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APPENDIX

MATHEMATICAL REVISION OF WEIGHTED ℓ_1 MINIMIZATION

This section is to rewrite weighted ℓ_1-minimization problem (WP1) in compressive sampling recovery. We consider a $N \times 1$ signal vector which is sparse. And also, the closest linear system of equations for searching the minimum cost of non-zero coefficients is in the ℓ_1-norm with its constraints as,

$$(\mathrm{WP}_1) \qquad \min \|Ws\|_1 \text{ sub. to } \Theta s = y .$$

There are various optimization techniques to solve this system as well as a convenient simplex linear programming. Firstly, we write this problem as a linear programming problem by replacing the objective function by $\|Ws\| = w_1|s_1| + \cdots + w_N|s_N|$, where W is the diagonal matrix with positive weights $w_1, \ldots, w_N$ as below,

$$\min \; w_1|s_1| + \cdots + w_N|s_N| \quad \text{sub. to } \; \Theta s = y . \tag{17}$$

Since the objective function of this optimization is not linear, its nonlinearities can be transformed into the set of constraints by adding the new variables $t_1, \ldots, t_N$,

$$\begin{array}{ll} \min & w_1 t_1 + \cdots + w_N t_N \\ \text{sub. to} & |s_1| \le t_1 \\ & \vdots \\ & |s_N| \le t_N \\ & \Theta s = y \end{array} . \tag{18}$$

Due to $|x_i| \le t_i$, this means that $-t_i \le x_i \le t_i$. Then, we can transform this problem to linear programming by adding the following n inequalities,

$$\begin{array}{ll} \min & w_1 t_1 + \cdots + w_N t_N \\ \text{sub. to} & s_1 \le t_1 \\ & s_1 \ge -t_1 \\ & \vdots \\ & s_N \le t_N \\ & s_N \ge -t_N \\ & \Theta s = y \end{array} . \tag{19}$$

After that, replace $s_i \le t_i$ by $I_N s \le I_N t$ and also $I_N x - I_N t \le 0$. Likewise, $s_i \ge -t_i$ implies $I_N x + I_N t \ge 0$,

$$\begin{array}{ll} \min & Wt \\ \text{sub. to} & I_N x - I_N t \le 0 \\ & I_N x + I_N t \ge 0 \\ & \Theta s = y \end{array} , \tag{20}$$

where I_N is an $N \times N$ identity matrix. Finally, the objective function and constraints are linear as the linear programming form which can be solved via simplex algorithm.

Furthermore, there is an available method to reduce size of this problem by examining the *dual problem* which can reform a standard linear program into the equivalent dual linear program,

$$\min \ y^T b \quad \text{sub. to} \ \Theta^T b + z = c, \tag{21}$$

We include the terms of dual vectors b, u, and v which correspond to the constraints from the primal problem with on restrictions on s or t. The form of optimization (20) can be rewritten as,

$$\begin{aligned}
&\min && [y^T \quad \overline{0} \quad \overline{0}]\begin{bmatrix} b \\ u \\ v \end{bmatrix} \\
&\text{sub. to} && \begin{bmatrix} \Theta^T & I_N & I_N \\ \overline{0}^T & -I_N & I_N \end{bmatrix}\begin{bmatrix} b \\ u \\ v \end{bmatrix} = \begin{bmatrix} \overline{0} \\ w \end{bmatrix}, \\
& && b \text{ unrestricted} \\
& && u \leq 0 \\
& && v \geq 0
\end{aligned} \tag{22}$$

where $\overline{0}^T$ is an zero vector or matrix and weighting vector $w = [w_1 \cdots w_N]^T$. We reform the equation (22) as,

$$\begin{aligned}
&\min && y^T b \\
&\text{sub. to} && \Theta^T b + I_N u + I_N v = 0 \\
& && -I_N u + I_N v = w \\
& && u \leq 0 \\
& && v \geq 0
\end{aligned} \quad . \tag{23}$$

Note that $u \leq 0$ implies $-u \geq 0$ .We replace u with $-u$,

$$\begin{aligned}
&\min && y^T b \\
&\text{sub. to} && \Theta^T b - u + v = 0 \\
& && u + v = w \\
& && u \geq 0 \\
& && v \geq 0 \\
& && w \geq 0
\end{aligned} \quad . \tag{24}$$

Then, $u + v = w$ implies $v = w - u$ so that $w - u$ is the substitute for v,

$$\begin{aligned}
&\min && y^T b \\
&\text{sub. to} && \Theta^T b - u + (w - u) = 0 \\
& && u \geq 0 \\
& && v \geq 0 \\
& && w - u \geq 0
\end{aligned} \quad . \tag{25}$$

Finally, we have the equivalent dual problem,

$$\min\ y^T b \quad \text{sub. to } \Theta^T b - 2u = -w, 0 \le u \le w. \tag{26}$$

Now, this is an achievable representation of weighted ℓ_1-minimization which is changed from the original problem with more constraints than variables (since $M < N$), into a problem with more variables than constraints. Also, this form can be conveniently solved by simplex linear programming algorithm.

RESEARCH PUBLICATIONS

1. **Kasidit Charunphaisan**, and Anupap Meesomboon. Automatic Adaptive Reweighted ℓ_1-Minimization for Compressive Sampling Recovery. The 14^{th} National Graduate Research Conference, King Mongkut's University of Technology North Bangkok, Thailand. September 10-11, 2009.
2. **Kasidit Charunphaisan**, and Anupap Meesomboon. Hard Selective Reweighted ℓ_1-Minimization for Compressive Sampling Recovery. 32^{nd} Electrical Engineering Conference, Tawaravadee Resort Hotel, Prachinburi, Thailand. October 28-30, 2009.
3. **Kasidit Charunphaisan**, and Anupap Meesomboon. New Reweighted ℓ_1-minimization Algorithms for Compressive Sampling Recovery. KKU Research Journal (Graduate Study) Vol. 10 No. 3 (in press) July-September 2010.

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การหาค่าต่ำสุดของนอร์มหนึ่ง โดยการหน่วงน้ำหนักแบบปรับค่าอัตโนมัติ สำหรับการสร้างสัญญาณย้อนกลับของคอมเพรสซีฟแซมปลิ่ง

Automatic Adaptive Reweighted ℓ_1-Minimization for Compressive Sampling Recovery

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บทคัดย่อ

ระบบที่มีจำนวนสมการน้อยกว่าตัวแปรเป็นปัญหาที่น่าสนใจ เนื่องจากมีคำตอบจำนวนมาก ที่สามารถเป็นคำตอบของระบบนี้ คอมเพรสซีฟแซมปลิ่ง (compressive sampling) เป็นแนวทางใหม่ที่ยืนยันความเป็นไปได้ในการหาคำตอบที่ถูกต้อง โดยการหาค่าต่ำสุดของนอร์มหนึ่ง (ℓ_1-minimization) จนถึงปัจจุบัน วิธีการหาค่าต่ำสุดของนอร์มหนึ่ง ได้ถูกพัฒนาไปเป็นระเบียบวิธีการคำนวณซ้ำและมีการถ่วงน้ำหนัก (iterative reweighted algorithm) ซึ่งคำตอบที่ได้มีความถูกต้องมากขึ้น แต่อย่างไรก็ตาม การเลือกค่าของการถ่วงน้ำหนักยังคงเป็นประเด็นที่ไม่ชัดเจน จากการทดสอบ ถ้าพิจารณาสัญญาณที่มีความเบาบาง (sparse signal) ซึ่งมีจำนวนของสมาชิกที่มีค่าเป็นศูนย์อยู่จำนวนมาก พบว่า กรณีที่หาคำตอบโดยการหาต่ำสุดของนอร์มหนึ่งแล้วได้คำตอบที่ไม่ถูกต้องนั้น มีความเป็นไปได้ว่า สมาชิกของคำตอบที่มีค่าเป็นศูนย์จะถูกเคลื่อนย้ายไปยังตำแหน่งอื่น ๆ ดังนั้น ระเบียบวิธีใหม่ที่นำมาเสนอ จึงถูกออกแบบมาเพื่อจัดการกับคำตอบที่ไม่ถูกต้อง โดยการเพิ่มค่าถ่วงน้ำหนักให้สูงขึ้น สำหรับสมาชิกของคำตอบที่มีค่าใกล้ศูนย์ และคำนวณกระบวนการหาค่าต่ำสุดของนอร์มหนึ่งซ้ำ ๆ จนกระทั่งได้คำตอบที่ถูกต้อง ผลจากการทดลองพบว่า วิธีที่นำเสนอสามารถหาคำตอบที่ถูกต้อง ดีกว่าการหาค่าต่ำสุดของนอร์มหนึ่งโดยประมาณร้อยละ 7.37

Abstract

An underdetermined system of linear equations which has fewer equations than unknowns is still an interesting problem because there are many possible solutions that ensure this system. Compressive sampling is a method that confirms the possibility to find the exact solutions via ℓ_1-minimization. Up to now this optimization has been developed into the iterative reweighted algorithm which presents the outstanding results, but there is not known the rule to select the weighting values. From our investigation, a considered signal is spare so that there is much zero information. In the case of poor reconstruction, the process of ℓ_1-minimization is possible to shift the solution entries from zero to the other values. Thus, a new proposed algorithm is designed to cope with the poor reconstructed solutions which are close to zero by multiplying the large weighting value and repeating ℓ_1-minimization until successive iterations. The numerical results show the percentage of exact reconstruction by the new proposed algorithm outperforming ℓ_1-minimization 7.37% approximately.

คำสำคัญ : คอมเพรสซีฟแซมปลิ่ง, คอมเพรสซีฟเซนซิ่ง, คอมเพรสเซนซิ่ง, ระบบสมการเส้นตรง, สัญญาณที่มีความเบาบาง, การหาจุดต่ำสุดของนอร์มหนึ่ง, การหาจุดต่ำสุดของนอร์มหนึ่งแบบถ่วงน้ำหนัก

Keywords : compressive sampling, compressive sensing, compressed sensing, system of linear equations, sparse signals, ℓ_1-minimization, reweighted ℓ_1-minimization

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Introduction

In engineering and scientific problems, an interesting system which has fewer equations than unknowns challenges the technique to recover exact solution. Consider a linear system $y = Ax$ where A is an matrix $M \times N$. This system can be sufficiently solved the solution via $x = A^{-1}y$ for which $M = N$, but in practical the matrix A has fewer rows than column; i.e. $M < N$ generating many solutions. In this case the true solution cannot be solved unless there are other conditions.

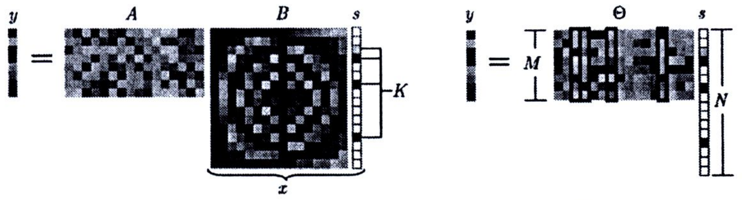
Compressive sampling is an alternative compression method already designed to tackle this problem. It confirms the possibility of exact recovery when a considered signal is sparse [1,2]. Let the signal x in R^N be represented in term of basis $B = [B_1|B_2|\ldots|B_N]$ and weighting coefficients s,

$$x = Bs \qquad \text{or} \qquad x = \sum_{i=1}^{N} s_i B_i \quad , \tag{1}$$

where s is the $N \times 1$ column vector in the $N \times N$ basis domain B. In the case of sparse signal, the signal x is a linear combination of barely K non-zero coefficients with $K << N$ and the value of $(N - K)$ entries are close to zero. Fig. 1(a) is a graphic example of the $M < N$ system; the non-zero and zero entries in any elements are represented as color points and white-color points respectively. Notice the color elements s_i in coefficient vector s, there are only 4 color points (non-zero entries) of 16 coefficient elements. Thus, the important parts of signal x are from the non-zero entries of coefficient vector s, while the basis B is an arbitrary transformed basis of signal x (such as Fourier transform, Discrete Cosine transform, Wavelets etc.). The equivalent equations of this system can be expressed as,

$$y = Ax = ABs = \Theta s \,. \tag{2}$$

Furthermore, compressive sampling needs sufficiently the rows of matrix A about $M \approx K$ or slightly more to collect almost significant coefficients s_i [2], because the locations of non-zero coefficients s_i are absolutely encoded by the transformation matrix A. From (2), this system combines the vector s with matrix product $\Theta = AB$. This means that the compressed signal y is the representation of non-zero coefficients s_i and these highlighted columns of matrix product Θ, as shown in Fig. 1(b).



(a) (b)

Fig. 1 Graphic example of $y = Ax$ system with $M < N$

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In reconstruction process, our goal is to solve vector s from the given vector y, but in framework the locations of K non-zero coefficients are not known. Thus, a sufficient and necessary condition required to define the matrix product Θ are the *restricted isometry property* (RIP) [2], as below,

$$1-\varepsilon \leq \frac{\|\Theta v\|_2}{\|v\|_2} \leq 1+\varepsilon \text{,} \qquad (3)$$

where some $\varepsilon > 0$ and the matrix Θ is well-conditioned for an arbitrary $3K$-sparse vector v [2]. So far there have been shown that Gaussian distribution matrix and Rademacher distribution matrix ensuring the RIP [2].

Due to the considered signal is sparse so that there are many zero entries in the signal vector x (we can realize this problem in any transformed domains because it is from the same signal). Thus, the objective of recovery process is to find the possible vector x which has a minimum number of non-zero entries. A suitable method to solve this problem that is ℓ_0-minimization,

$$\min\|x\|_0 \text{ sub. to } Ax = y \text{.} \qquad (4)$$

Define ℓ_0 as "zero norm", which is used to count a number of non-zero entries in signal vector x. The other forms of ℓ_p-norm are expressed as,

$$\|x\|_p = \begin{cases} \left| i \,|x_i \neq 0,\ i = 1,2,\dots,N \right| & ,p = 0 \\ \sum_{i=1}^{N}\left(|x_i|^p\right)^{\frac{1}{p}} & ,p > 0 \end{cases} . \qquad (5)$$

The minimum ℓ_0-norm reconstruction recovers a K-sparse signal exactly [1,2]. However, this optimization is hard to solve because an exhaustive enumeration for all possible solutions is NP-complete which requires C_K^N possible combinations for the locations of non-zero entries in signal vector x. Thus, the objective function is unavoidably changed into approximated ℓ_1-norm,

$$\min\|x\|_1 \text{ sub. to } Ax = y \text{.} \qquad (6)$$

The computation complexity of this optimization is in polynomial occasion about $O(N^3)$. It is conveniently solved by a linear programming recovering the solutions close to K-sparse. However, it sometimes cannot recover the exact solutions for all sparse signals because of the probability condition in [1, *Theorem 1.3*]. Thus, the way to change the objective function of problem (6) will be possible, if this optimization is still in convex [3]. A relaxation of weighted ℓ_1-minimization which employs this idea to enhance the objective function is expressed as,

$$\min \|Wx\|_1 \text{ sub. to } Ax = y, \tag{7}$$

where $W = diag([w_1\ w_2, \ldots, w_n]^T)$ is a weighting matrix with $N \times N$ size. For analytic geometry in Fig. 2(a), the ℓ_1 ball which has the radius of exact signal vector $\|x_0\|_1$ touches the hyperplane $Ax = y$ at correct vertex. While Fig. 2(b), the interior ℓ_1 ball spans to touch the hyperplane in the fault point for which $\|x\|_1 < \|x_0\|_0$. Thus, the enhancement of interior ℓ_1 ball to the correct vector x_0 will be available, if $\|Wx\|_1 < \|Wx_0\|_1$, as Fig. 2(c).

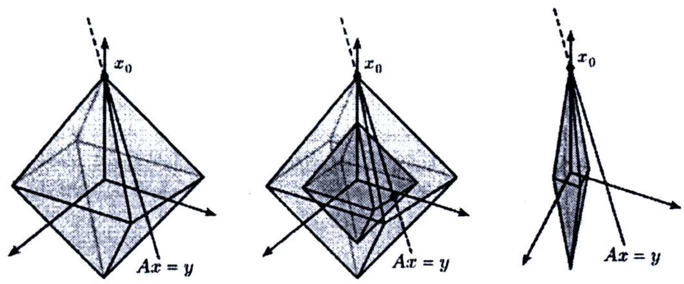

Fig. 2 Diversity of weighted ℓ_1 balls for sparse signal reconstruction

However, the corresponding ℓ_1 relaxations of weighted and un-weighted problems are possible to present different solutions [3]. One possible to control the weighting matrix should counteract the influence of signal penalty function. A recommended weighting function from [3] is defined as,

$$w_i = \begin{cases} \dfrac{1}{|x_{0,i}|} & \text{if } x_{0,i} \neq 0 \\ \infty & \text{if } x_{0,i} = 0 \end{cases}, \tag{8}$$

where $x_{0,i}$ is the true solution to each entry i. However, we could not define $w_i = \infty$ in the numerical computation. Thus the weighting function is basically changed into the equivalent form,

$$w_i = \frac{1}{|x_i| + \varepsilon}, \tag{9}$$

$\varepsilon > 0$ for ensuring the division of zero-value component in the reconstructed elements x_i which might estimate the weighting value w_i to infinite. Fig. 3 shows the distribution of weighting function (9) when varying the parameter ε. However, there is currently no smart and robust rule that would automatically select the suitable parameter ε [3]. Thus, the algorithm that ensures appropriate weighting value is still an open question.

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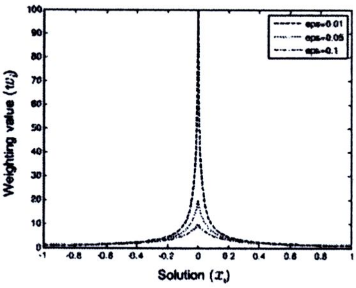

Fig. 3 Distribution of weighting function (9) when varying parameter ε in the period of possible solution between -1 and 1

Methods and solutions

The parameter $\varepsilon > 0$ in weighting function (9) is unbounded so that it is difficult to select its value. Additionally, if the parameter ε is varied, the distribution of weighting value will be changed obviously (Fig. 3). From our observation, there are many weighting values for each solution x_i followed condition (9). And their values are large for only the solutions close to zero. Thus, the way to change weighting function which has the similar characteristic will be fortunately available, if the optimization is still convex with the same constraints. A new proposed *hard selective weighting function* is expressed as,

$$w_i = \begin{cases} \beta_{\max} & , |x_i| < \tau \\ \beta_{\min} & , |x_i| \ge \tau \end{cases}, \qquad (10)$$

where a threshold τ is used to separate the weighting value $\beta_{\max}$ and $\beta_{\min}$ for the solutions x_i which are close to zero and the other solutions respectively. This threshold is bounded in the period of possible solutions so that it is easier to specify the position of threshold τ than the parameter ε in weighting function (9). The suitable weighting values of hard selective weighting function that are guarantee to find the true solution defined as $\beta_{\max} / \beta_{\min} = \infty$, but in numerical experiment the ratio cannot be infinity so that we introduce the new equivalent ratio as,

$$\frac{\beta_{\max}}{\beta_{\min}} = \frac{1}{\mu}, \qquad (11)$$

where μ is represented as the fixed-point accuracy. For example, if the considered signal is of the accuracy of floating fixed-point values at 0.01, the parameter $\mu \le 0.01$ will be the sufficient value, and the ℓ_1 ball will span more correctly when the limit of μ is close to zero. However, this ratio is typically set the parameters $\beta_{\min}, \beta_{\max} > 0$ because some important solutions x_i of summation $\|Wx\|_1$ might be ignored for which $\beta_{\min}, \beta_{\max} = 0$. Fig. 4 shows the distribution of hard selective weighting function where $\tau = 0.05$, $\beta_{\min} = 1$, and $\beta_{\max} = 100$ comparing with weighting function (9) where $\varepsilon = 0.01$.

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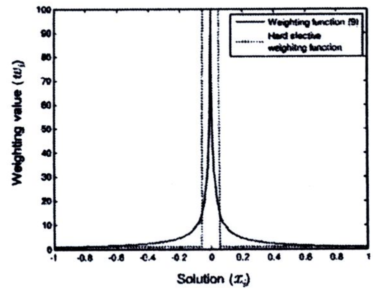

Fig. 4 Distribution of weighting function (9) comparing with hard selective weighting function in the period of possible solution between -1 and 1

The reweighted algorithm is applied the solution x_i from weighted ℓ_1-minimization (7) to update the weighting values w_i. Moreover, a new proposed algorithm (Fig. 5) includes the selecting threshold step for iteration $\gamma = 0$ described as follow. First, define the accuracy μ to be a period of frequency bins and compute a number of reconstructed elements in each frequency bin (that is histogram of reconstructed signal). Then, select the smallest index of histogram which contains zero element to be threshold τ. This idea presupposes the non-zero solutions which have value very close to zero are zero candidate solutions, so the weighting function is used to enhance the shape of interior ℓ_1 ball (Fig. 2(c)) for all zero candidates by multiplying large factor. For example, Fig. 4 shows the histogram of poor reconstructed signal of size 250-dimensional vector x after solving via ℓ_1-minimization. There are not only zero entries in zero index, but also the other zero entries might be shifted to the other values around zero index. These entries which are zero candidate solutions are multiplied by factor β_{max} and the other entries are multiplied by factor β_{min}. Thus, this algorithm selects the suitable threshold τ=0.07 for covering all zero candidates.

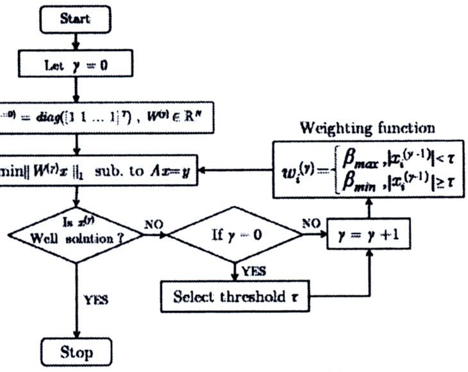

Fig. 5 New proposed reweighted algorithm

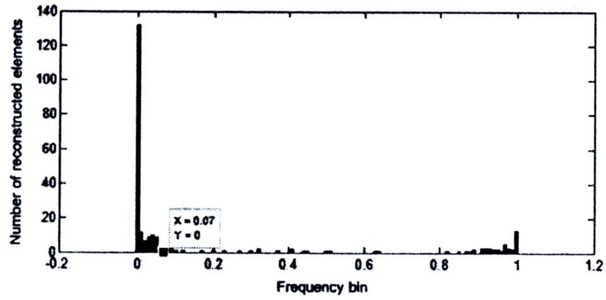

Fig. 6 Histogram of reconstructed signal example

Results and discussion

This section shows the numerical experiment which confirms the effectiveness of our idea in framework. Due to, the rule that would automatically select parameter ε in the weighting function (9) is not present [3] so that this experiments are enough to illustrate the new proposed reweighted algorithm (Fig. 3) comparing with ℓ_1-minimization. The considered signal is a normalized 250-dimensional vector x. Let the transformation matrix A be the zero-mean Gaussian distribution matrix of size 50×250, generated once for each signal. This experiments repeat 200 trails for each K-sparse signal and categorize the exact signal which has the peak signal to noise ratio, $\mathrm{PSNR} \geq 80$. Fig. 7 shows the 5 iterative numerical reweighted algorithm with the fixed-point accurancy $\mu = 0.01$ can recover the exact solutions outperforming ℓ_1-minimization about 7.37% in average.

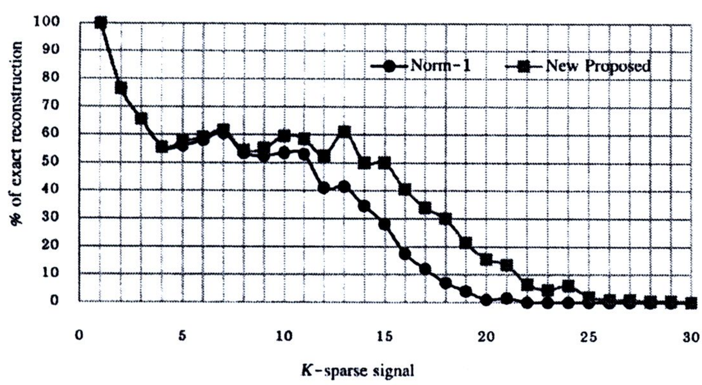

Fig. 7 Comparison of new proposed reconstruction algorithm and ℓ_1-minimization in compressive sampling

Conclusions

The new proposed algorithm is an alternative method to enhance the solutions of $y = Ax$ system. Although, our algorithm can do manifestly in practice, it is not enough for real-world conditions which need the algorithm that can recover exact solution with high probability. In further work, we will search out the other principles to enhance reconstruction algorithm and apply to some applications, such as Magnetic Resonance Imaging (MRI), image processing, etc.

Acknowledgements

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Hard Selective Reweighted ℓ_1-Minimization for Compressive Sampling Recovery

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Abstract

Sparsity is common in scientific and engineering problems. On technique as known compressive sampling, ℓ_1-minimization is a convenient reconstruction algorithm. However, this optimization could not recover the true signal for all sparsity. So far reweighted ℓ_1-minimization algorithm has been proposed to enhance signal recovery and showed a surprised result, but how to specify weighting value is still questionable. This problem is coped with by the new proposed *Hard Selective Reweighted* (HSR) algorithm which is an alternative algorithm to find appropriate weighting value for reweighted algorithm. The numerical results show that the percentage of exact reconstruction by HSR algorithm outperforming ℓ_1-minimization 8.73% on an average.

Keywords: compressive sampling, compressed sampling, compressive sensing, sparsity, reweighted ℓ_1-minimization, ℓ_1-minimization.

1. Introduction

Sparsity property notifies natural signals containing much repetitious information; this means that there are many zero entries in the transformed domain. Thus, it is not necessary to collect all information uniformly and the random sampling is also sufficient to keep essential information and possibly under Nyquist rate [1,2].

Compressive sampling, also known as compressed sampling and compressive sensing, is a newfangled method to compress signal ignoring sampling theorem and changing compression structure from traditional way. This principle applies to a sparse signal.

Let the sparse vector x be expressed in a linear combination with a basis $\Psi = [\psi_1|\psi_2|\cdots|\psi_N]$ as shown in (1) and (2)

$$x = \sum_{i=1}^{N} s_i \psi_i \,, \qquad (1)$$

$$x = \Psi s \,, \qquad (2)$$

where s is the $N \times 1$ column vector of coefficients s_i in the $N \times N$ basis domain Ψ. The signal x is sparse if it has only K non-zero entries and the value of $(N - K)$ entries are close to zero with $K << N$.

The traditional compression schemes transform the signal x to basis Ψ, and then select large-value coefficients s_i to be the compressed signal. Alternatively, compressive sampling transforms the signal x via the measurement matrix $\Phi_{M \times N}$ all at once as defined in (3) and shown in Fig. 1,

$$y = \Phi x = \Phi \Psi s = \Theta s \,. \qquad (3)$$

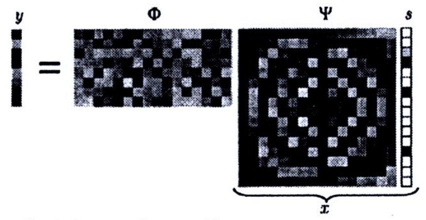

Fig. 1 Compressive sampling measurement process

In reconstruction process, our goal is to solve vector s from the given vector y but this system is a linear algebra problem with $M < N$, i.e. fewer equations than unknown, giving infinitely many solutions. This means that the reconstruction will not recover the true signal, unless there are other conditions.

However, compressive sampling process needs sufficiently measurement matrix rows $M \approx K$ to keep almost significant coefficients [2]. From (3), the measurement process combines the vector s with matrix product Θ which its columns correspond to nonzero coefficients s_i. In this case the compressed signal y is a linear combination of these some columns as shown in Fig. 2.

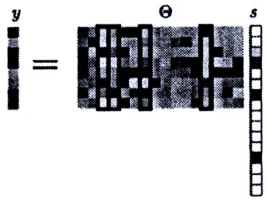

Fig. 2 Compressed signal from the linear combination of highlighted columns

Inpractice, the locations of the K non-zero entries in vector s are not known so that the matrix product Θ must satisfy the *restricted isometry property* (RIP) [2].

$$1-\varepsilon_{\mathrm{RIP}} \leq \frac{\|\Theta v\|_2}{\|v\|_2} \leq 1+\varepsilon_{\mathrm{RIP}}\,. \qquad (4)$$

where some $\varepsilon_{\mathrm{RIP}} > 0$ and the matrix Θ is well-conditioned for an arbitrary $3K$-sparse vector v [2]. So far there have been shown that Gaussian distribution matrix and Rademacher distribution matrix satisfy the RIP [2].

2. Reconstruction algorithms

Since $M < N$, there are many solution sets that satisfy the constraint $\Theta s = y$ and the possible solution vectors s lie on the $(N-M)$-dimensional hyperplane $\mathcal{H} = \mathcal{N}(\Theta) + s$ in R^N. The hyperplane corresponds to the null space $\mathcal{N}(\Theta)$ of matrix Θ which is translated to the exact sparse vector s. Because if $\Theta s = y$, then $\Theta(s+r) = y$ for any vector r in the null space. Thus, our objective is to find non-zero coefficient vector s in the hyperplane which has the minimum number of non-zero entries. An achievable way to search the sparest vector in the null space $\mathcal{H}$ is ℓ_0-minimization,

$$\min\|s\|_0 \ \text{sub. to } \Theta s = y\,. \qquad (5)$$

Define ℓ_0 as "zero norm" which is used to count a number of non-zero entries in vector s. The other forms of ℓ_p-norm for the vectors s are defined as,

$$\|s\|_p = \begin{cases} \left| i \,|\, s_i \neq 0,\ i = 1,2,\ldots,N \right| & ,p = 0 \\ \sum_{i=1}^{N} \left(|s_i|^p\right)^{\frac{1}{p}} & ,p > 0 \end{cases}. \qquad (6)$$

The minimum ℓ_0-norm reconstruction will recover a K-sparse signal exactly [1,2]. However, this optimization is hard to solve because an exhaustive enumeration for all possible solutions is NP-complete which requires C_K^N possible combinations for the locations of the non-zero entries in vector s. Thus, the objective is unavoidably changed to approximated ℓ_1-norm,

$$\min\|s\|_1 \ \text{sub. to } \Theta s = y\,. \qquad (7)$$

The solutions from ℓ_1-minimization are close to K-sparse with the polynomial computational complexity about $O(N^3)$ and this optimization is conveniently solved by a linear programming. It is proved in [1,*Theorem 1.3*] that signal s can be recovered with high probability by solving ℓ_1-minimization.

3. Reweighted ℓ_1-minimization algorithm

The ℓ_1-minimization problem in the previous section is a convex optimization problem. The objective function $\|s\|_1$ from this optimization problem can be changed to a relaxation of weighted ℓ_1-minimization problem with the same constraints,

$$\min\|Ws\|_1 \ \text{sub. to } \Theta s = y\,, \qquad (8)$$

where $W = diag([w_1\ w_2,\ldots,w_n]^T)$ is a weighting matrix with size $N \times N$. This optimization can be solved via the linear programming as problem (7). However, the corresponding ℓ_1 relaxations of weighted and un-weighted problems are possible to present different solutions [3]. Thus, the idea to choose weighting function should counteract the influence of signal penalty function. A recommended weighting function from [3] is defined as,

$$w_i = \begin{cases} \frac{1}{|s_{0,i}|} & \text{if } s_{0,i} \neq 0 \\ \infty & \text{if } s_{0,i} = 0 \end{cases}, \qquad (9)$$

which $s_{0,i}$ is the true solution to each entry i. If the original signal is K-sparse and $\|s_0\|_0 \leq K$, then reweighted ℓ_1-minimization algorithm will guarantee to find the correct solution [3]. However, we could not define $w_i = \infty$ in the numerical computation. Thus, the weighting function is basically transformed into the equivalent form,

$$w_i = \frac{1}{|s_i| + \varepsilon}\,, \qquad (10)$$

where $\varepsilon > 0$ for ensuring the division of zero-value component in the reconstructed vector $s^{(\gamma)}$ which might estimate the weighting value to infinite.

For the iterative algorithm, the first knowing solution vector $s^{(0)}$ is used to construct a range of possible weighting matrices $W^{(\gamma=1)}$ before solving the problem (8). Then, the next weighting matrix will be estimated by the previous solution vector $s^{(\gamma-1)}$. A simple reweighted algorithm is shown in Fig. 3.

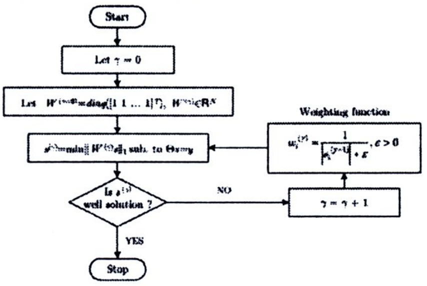

Fig. 3 Iterative reweighted algorithm

However, so far there is no smart and robust rule that would automatically select the suitable parameter ε

in the weighting function (10) [3]. Thus, the algorithm that ensures appropriate weighting function is still an open question.

4. Weighted ℓ_1-ball analysis

The weighting function from reweighted algorithm is not flexible to control its characteristic. If parameter ε is varied, the distribution of weighting value will be changed obviously. Thus, the adapted weighting function will be fortunately available, if the new weighting function remains convex.

Consider the ℓ_1-ball geometry in Fig. 4(a), the ℓ_1-minimization solutions are on the vertex of ℓ_1-ball with radius $\|s_0\|_1$ and touch the hyperplane at the true vector s_0. In some condition, the interior ℓ_1-ball spans to touch the hyperplane in the fault point for which $\|s\|_1 < \|s_0\|_0$, as Fig. 4(b). Thus the weighting function is used to improve the shape of interior ℓ_1-ball for which $\|Ws\|_1 \le \|Ws_0\|_1$. It is possible that the ℓ_1-ball will touch the hyperplane on the correct point, as Fig. 4(c).

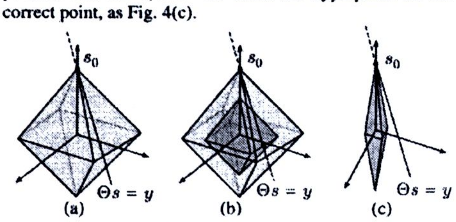

Fig. 4 Enhancing weighted interior ℓ_1-ball for reconstructed signal

From our observation, the weighting function from reweighted algorithm spans the interior ℓ_1-ball with various sizes, followed condition (10). And the weighting value is too large for only the solutions s_i which their values are close to zero. Thus, an alternative weighting function which has the similar characteristic, is the proposed *hard selective weighting function*,

$$w_i = \begin{cases} \beta_{max} & , |s_i| \le \tau \\ \beta_{min} & , |s_i| > \tau \end{cases}, \qquad (11)$$

where threshold τ is used to classify the weighting value for each solution s_i. Consider a 3-D example in Fig. 5, the spanning sizes of hard selective weighted ℓ_1-ball are inversely weighting values. This is because changing the weighting w_i effects the solutions s_i by factor $1/\beta_{max}$ and $1/\beta_{min}$ for the weighting values β_{max} and β_{min} respectively.

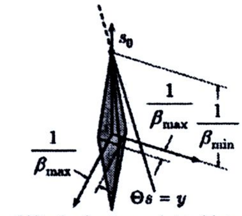

Fig. 5 Hard selective weighted interior ℓ_1-ball

One possible to control the weights, the right ℓ_1-ball has to span the vertex on non-zero solution axes, intersecting the hyperplane at the true point before the other vertices touch the hyperplane. Suppose, the hard selective ratio of weighted ℓ_1-ball is guaranteed to find the correct solutions, defined as,

$$\frac{\beta_{max}}{\beta_{min}} = \infty. \qquad (12)$$

In the numerical experiment, we cannot define the ratio (12) as infinity so that the recommendation of the suitable ratio depends on the accuracy of signal as below,

$$\frac{\beta_{max}}{\beta_{min}} = \frac{1}{\mu}, \qquad (13)$$

where μ is represented as the fixed-point accuracy. For example, if a considered signal is of the accuracy of floating fixed-point values at 0.01, the parameter $\mu \le 0.01$ will be the sufficient value, and the ℓ_1-ball will span more correctly when the limit of μ close to zero. However, we usually define $\beta_{min}, \beta_{max} > 0$ because if $\beta_{min}, \beta_{max} = 0$, some solutions s_i, which are the important part of summation $\|Ws\|_1$, will be ignored.

5. Hard selective reweighted algorithm

Compressive sampling considers the signal that is sparse. After recovery, a number of zero entries of reconstructed signal should be equal to original signal. However, ℓ_1-minimization sometimes could not recover the true signal; this means that some original zero entries shift to the other entries of reconstructed signal.

HSR is designed to cope with hard selective weighted ℓ_1-ball in the case of knowing a number of original zero entries. If un-weighted ℓ_1-minimization in the reweighted algorithm iteration $\gamma = 0$ gives the poor solutions, this algorithm will select the proper threshold τ which is described as follow. First, the algorithm computes a number of reconstructed elements in each frequency bin (that is histogram of reconstructed signal). After, it is to count the cumulative sum along different frequency index until the sum is greater than or equals to a number of original zero entries. Then, the last cumulative sum index will be the value of threshold τ. For example in cumulative sum of histogram (Fig. 6), suppose the original signal containing 70 zero entries, and the zero entries of reconstructed signal is only about 60 so that the threshold r should be 0.32 for covering 70 zero candidate solutions.

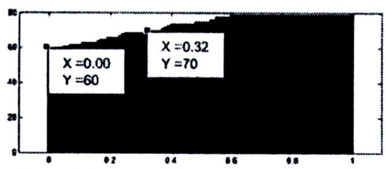

Fig. 6 Cumulative sum of histogram values of frequency components

Finally, the weighting values of solutions $s_i^{(r-1)}$ are β_{max} for the zero candidate solutions and β_{min} for the other solutions. The iterative hard selective reweighted algorithm is shown in Fig. 7.

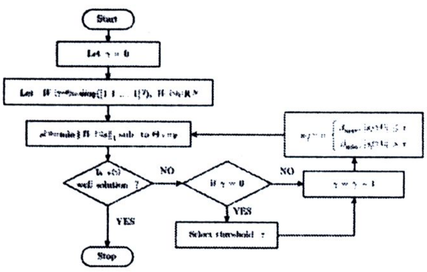

Fig. 7 Iterative hard selective reweighted algorithm

6. Numerical results

In this section we present numerical experiments that illustrate our idea derived in the geometry. Nowadays the rule for selecting the value of parameter ε in reweighted algorithm is not present [3]. Thus, these experiments are enough to show the percentage of successful recovery via HSR algorithmcomparing with ℓ_1-minimization. The considered sparse signal is a 250-dimesional vector s with normalized-value. Let matrix product Θ be the zero-mean standard (or normal) Gaussian measurement matrix of size 50×250, generated once for each K-sparse signal. The experiments employ 100 set of signals for each K-sparse signal and classify the exact signal which has PSNR ≥ 80 (The peak signal to noise ratio (PSNR) is an engineering term for the ratio between the maximum possible power of a signal and the power of corrupting noise that affects the fidelity of its representation [3]). Fig. 8 is the comparison of ℓ_1-minimization and HSR algorithm when setting 10 iterations and $\mu = 0.01$. It shows the percentage of HSR reconstruction algorithm outperforming ℓ_1-minimization 8.73% approximately.

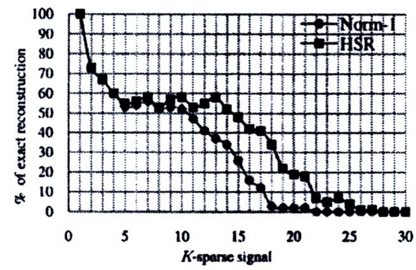

Fig. 8: Comparison of ℓ_1-minimization and HSR reconstruction algorithm in compressive sampling

7. Conclusions

HSR algorithm is a new desirable way to enhance the sparse signal reconstruction. Although, this algorithm needs to know a number of original zero entries, the parameter r is bounded in the period of possible solutions (because signal can be normalized) while the parameter $\varepsilon > 0$ in reweighted algorithm is still unbounded. For further work, we are searching out smart techniques for selecting all parameters without knowing original zero entries. In addition, we will look for the other principles to enhance reconstruction algorithm for compressive sampling and apply to some applications, such as Magnetic Resonance Imaging (MRI), image processing, signal recognition etc.

8. Acknowledgements

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New Reweighted ℓ_1-minimization Algorithms for Compressive Sampling Recovery
การหาค่าต่ำสุดของนอร์มหนึ่งแบบถ่วงน้ำหนักรูปแบบใหม่ สำหรับการสร้างสัญญาณย้อนกลับของคอมเพรสซีฟแซมปลิ่ง

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ABTRACT

A recent compression method which overlooks the classical Shannon-Nyquist theorem is called *compressive sampling*, also known as *compressed sensing*. The reconstruction of this new compression method is proved to be done with high probability of success by performing ℓ_1-minimization problem. The ℓ_1-minimization reconstruction has been developed to the reweighted algorithm which recovers closely approximate sparse solutions. However, there is no rule that automatically selects the appropriate weighting values. This paper proposes the enhancements of reweighted ℓ_1-minimization by indicating the choice of weighting functions and the suggestion to find the weighting values.

In reconstruction process, the approximate ℓ_1-minimization might recover the fault signal by shifting the zero solutions to the other values. Thus, the *hard selective reweighted* (HSR) algorithm is designed to increase the importance of zero candidates by selecting the near-zero solutions whose numbers are equal to a number of original zero entries scaled by greater weighting value. In general, the locations of zero entries are not known so that the HSR algorithm could not apply to the real-world problems. This problem is coped with by the second proposed *automatic adaptive reweighted* (AAR) algorithm which is used to predict the locations of zero entries without knowing a number of original entries. The idea is to find the smallest frequency bin of solutions which contains empty member then set it to be the threshold and the solutions which are close to zero and the others scaled by larger and smaller weighting values, respectively. The numerical results show comparatively that HSR and AAR algorithms outperform ℓ_1-minimization. Furthermore, both of these algorithms are demonstrated to be applied to manmade and magnetic resonance imaging (MRI) images.

บทคัดย่อ

แนวทางการบีบอัดสัญญาณแบบใหม่ ซึ่งไม่ขึ้นอยู่กับทฤษฎีชานนอน และไนควิสท์ (Shannon-Nyquist theorem) เรียกว่าคอมเพรสซีฟแซมปลิ้ง (compressive sampling) หรือคอมเพรสเซนซิ่ง (compressed sensing) การสร้างสัญญาณย้อนกลับของแนวทางนี้ได้พิสูจน์ และสามารถทำได้โดยการหาค่าต่ำสุดแบบนอร์มหนึ่ง (ℓ_1-minimization) ซึ่งต่อมาได้รับการพัฒนามาเป็นระเบียบวิธีการแบบถ่วงน้ำหนัก (reweighted algorithm) และสามารถสร้างสัญญาณกลับได้ใกล้เคียงกับสัญญาณที่มีความเบาบาง (sparse signal) แต่อย่างไรก็ตาม ยังไม่มีแนวทางการเลือกค่าถ่วงน้ำหนัก แบบอัตโนมัติ ด้วยเหตุนี้ บทความฉบับนี้จึงเสนอทางเลือกของฟังก์ชันถ่วงน้ำหนักแบบใหม่ และมุ่งเสนอแนวทางการเลือกค่าถ่วงน้ำหนักที่เหมาะสม

ในกระบวนการสร้างสัญญาณย้อนกลับโดยการหาค่าต่ำสุดแบบนอร์มหนึ่งนั้น มีโอกาสผิดพลาด เนื่องจากมีการย้ายคำตอบที่มีค่าเป็นศูนย์ไปยังตำแหน่งอื่นๆ ดังนั้นระเบียบวิธีการถ่วงน้ำหนักแบบเลือกค่าอย่างฉับพลัน (hard selective reweighted, HSR) จึงถูกออกแบบเพื่อเพิ่มความสำคัญของตัวแทนคำตอบศูนย์ ซึ่งเลือกจากคำตอบที่มีค่าใกล้ศูนย์มาเป็นจำนวนเท่ากับจำนวนของคำตอบศูนย์ของสัญญาณดั้งเดิม โดยจะทำการเพิ่มขนาดของคำตอบให้ใหญ่ขึ้น แต่โดยทั่วไปตำแหน่งของคำตอบศูนย์นั้นไม่สามารถรู้ได้ จึงส่งผลให้ระเบียบวิธีการถ่วงน้ำหนักแบบเลือกค่าอย่างฉับพลันไม่สามารถประยุกต์ใช้ในปัญหาในโลกความเป็นจริง เพื่อที่จะจัดการกับปัญหานี้จึงได้มีการเสนอเพิ่มเติมระเบียบวิธีการถ่วงน้ำหนักแบบเลือกค่าได้อัตโนมัติ (automatic adaptive reweighted, AAR) ซึ่งใช้สำหรับทำนายหาตำแหน่งของค่าคำตอบศูนย์ โดยไม่จำเป็นต้องทราบจำนวนของคำตอบศูนย์ของสัญญาณดั้งเดิม แนวคิดนี้ใช้วิธีการแบ่งคำตอบออกเป็นช่วงความถี่ต่างๆ และตรวจสอบว่าช่วงความถี่ที่มีค่าน้อยที่สุดใดไม่มีสมาชิกอยู่ จะถูกกำหนดให้เป็นตำแหน่งจุดของการเปลี่ยนแปลง และกำหนดให้คำตอบที่มีค่าใกล้ศูนย์ และคำตอบอื่นๆถูกเพิ่มขนาดให้ใหญ่ขึ้น และลดขนาดให้เล็กลงตามลำดับ จากผลการทดสอบเชิงตัวเลข พบว่า ผลการสร้างสัญญาณย้อนกลับที่ถูกต้อง โดยระเบียบวิธีการถ่วงน้ำหนักแบบเลือกค่าอย่างฉับพลัน และระเบียบวิธีการถ่วงน้ำหนักแบบเลือกค่าได้อัตโนมัติ ดีกว่าวิธีการหาค่าต่ำสุดแบบนอร์มหนึ่ง นอกจากนั้นระเบียบวิธีทั้งสองยังถูกนำไปประยุกต์กับภาพที่ถูกสังเคราะห์จากมนุษย์ และภาพที่สร้างจากเรโซแนนซ์แม่เหล็ก (magnetic resonance imaging, MRI)

Key Words : Compressive sampling, Compressed sampling, Compressive sensing, Sparsity, Reweighted ℓ_1-minimization, ℓ_1-minimization.

คำสำคัญ : คอมเพรสซีฟแซมปลิ้ง คอมเพรสแซมปลิ้ง คอมเพรสซีฟเซนซิ่ง สัญญาณที่มีความเบาบาง การหาค่าต่ำสุดของนอร์มหนึ่งโดยถ่วงน้ำหนัก การหาค่าต่ำสุดของนอร์มหนึ่ง

Introduction

A traditional compression method usually applies a transformation to a sampled signal and then truncates most of coefficients but significant ones. This means that the quality of signal compression depends on type and efficiency of transformation. Up until now there are many transformations which have been currently used such as *Fourier transformation*, *discrete cosine transformation*, *wavelets*, etc. However, the way to choose the transformation is specifically designed for each application so that there are no suitable transformations for all signals. A brief traditional compression process is shown in Figure 1.

Compressive sampling, also known as compressed sensing, is a new founded method to compress signal by exploiting its compressibility. A conventional sampled signal depends on Nyquist-bandlimited sampling rate but this method ignores the Shannon-Nyquist sampling theorem. This new idea was motivated in 2006 (Candès, Romberg and Tao, 2006). They assume that several signals in the world are sparse, i.e. they contain much repetitious information. Thus, it is not necessary to sample the signal following Shannon-Nyquist theorem and random sampling is sufficiently allowed for a considered sparse signal.

Let x be a real-value, finite-length, discrete-time, one-dimensional signal, which can be viewed as an $N \times 1$ column vector in R^N. A signal in R^N can be represented in term of a $N \times N$ basis matrix $\Psi = [\psi_1 | \psi_2 | \ldots | \psi_N]$ by stacking the basis vector ψ_i as columns, i.e.

$$x = \sum_{i=1}^{N} s_i \psi_i , \tag{1}$$

$$x = \Psi s . \tag{2}$$

where s is the $N \times 1$ column vector of weighting coefficients s_i. This operation is the first stage in Figure 1.

A sparse representation focuses on the elements of non-zero entries in coefficient vector s. If there are K non-zero entries with $K \ll N$ – only K of the s_i in (1) are non-zero and $(N-K)$ are zero, the signal will be considered as a K-sparse signal. In fact, many natural and manmade signals are sparse with a few large coefficients and many small coefficients (Baraniuk, 2007).

The idea of compressive sampling is to transform the signal x via a measurement matrix $\Phi_{M \times N}$ as defined in (3) and shown in Figure 2.

$$y = \Phi x = \Phi \Psi s = \Theta s . \tag{3}$$

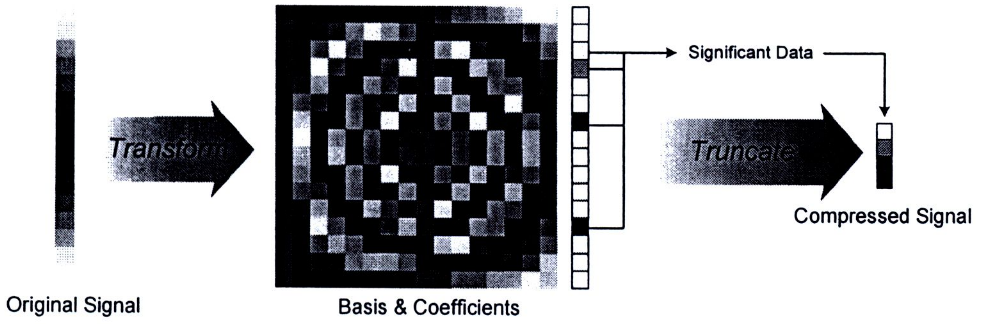

Figure 1 Traditional compression process

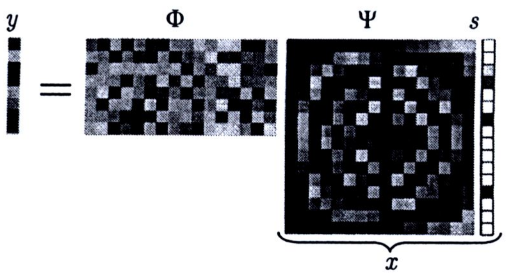

Figure 2 Compressive sampling measurement process (Baraniuk, 2007)

From (3), the matrix product Θ is the representation of measurement matrix Φ and basis matrix Ψ whose columns correspond to non-zero coefficients s_i; i.e. the compressed signal y is a linear combination of K columns of matrix product Θ as shown in Figure 3.

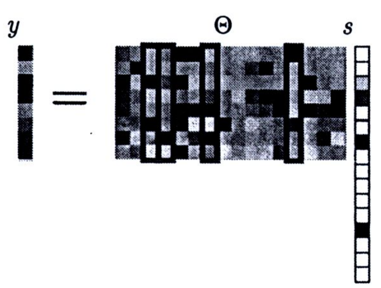

Figure 3 Compressed signal from the linear combination of highlighted columns (Baraniuk, 2007)

It is surprising that with $M < N$ the signal x can be reconstruction from the compressed signal y (Candès, Romberg and Tao, 2006). This is to find a solution of system of linear equations with fewer equations than unknowns. As known from linear algebra there are infinitely possible solutions. Thus, this problem cannot be solved unless there are other imposed conditions.

The reconstruction algorithm for K-sparse requires sufficiently $M \approx K$ measurement matrix or slightly more measurements to collect significant coefficients. Because there are K non-zero entries which cause this system likely that there are K unknowns so that the measurement rows $M \approx K$ which approximate to K equations are enough to solve this problem (Baraniuk, 2007).

In actual fact, the locations of non-zero coefficients s_i in K-sparse signal are not known although a number of equations M equal or exceed a number of unknowns K. Thus, a necessary and sufficient condition that ensures the solution of $M \approx K$ system is that the vector $\bar{s}$ must share the same locations as K non-zero entries. The matrix Θ which applies this ideal preserving the lengths of these particular K-sparse vectors is said to have *restricted isometry property* (RIP),

$$1-\varepsilon_{\mathrm{RIP}} \leq \frac{\|\Theta \bar{s}\|_2}{\|\bar{s}\|_2} \leq 1+\varepsilon_{\mathrm{RIP}}, \tag{4}$$

for some $\varepsilon_{\mathrm{RIP}} > 0$. So far there are independent and identically distributed (i.i.d.) random Gaussian distribution and Rademacher distribution satisfying the RIP property (Baraniuk, 2007).

Reconstruction algorithms

The $M < K$ system generates infinitely many possible solutions which all lie on the $(N-M)$-dimensional hyperplane $\mathcal{H} = \mathcal{N}(\Theta) + s$ in R^N. The true solution vector s is also sparse corresponding to the constraint $\Theta(s+r) = y$ for any vector r in null space $\mathcal{N}(\Theta)$. Thus, our goal is to find the sparsest coefficient vector s in the translated null space.

A suitable method that counts the smallest number of non-zero in coefficient vector s is ℓ_0-minimization,

$$\min \|s\|_0 \text{ sub. to } \Theta s = y , \tag{5}$$

which ℓ_0 is well-called "zero norm". The other form of ℓ_p-norm for the vector s is defined as,

$$\|s\|_p = \sum_{i=1}^{N}\left(|s_i|^p\right)^{\frac{1}{p}} ,p > 0 . \tag{6}$$

However, the ℓ_0-minimization is hard to solve because an exhaustive enumeration which is NP-complete requires C_K^N possible combinations for all locations of non-zero entries in vector s. Thus, this optimization is adapted to the approximate ℓ_1-minimization,

$$\min \|s\|_1 \text{ sub. to } \Theta s = y . \tag{7}$$

This is a convex optimization problem that conveniently reduces to a linear programming which requires computational complexity about $O(N^3)$ (Boyd and Vandenberghe, 2004). It is proved that this optimization can exactly reconstruct K-sparse signal vectors with high probability (Candès, Romberg and Tao, 2006).

Reweighted ℓ_1-minimization

The differences of solutions between ℓ_0 and ℓ_1 norms are the locations of K non-zero entries. Although ℓ_1-minimization can search the locations of K non-zero entries, a number of reconstructed zero entries are not equal to the original. Because the objective function of ℓ_1-minimization is designed specifically for a symmetric ℓ_1-ball (the possible capacity cost of objective function $\|s\|_1$) which has probability to touch the hyperplane in a wrong position for which $\|s\|_1 < \|s_0\|_0$, shown as interior ℓ_1-ball (Figure 4(b)) while the ℓ_1-ball of radius $\|s_0\|_1$ touches the true solution at the vertex containing more zero entries and close to K-sparse (Figure 4(a)).

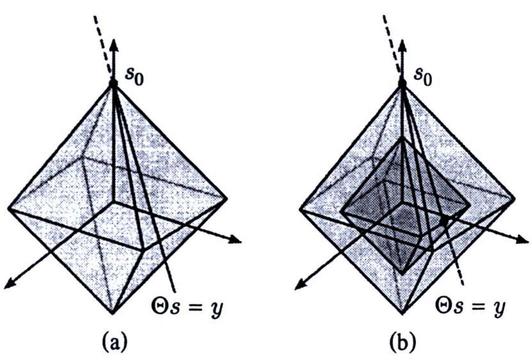

Figure 4 ℓ_1-ball for reconstructed signal (Candès, Wakin, and Boyd, 2007)

The enhancement of ℓ_1-minimization which reshapes the ℓ_1-ball to counteract the function penalty is possible for a convex optimization. A weighted relaxation ℓ_1-minimization which employs this idea to formulate the objective function $\|Ws\|_1$ with the same constraints is expressed as,

$$\min \|Ws\|_1 \text{ sub. to } \Theta s = y , \tag{8}$$

where $W = diag([w_1\ w_2,\ldots,w_N]^T)$ is a weighting diagonal matrix of size $N \times N$. The weighted ℓ_1-minimization can be solved via linear programming as same as ℓ_1-minimization. However, the solutions from the convex optimization might present the different solutions. Thus, the suitable weighting matrix can reshape the ℓ_1-ball in order to avoid the fault solution $s \neq s_0$ for which $\|Ws\|_1 \leq \|Ws_0\|_0$ as illustrated in Figure 5.

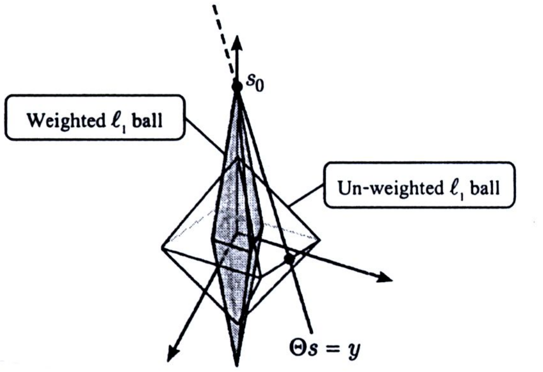

Figure 5 Weighted & un-weighted ℓ_1-ball for coefficient signal vector (Candès, Wakin, and Boyd, 2007)

The conceptual weighting function is designed to control the true solution by counteracting the influence of signal magnitude on the ℓ_1 penalty function. The recommended weights are inversely to the true signal magnitude (Candès, Wakin, and Boyd, 2007),

$$w_i = \begin{cases} \dfrac{1}{|s_{0,i}|} & \text{if } s_{0,i} \neq 0 \\ \infty & \text{if } s_{0,i} = 0 \end{cases}, \qquad (9)$$

where $s_{0,i}$ is the true solution to each entry i. This weighting function guarantees to find the true solution but the locations of K non-zero entries are not already known. Thus, this weighting function is unavoidably applied the solution from ℓ_1-minimization to construct the weights. Otherwise, the numerical computation cannot be defined when $w_i = \infty$ so that the weighting function are revised to the equivalent form,

$$w_i = \frac{1}{|s_i| + \varepsilon}, \qquad (10)$$

where $\varepsilon > 0$ for ensuring the division of zero-value component in the reconstructed vector s_i which might estimate the weighting value w_i to infinity.

The equivalent weighting function (10) is not required the location of the true solutions so that the efficiency to find the true solutions might be generally decreased. The achievable process which enhances weighted ℓ_1-minimization is to repeat the algorithm iteratively. A simple reweighted algorithm is shown in Figure 6.

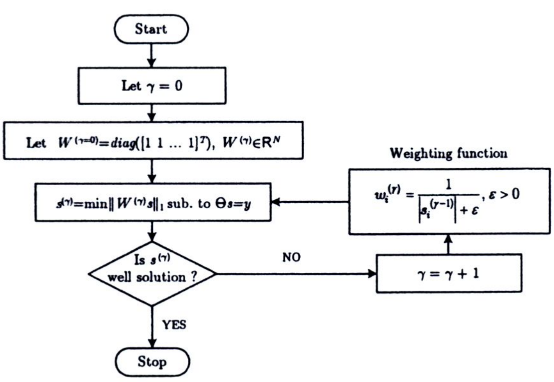

Figure 6 Iterative reweighted ℓ_1-minimization

The first knowing solution vector $s^{(0)}$ obtained from un-weighted ℓ_1-minimization is used to construct a range of weighting matrix $W^{(1)}$. Then, the next weighting matrix will be estimated by the previous solution vector $s^{(\gamma-1)}$. This algorithm repeats until the solutions terminate on convergence or when iteration γ attains a specified maximum number of iterations γ_{max}.

In general, the weighting magnitude w_i depends on the choice of parameter ε which controls its distribution changed obviously when varying the value of ε.

However, there are currently no smart and robust rules that would automatically select the parameter ε adapting the dynamic range of weighting values (Candès, Wakin, and Boyd, 2007). Thus, the algorithm that ensures the appropriate weighting magnitude w_i is still an open question.

Methods and Solutions

An innovative proposed idea is to design new weighting function which is more general to select the appropriate weighting value. Since the parameter ε in weighting function is unbounded for some $\varepsilon > 0$ so that it is difficult to vary its value when undergoing the experiment. The way to adapt to the new weighting function is possible for convex optimization. From weighting function (9), the weighting values are mostly to scale the solutions which their values are close to zero by the large factors. Thus, the achievable weighting function which has the similar characteristic designed to reduce the complexity of function is proposed as *hard selective weighing function* (Charunphaisan and Meesombon, 2009A),

$$w_i = \begin{cases} \beta_{\max} & ,|s_i| \le \tau \\ \beta_{\min} & ,|s_i| > \tau \end{cases}, \tag{11}$$

where τ is the threshold which its value is in the period of possible solutions divided the solutions into 2 groups; there are the solutions which are close to zero multiplied by $\beta_{\max}$ and the other solutions are multiplied by $\beta_{\min}$. Furthermore, the parameters $\beta_{\max}, \beta_{\min} > 0$ are well-defined for the reason that if $\beta_{\max}, \beta_{\min} = 0$, some solutions which are in the summation $\|Ws\|_1$ will be ignored.

According to Figure 5, the vertex of weighted ℓ_1-ball is extended to intersect the hyperplane at the true solution which contains many zero entries. Suppose, the solution s_i which are multiplied by $\beta_{\max}$ are zero candidates so that the weighting value $\beta_{\max}$ should be infinitely larger than $\beta_{\min}$ as a *hard selective ratio*,

$$\frac{\beta_{\max}}{\beta_{\min}} = \infty, \tag{12}$$

However, this ratio cannot be defined as infinity in numerical experiments in order that the infinite ratio is alternatively changed to the suggested formulation,

$$\frac{\beta_{\max}}{\beta_{\min}} = \frac{1}{\mu}, \tag{13}$$

where $\mu \le 1$ is represented as the expanding rate. The ℓ_1-ball can be spanned more elaborately touching the hyperplane when the limit of μ is closer to zero. Additionally, the recommended value of parameter μ is equal to the floating fixed-point accuracy. For example, if the considered signal is of the accuracy at 0.01, the parameter $\mu \le 0.01$ will be the sufficient value. The *hard selective reweighted* (HSR) algorithm, is shown in Figure 7.

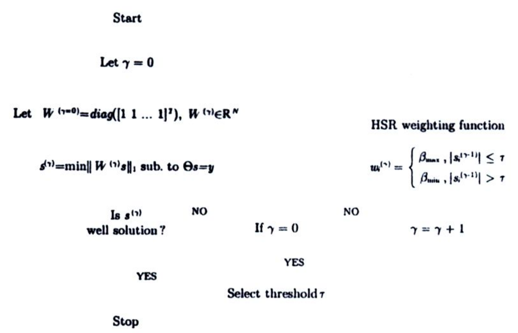

Figure 7 Iterative HSR algorithm including selecting threshold τ process

In reconstruction process, the ℓ_1-minimization is possible to recover the fault signal. For example in Figure 9(b), it shows the scatter plot of reconstructed coefficient vector $s^{(0)}$. As we notice these points spreading widely the diagonal line so that, in this case, the un-weighted ℓ_1 minimization does not offer the exact solution.

Another point of view, this means that some zero entries in the reconstructed coefficient vector $s^{(0)}$ might spread around to the other solutions. One possibility to recall the original zero entries is to find the locations of all zero candidate entries. In this paper, we assume the locations of solutions which their values are close to zero will be set as zero entries. Moreover, if a number of zero entries in the original coefficient vector $s_0 = \Psi^{-1} x_0$ are correctly known and the ℓ_1-minimization presents the fault reconstructed vector $s^{(0)}$, the threshold τ in reweighted ℓ_1-minimization will be computed as follows. Firstly, calculate a number of absolute elements of fault reconstructed vector $s^{(0)}$ in each frequency bin (that is histogram of absolute reconstructed vector $|s^{(0)}|$). After that, count the cumulative sum along difference frequency index until the sum is greater than or equals to a number of original zero entries. Finally, the last cumulative sum is the value of the threshold τ.

For example, Figure 8 shows the cumulative sum of histogram of absolute coefficient vector $|s^{(0)}|$ via ℓ_1-minimization. We know that the original coefficient vector s_0 contains 455 zero entries while the zero entries of absolute coefficient vector $|s^{(0)}|$ are merely about 370 so that the threshold is defined as 0.05 for containing equally 455 zero candidate solutions.

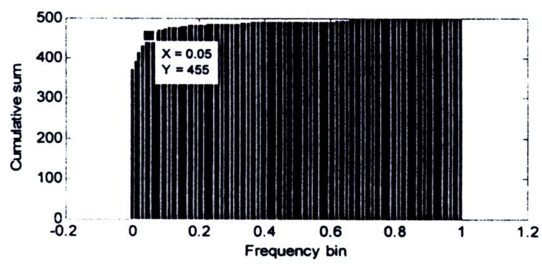

Figure 8 Cumulative sum of histogram values of absolute reconstructed vector $|s^{(0)}|$ using unweighted ℓ_1-minimization

However, a number of original zero entries are not known in practical so that an alternative simple algorithm without knowing a number of original zero entries is also proposed by finding only the smallest frequency bin of histogram of absolute coefficient vector $|s^{(0)}|$ which is empty, called *automatic adaptive reweighted* (AAR) algorithm (Charunphaisan and Meesombon, 2009B). A brief concept of algorithm is to assume the near-zero solutions being the zero candidates and their entries are punctuated by the

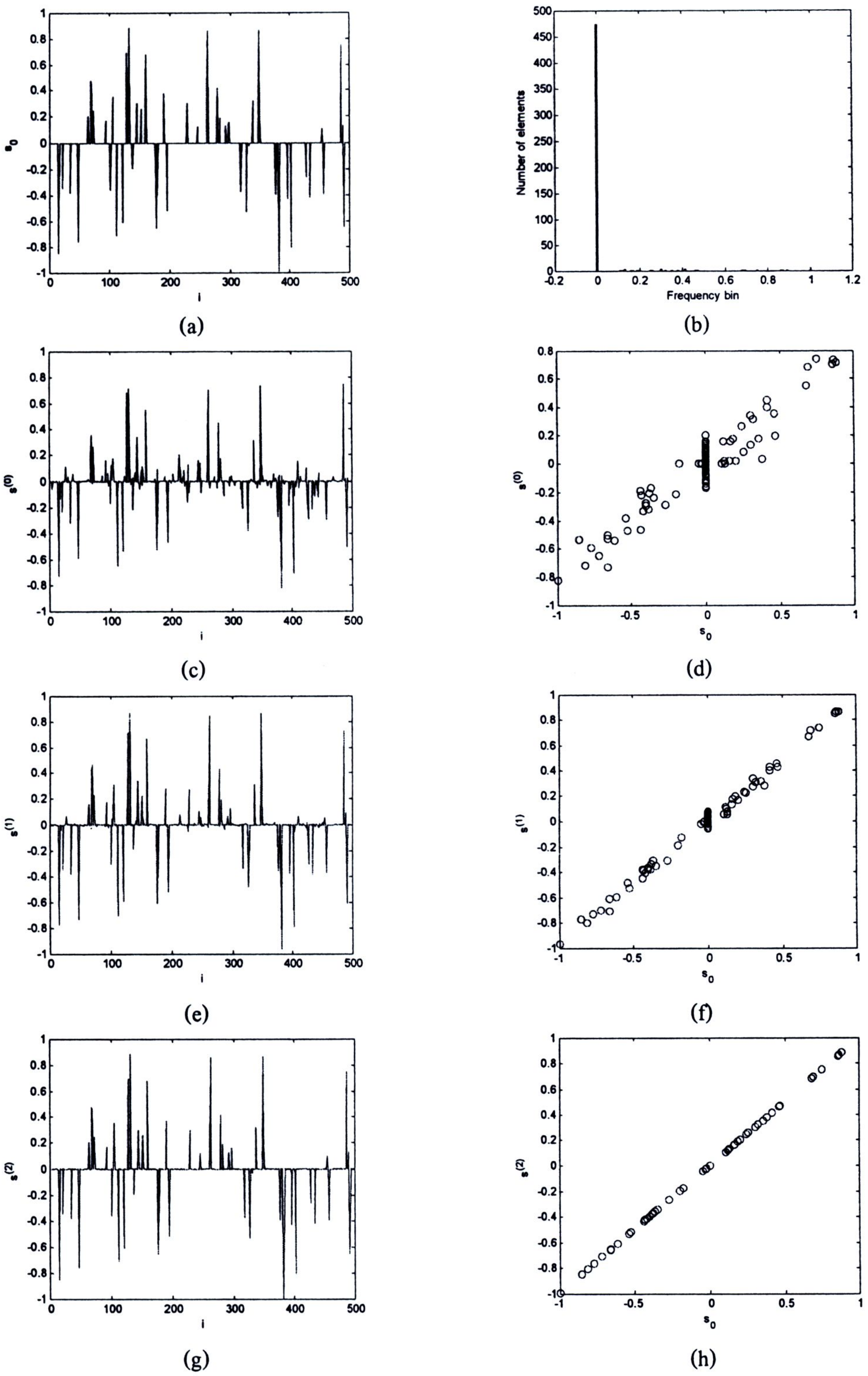

Figure 9 Sparse signal recovery using HSR algorithm. (a) Original cofficient vector s_0 on the interval [-1,1], length $N = 500$, with 45 spikes and (b) its histogram. (c) Reconstructed cofficient vector $s^{(0)}$ and (d) scatter plot (coefficient-by-coefficient of s_0 versus its reconstruction) using unweighted ℓ_1-minimization. (e) Reconstructed cofficient vector $s^{(1)}$ after the first reweighted iteration and (d) its scatter plot. (g) Reconstructed cofficient vector $s^{(2)}$ after the second reweighted iteration and (h) its scatter plot.

nearest empty frequency bin of histogram of absolute coefficient vector $|s^{(0)}|$. For example, Figure 10 shows the histogram of fault absolute coefficient vector $|s^{(0)}|$. There are not only zero entries in zero index, but also the other zero entries might be shifted to the other values around zero index. The nearest empty frequency bin of this example is about 0.08 so that the algorithm has decided to select the threshold τ=0.08 for containing the near-zero solutions to be zero candidate entries.

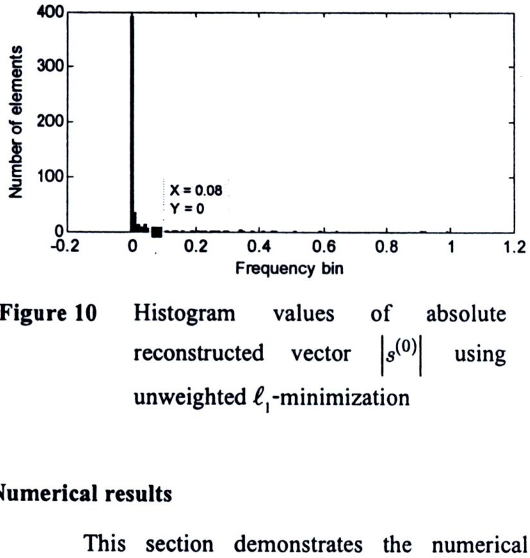

Figure 10 Histogram values of absolute reconstructed vector $|s^{(0)}|$ using unweighted ℓ_1-minimization

Numerical results

This section demonstrates the numerical experiments that the concept of ℓ_1-ball analysis can do manifestly in practical. However, reweighted ℓ_1-minimization is currently not known the method to define appropriate parameter ε in weighting function (10). Thus, the results can present sufficiently the percentage of successful reconstruction via ℓ_1-minimization, HSR and AAR algorithms.

The first experiment considers a normalized uniformly distribution K-sparse signal vector of size 250-dimensional and defines the measurement matrix Φ as a zero-mean normal (Gaussian) matrix of size 50×250, generated once for 200 trails of each K-sparse. This experiment categorizes the reconstructed signal which has PSNR ≥ 80 as the exact reconstructed signal. Let the expanding rate $\mu = 0.01$ and set 5 iterations for all computational experiments.

In Figure 11, the comparison graph shows that the percentages of HSR and AAR algorithms outperform ℓ_1-minimization about 12.63% and 13.38%, respectively.

The second experiment applies these algorithms to recover an example of manmade image of size 25×25 with 34.50% K-sparse. The numerical result shows that PSNRs of its reconstructions via ℓ_1-minimization, HSR and AAR algorithms as shown in Figure 12.

In the last experiment, an angiogram MRI image of size 432×338 (Figure 13(a)) with 58.14% K-sparse and a throat MRI image of size 336×337 (Figure 14(a)) with 60.29% K-sparse are cropped to the undersized image of size 25×25 which are the representation of original image with 64.80% K-sparse and 59.04% respectively.

Figure 13 and Figure 14 show the reconstructions of angiogram and neck MRI images respectively, via ℓ_1-minimization, HSR and AAR algorithms.

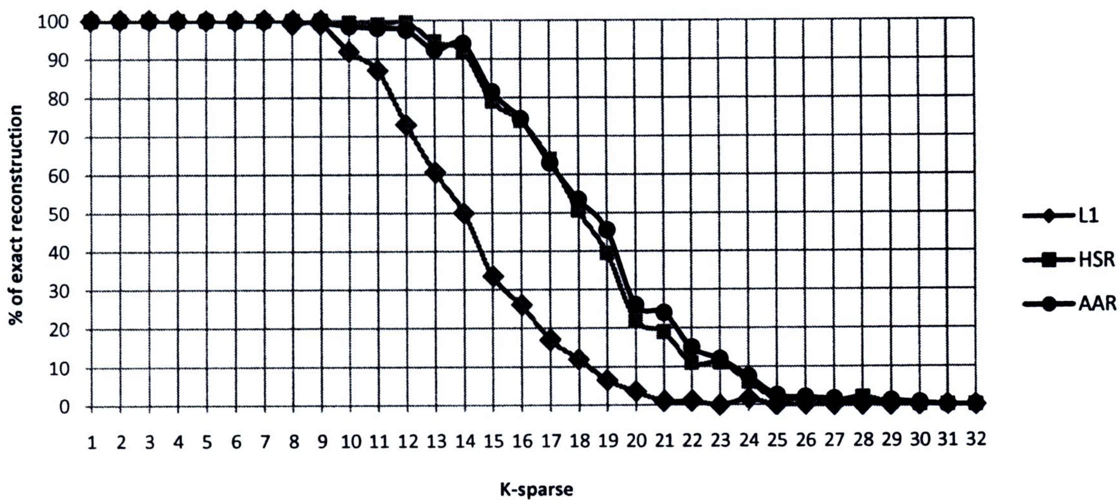

Figure 11 Comparison of ℓ_1-minimization, HSR and AAR algorithms in compressive sampling reconstruction

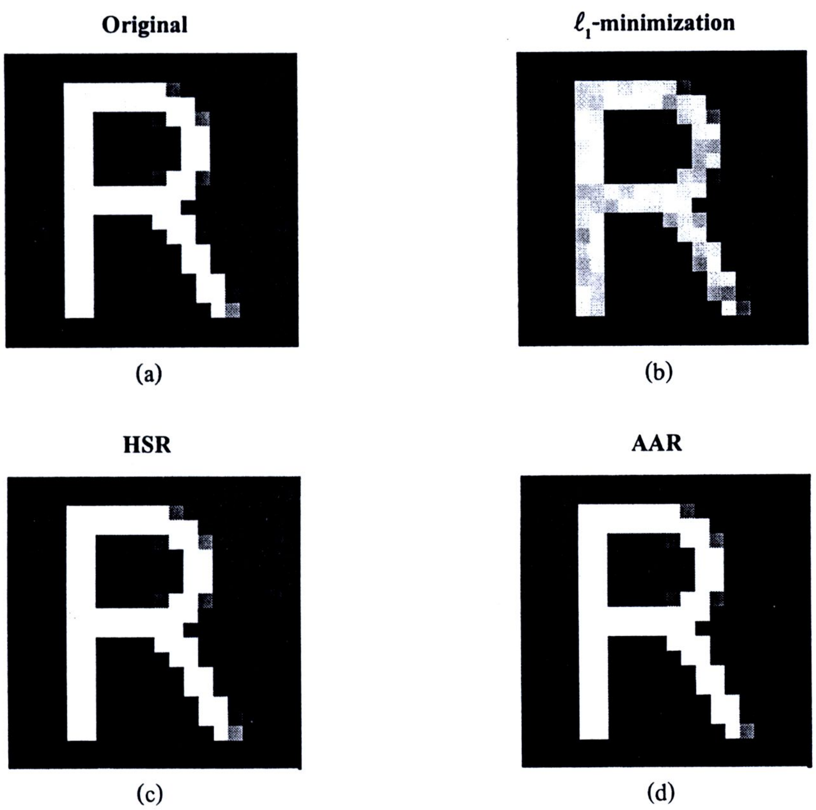

Figure 12 Example of manmade images in compressive sampling: (a) original image (b) ℓ_1-minimization, PSNR=61.83 dB (c) reconstructed image when using HSR algorithm, PSNR=539.21 dB and (d) reconstructed image when using AAR algorithm, PSNR=545.02 dB

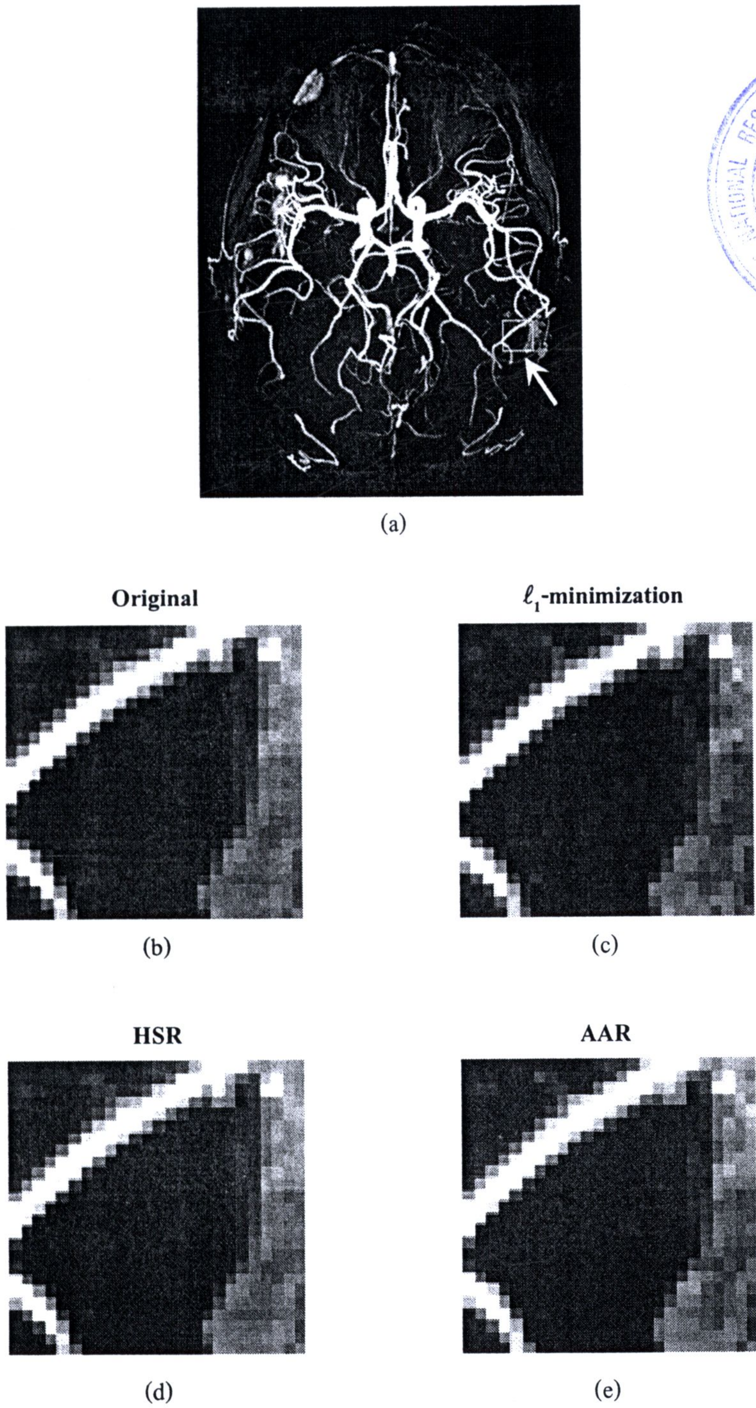

Figure 13 Angiogram MRI images in compressive sampling reconstruction: (a) original MRI image of size 432×338 (Dyck and Wilson, 2006) (b) original cropped MRI image of size 25×25 (framed in Figure 16 (a)) (c) reconstructed cropped MRI image when using ℓ_1-minimization, PSNR=71.30 dB (d) HSR algorithm, PSNR=597.56 dB and (e) AAR algorithm, PSNR=110.88 dB

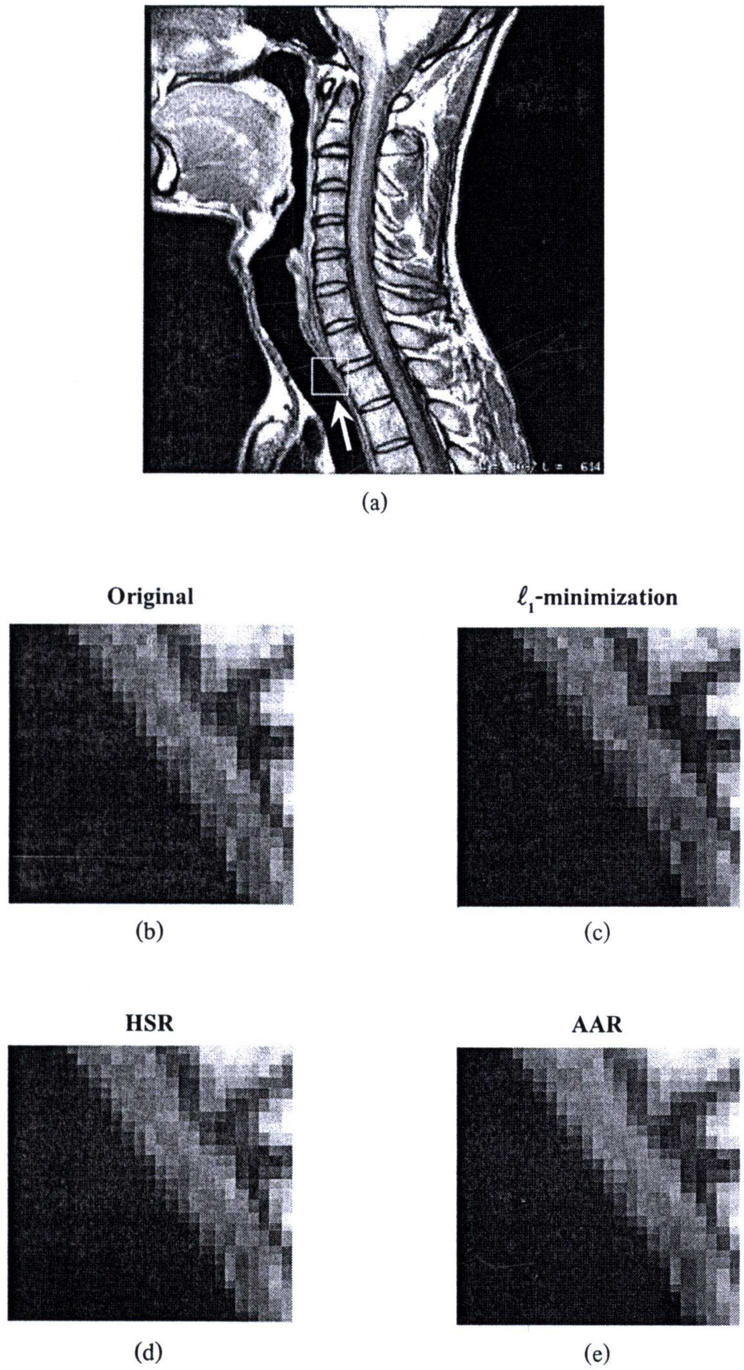

Figure 14 Throat MRI images in compressive sampling reconstruction: (a) original MRI image of size 336×337 (Slocum, 2009) (b) original cropped MRI image of size 25×25 (framed in Figure 16 (a)) (c) reconstructed cropped MRI image when using ℓ_1-minimization, PSNR=69.26 dB (d) HSR algorithm, PSNR=574.92 dB and (e) AAR algorithm, PSNR=585.90 dB

Results and Discussions

HSR algorithm can recover the exact signal with high probability but this algorithm needs to know a number of original zero entries. However, AAR algorithm is designed to cope with this problem although the percentage of exact reconstruction is little lower than HSR algorithm. This means that AAR algorithm can supersede HSR algorithm in case that a number of original zero entries are not known.

Even though, from the results, the HSR sometimes yields higher PSNR than that of AAR (sometimes it is the other way round), the main purpose of this paper is to provide methods that give better performance than ℓ_1-minimization reconstruction.

It is also an open question in compressive sampling that when and what kind of signals for the compressive sampling reconstruction process which should work since the main result (Candès, Romberg and Tao, 2006) is only proved with high probability of success when the signal is sparse enough.

In this paper, the enhancements of reweighted algorithms are proposed to find the appropriate weighting matrix. As yet there are many undetermined questions for some properties as below,

- What condition does the algorithm converge?
- How many iterations do the solution converge?
- Can these weighting functions and algorithms apply to the other applications?

Furthermore, since our experiments are only applied by the linear programming so that there are many interesting tools to solve this optimization problem such as Dantzig selector, basis pursuit and total variance minimization in addition to ℓ_1-Magic (Candès and Romberg, 2005). For further works, we will search out the theoretical supports for above questions and apply to the other applications such as signal recognition, remote sensing image, image processing, etc.

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CURRICULUM VITAE

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