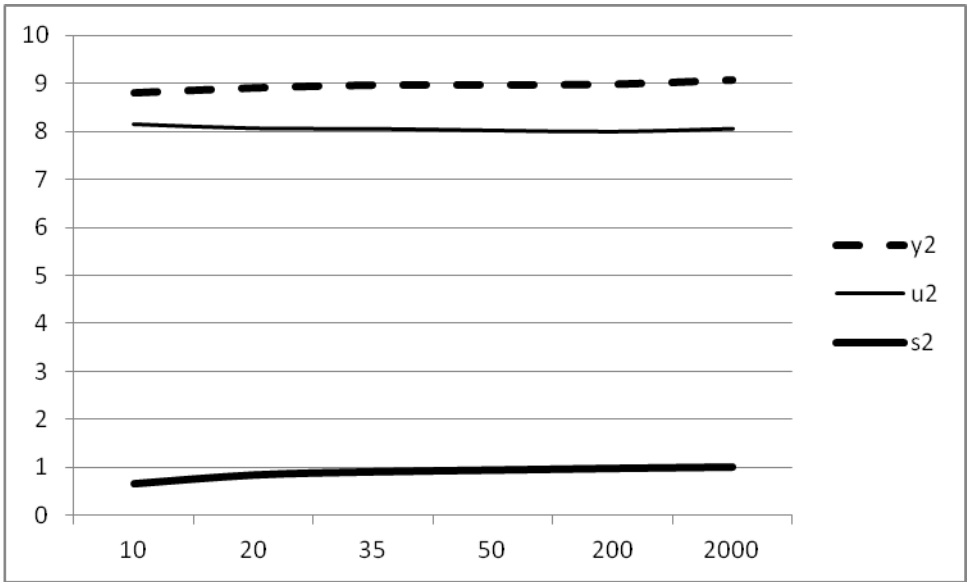
and

$$S_n^2 = \frac{4}{\left(4 - M\sum_{i=1}^{M} v_i^2\right)\sum_{i=1}^{M} v_i^2}\left(\sum_{i=1}^{M} \tilde{N}_i v_i\right)^2 .$$

(14)

It is well known (see, e.g., Greenwood and Nikulin, 1996) that chi-squared type statistics attain their corresponding limit distributions if expected cell counts are more than or equal 5. To check this for our case we simulated expected values of statistics (6), (13), and (14) for expected cell counts from 1 to 20 (M = 10) (see Fig 1).

FIGURE 1: AVERAGES OF *Y*2 (10), *U*2 (13), AND *S*2 (14) AS FUNCTIONS OF *n*



From this figure we see that if n is more than or equal 35 (expected cell count equals 3.5) then the expected values of tests (6), (13), and (14) approximately equal to 9, 8, and 1, as it should be for M=10 if the theory is valid.

To investigate power of tests (6), (13), and (14) consider first their simulated distributions under the null hypothesis (2). Our simulation shows that probabilities to fall into rejection region of the level $\alpha = 0.05$ differ from Type 1 error of the same level no more than on one simulated standard deviation (tested for samples of size n = 200 and for the number M of grouping intervals such that expected cell frequencies are more than or equal to 5).

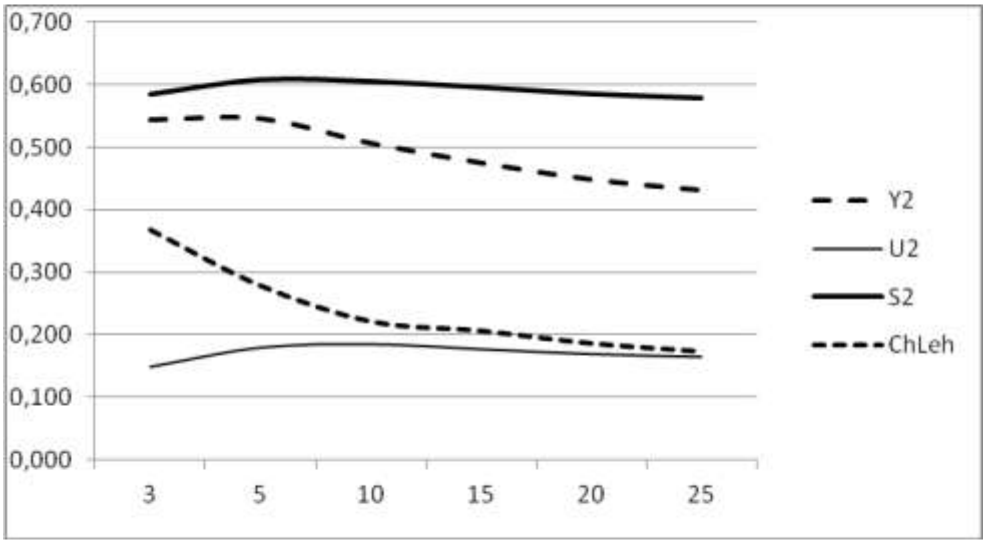
The same is true for $n = 100$ if $M \leq 10$. This permitted us, when simulating power under different alternatives, to define rejection regions using critical values of corresponding chi-squared distributions.

Let the alternative be the two-dimensional standard logistic distribution with independent components distributed as

$$l(x) = \frac{\frac{\pi}{\sqrt{3}} \exp\left(-\frac{\pi x}{\sqrt{3}}\right)}{\left[1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right]^2}, \quad x \in R^1 .$$

Simulated powers of tests (6), (13), (14), and $ChLeh = \sum_{i=1}^{M} \tilde{N}_i^2$ (see Chernoff and Lehmann, 1954; Moore and Stubblebine,1981, p.721) are shown on Figure 2.

FIGURE 2: SIMULATED POWERS OF TESTS (10), (13), (14), AND $ChLeh = \sum_{i=1}^{M} \tilde{N}_i^2$ AS FUNCTIONS OF *M*



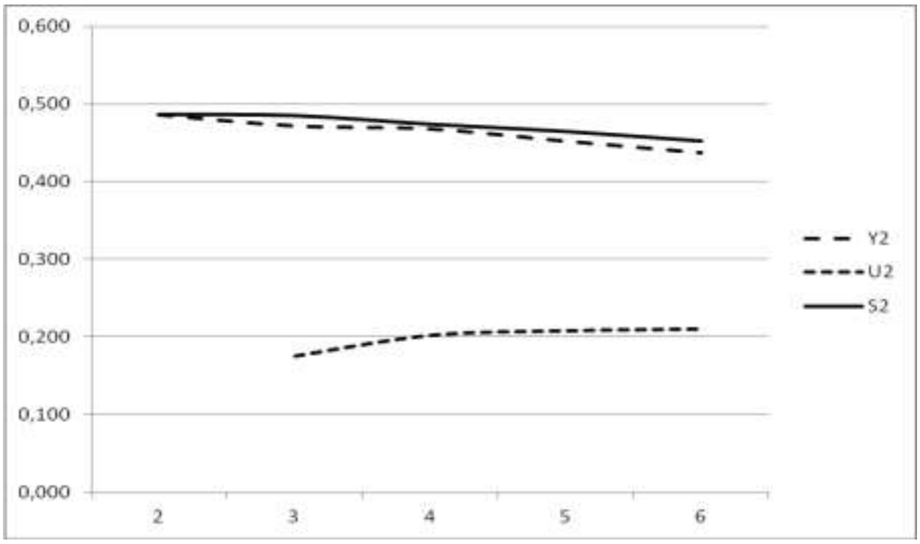
What do we see from this figure? First of all we see that power of S_n^2 is the highest one and, what is of importance, almost does not depend on the number *M* of grouping intervals. Secondly, because of the decomposition $Y_n^2 = U_n^2 + S_n^2$, power of Y_n^2 is more than that of U_n^2 and less than power of S_n^2. The same situation was observed in the univariate case (see Voinov et al, 2009, 2012). Concerning the Chernoff and Lehmann

(1954), and Dzhaparidze and Nikulin (1974) statistics, *ChLeh* and DN, the following can be said. Since the famous result of Chernoff and Lehmann (1954) we know that the Pearson-Fisher statistic $\sum_{i=1}^{M}\tilde{N}_i^2$ (we denoted it in this paper as *ChLeh*) does not follow in the limit the χ^2_{M-1} distribution and may depend on unknown parameters. Roy (1956), Watson (1958), and Dahiya and Gurland (1972) showed that for a location and scale family, if one will use random intervals, the statistic $\sum_{i=1}^{M}\tilde{N}_i^2$ will be distribution free following in the limit the chi-squared distribution, but the number of degrees of freedom will depend on the null hypothesis. This is exactly the case considered by Moore and Stubblebine (1981) who proved that under two-dimensional circular normal distribution the limit distribution of $\sum_{i=1}^{M}\tilde{N}_i^2$ does not depend on parameters and follows in the limit the chi-squared distribution with the number of degrees of freedom that falls between *M*-2 and *M*-1. From Figure 2 we see that power of *ChLeh* is indeed between powers of Y_n^2 (*M*-1 d.f.) and U_n^2 (*M*-2 d.f.). McCulloch (1985) and Mirvaliev (2001) showed that "the DN statistic U_n^2 behaves locally like the Pearson-Fisher statistic (*ChLeh*)". In the univariate case we did not see the essential difference between powers of the DN and Pearson-Fisher's $\sum_{i=1}^{M}\tilde{N}_i^2$ (Voinov et al, 2009), but in our two-dimensional case the power of *ChLeh* is noticeably higher than that of DN statistic. Still power of those statistics is much lower than that of S_n^2. This suggests to use S_n^2, if one intends to test for the two-dimensional circular normal distribution w.r.t. two-dimensional logistic distribution with independent components using chi-squared type tests.

As another alternative consider the two-dimensional normal distribution with correlated components. Let the alternative hypothesis be such that $\mu = (0,0)^T$ and

$$\Sigma_{01} = \begin{pmatrix} 2 & 1.4 \\ 1.4 & 2.5 \end{pmatrix}.$$

FIGURE 3: POWERS OF *Y*2 (10), *U*2 (13), AND *S*2 (14) AS FUNCTIONS OF *M*



From this Figure we see that powers of S_n^2 and Y_n^2, being high enough, are almost the same (still power of S_n^2 is slightly higher than that of Y_n^2), and that power of U_n^2 is much lower as it was expected. We see also that powers of both S_n^2 and Y_n^2 are the highest and the same is true for M = 2. The same results are obtained if

$$\Sigma_{02} = \begin{pmatrix} 2 & -1.4 \\ -1.4 & 2.5 \end{pmatrix}.$$

A COMPARISON OF DIFFERENT TESTS

Let us test the null hypothesis (2) using the modified chi-squired test S_n^2 (14), the multivariate skewness test (Sk) $b_{1,2}$ of Mardia (1970), p.523, the kurtosis test (Kur) $b_{2,2}$ of Mardia (1970), p.525, the Anderson-Darling A^2, and Cramer-von Mises W^2 EDF-tests (see Henze, 2002, p.483). Considering the two-dimensional standard logistic distribution with independent components as an alternative, the following results (see Table 1) were obtained.

Table 1: POWERS OF S_n^2, Sk, Kur, A^2 AND W^2 FOR THE CIRCULAR TWO-DIMENSIONAL NORMAL NULL AGAINST TWO-DIMENSIONAL STANDARD LOGISTIC DISTRIBUTION WITH INDEPENDENT COMPONENTS. ERRORS ARE CONSIDERED TO BE ± ONE SIMULATED STANDARD DEVIATION

Test	Power
S_n^2	0.609 ± 0.005
Sk	0.357 ± 0.006
Kur	0.832 ± 0.005
A^2	0.668 ± 0.009
W^2	0.671 ± 0.010

From Table 1 we see that the Mardia's test $b_{2,2}$ is the most powerful for the particular null and alternative hypotheses. At the same time we have to note that the application of A^2 and W^2 is more difficult, because one has to define critical values of those tests by simulation.

Consider now power of different tests for the circular two-dimensional normal null distribution against two-dimensional alternative with correlated components. Let alternatives be two-dimensional normal distributions with the zero vector of means and covariance matrices $\mathbf{\Sigma}_{01}$ and $\mathbf{\Sigma}_{02}$. Results of our simulation study are summarized in Tables 2 and 3.

TABLE 2: POWERS OF S_n^2, Sk right, Sk left, Kur, A^2 AND W^2 FOR THE CIRCULAR TWO-DIMENSIONAL NORMAL NULL AGAINST TWO-DIMENSIONAL NORMAL DISTRIBUTION WITH THE COVARIANCE MATRIX $\mathbf{\Sigma}_{01}$. ERRORS ARE CONSIDERED TO BE $\pm$ ONE SIMULATED STANDARD DEVIATION

Test	Power
S_n^2	0.485 ± 0.008
Sk right	0.027 ± 0.002
Sk left	0.813 ± 0.005
Kur	0.182 ± 0.005
A^2	0.622 ± 0.013
W^2	0.622 ± 0.012

Table 3: POWERS OF S_n^2, Sk right, Sk left, Kur, A^2 AND W^2 FOR THE CIRCULAR TWO-DIMENSIONAL NORMAL NULL AGAINST TWO-DIMENSIONAL NORMAL DISTRIBUTION WITH THE COVARIANCE

MATRIX Σ_{02}. ERRORS ARE CONSIDERED TO BE $\pm$ ONE SIMULATED STANDARD DEVIATION

Test	Power
S_n^2	0.486 ± 0.008
Sk right	0.026 ± 0.002
Sk left	0.817 ± 0.008
Kur	0.180 ± 0.006
A^2	0.617 ± 0.010
W^2	0.618 ± 0.010

In those Tables by "Sk right" and "Sk left" we mean the skewness test $b_{1,2}$ of Mardia (1970) applied for right-tailed and left-tailed rejection regions correspondingly. From Tables 2 and 3 we see, first, that all tests considered make no difference between alternatives with positive (Σ_{01}) and negative (Σ_{02}) correlation. Secondly, tests "Sk right" are biased for right-tailed rejection region. This contradicts to the opinion of Mardia (1970), p.523. See also Henze (2002), p.474. We see that, if considered alternatives are fixed in advance, then power of "Sk left" will be the highest, but one has to know in advance that test $b_{1,2}$ is biased for the right-tailed rejection region. One sees also that power of "Kur" is noticeably lower than that of the chi-squared type test S_n^2, and that powers of A^2 and W^2 are higher but comparable with that of S_n^2.

CONCLUSION

Among all modified chi-squared tests for the two-dimensional circular normality considered in Section 3 the test S_n^2 possesses the large enough power that is more than that of NRR Y^2 and DN U^2 tests. The same results were observed in the univariate case (Voinov et al., 2009, 2012), but in the two-dimensional case of the circular null normality power of S_n^2 almost does not depend on the number of equiprobable grouping intervals. This is an evident advantage because in the univariate case it is not easy to define the optimal number of grouping intervals (see, e.g., Greenwood and Nikulin, 1996).

To characterize anyhow the dependence of tests' power on an alternative hypothesis consider ratios of powers of tests considered in Section 4. By Ratio we shall mean, e.g., the ratio of power of S_n^2 for logistic alternative to that for the two-dimensional normal distribution (we always divide the large value by a small one). Using Tables 1, 2 and 3, the following Table 4 can be constructed.

TABLE 4: RATIOS OF POWERS FOR S_n^2, Sk, Kur, A^2 and W^2

Test	Power
S_n^2	0.609/0.485=1.25
Sk	0.815/0.357=2.28
Kur	0.832/0.181=4.6
A^2	0.668/0.619=1.08
W^2	0.671/0.620=1.08

From Table 4 we see that the S_n^2, A^2, and W^2 tests are, in some sense, much less sensitive (Ratio = 1.08, and 1.25) to alternatives considered. We cannot conclude that S_n^2, A^2 and W^2 tests are the “omnibus" tests because many other alternatives have to be considered, but w.r.t. the logistic and two-dimensional normal alternatives they look more preferable than other tests, if, of course, those alternatives are considered to be unknown.

At this point we also would like to mention the following. The standard deviations of power estimates of S_n^2, Sk and Kur for the logistic alternatives are comparable, and those deviations for A^2 and W^2 are approximately twice larger. For the two-dimensional normal alternative with Σ_{02} the standard deviations of power estimates of S_n^2 are even 1.56 times less than those for A^2 and W^2.

As an overall conclusion we would like to say that S_n^2 test is quite comparable with EDF tests and even possesses some advantages. One does not need to simulate critical values to define rejection region, because for S_n^2 test these are critical values of the chi-squared probability distribution with 1 d.f. Another advantage is that there is no problem with defining the optimal number of grouping intervals. Concerning Mardia's skewness and kurtosis tests, we do not recommend them for applications because of biasedness of the tests based on multivariate skewness.

REFERENCES

Balakrishnan, N., Brito, M.R., Quiroz, A.J. (2007). "A vectorial notion of skewness and its use in testing for multivariate symmetry", *Communications is Statistics- Theory and Methods*, 36, 1757-1767.

Balakrishnan, N., Scarpa, B. (2012). "Multivariate measures of skewness for the skew-normal distribution", *Journal of Multivariate Analysis*, 104, 73-87.

Chernoff, H. and Lehmann, E.L. (1954). "The use of maximum likelihood estimates in χ^2 tests for goodness of fit", *Annals of Mathematical Statistics*, 25, 579-586.

Dahiya, R.C., Gurland, J. (1972). "Pearson chi-squared test of fit with random intervals", *Biometrica*, 59, 147-153.

Doornik, J.A., Hansen, H. (2008). "An omnibus test for univariate and multivariate normality", *Oxford Bulletin of Economics and Statistics*, 70, 927-939.

Dzhaparidze, K.O., Nikulin, M.S. (1974). "On a modification of the standard statistic of Pearson", *Theory of Probability and Applications*, 19, 851-853.

Greenwood, P. E., Nikulin, M.S. (1996). *A Guide to Chi-squared Testing*, New York: John Wiley and Sons.

Gumbel, E.J. (1954). Applications of the circular normal distribution. Journal of the American Statistical Association, 49, 267-297.

Henze, N. (2002). "Invariant tests for multivariate normality: a critical review", *Statistical Papers*, 43, 467-506.

Henze, N.,and Wagner, T. (1997). "A new approach to the BHEP tests for multivariate Normality", *Journal of multivariate analysis*, 62, 1-23.

Khatri, C.G. (1968). "Some results for the singular normal multivariate regression", *Sankhyā, Series A*, 30, 267-280.

Kotz, S., Balakrishnan, N., and Johnson, N.L. (2000). *Continious Multivariate Distributions-Vol.1*, Second edition, New York: John Wiley and Sons.

Kowalski, C.J. (1970). "The performance of some rough tests for bivariate normality before and after coordinate transformations to normality", *Technometrics*, 12, 517-544.

Looney, S.W. (1995). "How to use tests for univariate normality to assess multivariate

Normality”, *The American Statistician,* 49, 64-70.

Malkovich, J.F., Afifi, A.A. (1973). “On the test for multivariate normality”, *Journal of the American Statistical Association,* 68, 176-179.

Mardia, K.V. (1970). “Measures of multivariate skewness and kurtosis with applications”, *Biometrika,* 57, 519-530.

McCulloch, C.E. (1980). “Symmetric matrix derivatives with applications”, *FSU Statistics Report M565,* Department of Statistics, Florida State University.

McCulloch, C.E. (1985). “Relationships among some chi-square goodness of fit statistics”, *Communications is Statistics- Theory and Methods*, 14, 593-603.

Mecklin, C.G., and Mundfrom, D.J. (2004). “An appraisal and bibliography of tests for multivariate normality”, *International Statistical Reviews,* 72, 123-138.

Mirvaliev, M. (2001). *An investigation of generalized chi-squared type statistics,* Doctoral dissertation, Academy of Sciences of the Republic of Uzbekistan, Tashkent (unpublished).

Moore, D.S. (1971). “A chi-square statistic with random cell boundaries”, *Annals of Mathematical Statistics,* 42, 147-156.

Moore, D., Stubblebine, J.B. (1981)). “Chi-square tests for multivariate normality with application to common stock prices”, *Communications is Statistics- Theory and Methods,* A(10), 713-738.

Nikulin, M.S. (1973). “Chi-square test for continuous distributions”, *Theory of Probability and its Applications*, 18, 638-639.

Press, S.J. (1972). *Applied Multivariate Analysis,* New York: Holt, Rinehart and Winston.

Rao, K.C., and Robson, D.S. (1974). “A chi-squared statistic for goodness-of-fit tests within the exponential family”, *Communications in Statistics*, 3, 1139-1153.

Roy, A.R. (1956). “On χ^2 statistics with variable intervals”, *Technical Report N1,* Stanford University, Statistics Department.

Royston, J.P. (1983). “Some techniques for assessing multivariate normality based on the Shapiro-Wilk W”, *Applied Statistics,* 32, 121-133.

Shapiro, S.S., Wilk, M.B. (1965). “An analysis of variance test for normality”, *Biometrika,* 52, 591- 611.

Srivastava, M.S., and Hui T.K. (1987). “On assessing multivariate normality based on Shapiro-Wilk W statistic”, *Statistics and Probability Letters*, 5, 15-18.

Tserenbat, O. (1990). “On one modified chi-squared statistic for multivariate normality testing”, *Izvestiya. Academy of Science of Uzbekistan, Physical and Mathematical Series*, 4, 40-47.

Voinov, V., Nikulin, M. (2011). “Chi-Square Goodness-of-fit Tests: Drawbacks and Improvements”, In: *International Encyclopedia of Statistical Science*, Ed: Lovric Miodrag, Springer, pp. 246-250.

Voinov, V., Pya, N., Alloyarova, R. (2009). “A comparative study of some modified chi-squared tests”, *Communications in Statistics-Simulation and Computation*, 38, 355-367.

Voinov, V., Pya, N., Shapakov, N., Voinov, Y. (2012). “Goodness of fit tests for the Power Generalized Weibull probability distribution”, *Communications in Statistics-Simulation and Computation*, (to appear).

Watson, G.S. (1958). “On chi-square goodness-of-fit tests for continuous distributions”, *Journal of the Royal Statistical Society*, B20, 44-61.