

CHAPTER III

MAIN RESULTS

In this chapter, we study the ruin probability of the discrete-time surplus process under the claim arrival times $T_n = n, n \in \mathbb{N}$ and $\{X_n K_n, n \in \mathbb{N}\}$ is an independent and identically distributed claim size process where X_n be the positive random variable and K_n be the random variable has bernoulli distributed.

3.1 Model Descriptions

We consider the discrete-time surplus process at the situation that the possible insolvency can occur only at claim arrival times $T_n = n, n \in \mathbb{N}$. The n th claim arriving at time n can cause the insurance company has to pay $X_n K_n$. where X_n be the positive random variable and K_n be the random variable has bernoulli distributed defined in the probability space (Ω, \mathcal{F}, P) . The random variable K_n describes the situation that the claim size at time n can be occur or not. If K_n takes value 1 with probability p , then the n th claim arriving at time n cause the payment X_n . On the other hand, K_n takes value 0 with probability $1 - p$, then the n th claim arriving at time n is zero.

We assume that $\{X_n, n \in \mathbb{N}\}$ and $\{K_n, n \in \mathbb{N}\}$ are independent and identically distributed (i.i.d.) process. Throughout this thesis, for the sake of convenience, we assume that the process $\{X_n K_n\}$ is i.i.d., so we make the following assumption;

Assumption 1: Independent Assumption (IA)

$\{X_n K_n, n \in \mathbb{N}\}$ is an mutually independent and identically distributed (i.i.d.) process.

Next, let the constant $c_0 > 0$ represent the premium rate for one unit time which is calculated by the expected value principle, i.e.,

$$c_0 = (1 + \theta_0)E(X_1) \tag{3.1.1}$$

where $\theta_0 > 0$ is the safety loading of the insurer. Thus, the quantity $c_0 n$ describes inflow of capital into the business by time n , and the random variable $\sum_{i=1}^n X_i K_i$ describes the outflow of capital due to payments for claims occurring in $\{1, 2, 3, \dots, n\}$. Therefore, the quantity

$$U_n = u + c_0 n - \sum_{i=1}^n X_i K_i \quad (3.1.2)$$

is the surplus at time n with the constant $U_0 = u > 0$ as the initial capital.

3.2 Ruin Probability Behaviors

Let $\{U_n, n \in \mathbb{N}\}$ be the discrete-time surplus process and $\{X_n K_n, n \in \mathbb{N}\}$ be an i.i.d. claim size process as in section 3.1. We consider the finite time ruin probabilities of the process $\{U_n, n \in \mathbb{N}\}$ which is driven by the i.i.d. claim size process $\{X_n K_n, n \in \mathbb{N}\}$ and the premium rate $c_0 > 0$.

Let $u \geq 0$ be an initial capital. For each $n \in \mathbb{N}$, we let

$$\varphi_n(u) := P(U_1 \geq 0, U_2 \geq 0, U_3 \geq 0, \dots, U_n \geq 0 | U_0 = u) \quad (3.2.1)$$

denote the *survival probability* at times n . Thus, the *ruin probability* at one of times $1, 2, 3, \dots, n$ is denoted by

$$\begin{aligned} \Phi_n(u) &= 1 - \varphi_n(u) \\ &= P(U_i < 0 \text{ for some } i \in \{1, 2, 3, \dots, n\} | U_0 = u). \end{aligned} \quad (3.2.2)$$

3.2.1 Ruin and Survival Probability

We define the *total claim size process* $\{\Lambda_n, n \in \mathbb{N}\}$ by

$$\Lambda_n := X_1 K_1 + X_2 K_2 + X_3 K_3 + \dots + X_n K_n \quad (3.2.3)$$

for all $n \in \mathbb{N}$. Next, we let $N \in \mathbb{N}$ be given, the survival probability at time N (as mentioned in (3.2.1)) can be expressed as follows:

$$\begin{aligned} \varphi_N(u) &= P(\Lambda_1 \leq u + c_0, \Lambda_2 \leq u + 2c_0, \dots, \Lambda_N \leq u + Nc_0) \\ &= P\left(\bigcap_{k=1}^N (\Lambda_k \leq u + kc_0)\right). \end{aligned} \quad (3.2.4)$$

From equation (3.2.4) , we have

$$\varphi_N(u) = E \left[\prod_{k=1}^N I_{(-\infty, 0]}(\Lambda_k - kc_0 - u) \right] \quad (3.2.5)$$

where

$$I_A(u) = \begin{cases} 1, & u \in A, \\ 0, & u \notin A \end{cases}$$

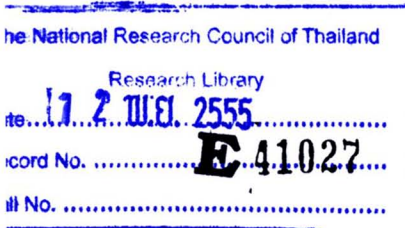
for all $A \subset \mathbb{R}$. For each $a \in \mathbb{R}$, we obtain

$$I_{(-\infty, 0]}(a - u) = \begin{cases} 1, & a \leq u, \\ 0, & u < a. \end{cases}$$



Then $I_{(-\infty, 0]}(a - u)$ is increasing and right continuous in x . This implies that $\prod_{k=1}^N I_{(-\infty, 0]}(a_k - u)$ is also increasing and right continuous in x where $a_k \in \mathbb{R}$ and $k = 1, 2, 3, \dots, N$. By the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \lim_{v \rightarrow u^+} \varphi_N(v) &= \lim_{v \rightarrow u^+} E \left[\prod_{k=1}^N I_{(-\infty, 0]}(\Lambda_k - kc_0 - v) \right] \\ &= E \left[\lim_{v \rightarrow u^+} \prod_{k=1}^N I_{(-\infty, 0]}(\Lambda_k - kc_0 - v) \right] \\ &= E \left[\prod_{k=1}^N I_{(-\infty, 0]}(\Lambda_k - kc_0 - u) \right] \\ &= \varphi_N(u) \quad \text{for all } u \in \mathbb{R}. \end{aligned} \quad (3.2.6)$$



Therefore, $\varphi_N(u)$ is increasing and right continuous. Moreover, we can conclude that $\Phi_N(u) = 1 - \varphi_N(u)$ is decreasing and also right continuous. Let $F_{X_1 K_1}(u)$ be the distribution function of $X_1 K_1$, i.e.,

$$F_{X_1 K_1}(u) = P(X_1 K_1 \leq u)$$

for all $u \in \mathbb{R}$. Since the claim size process $\{X_n K_n, n \in \mathbb{N}\}$ is i.i.d., we obtain

$$F_{X_N K_N}(u) = F_{X_1 K_1}(u)$$

for all $u \in \mathbb{R}$ and all $N \in \mathbb{N}$.

Next, we shall find the recursive formula for the survival probability in Lemma (3.2.2). In order to prove this lemma, we shall use the property which is given as follows:

Proposition 3.2.1 *Let (Ω, \mathcal{F}, P) be a probability space and $u, c \geq 0$ be given. If X and K are independent, then*

1. $(XK \leq u + c) \cap (K = 1) = (X \leq u + c) \cap (K = 1)$
2. $(XK \leq u + c) \cap (K = 0) = (K = 0)$.

Proof Let $\omega \in \Omega$, we have

$$\begin{aligned} \omega \in (XK \leq u + c) \cap (K = 1) &\iff X(\omega)K(\omega) \leq u + c \text{ and } K(\omega) = 1 \\ &\iff X(\omega) \leq u + c \text{ and } K(\omega) = 1 \\ &\iff \omega \in (X \leq u + c) \cap (K = 1). \end{aligned} \quad (3.2.7)$$

This means that

$$(XK \leq u + c) \cap (K = 1) = (X \leq u + c) \cap (K = 1).$$

Similarly, we obtain that

$$\begin{aligned} \omega \in (XK \leq u + c) \cap (K = 0) &\iff X(\omega)K(\omega) \leq u + c \text{ and } K(\omega) = 0 \\ &\iff 0 \leq u + c \text{ and } K(\omega) = 0 \\ &\iff K(\omega) = 0 \\ &\iff \omega \in (K = 0). \end{aligned} \quad (3.2.8)$$

Therefore,

$$(XK \leq u + c) \cap (K = 0) = (K = 0).$$

This completes the proof. □

Lemma 3.2.2 *Under Assumption 1, let $c_0 > 0, u \geq 0$ and $N \in \mathbb{N}$ be given. If $\{X_n K_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process and $P(K_1 = 1) = p = 1 - P(K_1 = 0)$, then the survival probability at one of the times $1, 2, 3, \dots, N$ satisfies the following equation*

$$\varphi_N(u) = (1 - p)\varphi_{N-1}(u + c_0) + p \int_{-\infty}^{u+c_0} \varphi_{N-1}(u + c_0 - x) dF_{X_1}(x) \quad (3.2.9)$$

where

$$\varphi_0(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Proof We prove (3.2.9) by induction. We start with $N = 1$. Since $\varphi_0(u) = 1$ for all $u \geq 0$, we have

$$\begin{aligned}
 \varphi_1(u) &= P(X_1 K_1 \leq u + c_0) \\
 &= P\left((X_1 K_1 \leq u + c_0) \cap [(K_1 = 0) \cup (K_1 = 1)]\right) \\
 &= P\left([(X_1 K_1 \leq u + c_0) \cap (K_1 = 0)] \cup [(X_1 K_1 \leq u + c_0) \cap (K_1 = 1)]\right) \\
 &= P\left((X_1 K_1 \leq u + c_0) \cap (K_1 = 0)\right) + P\left((X_1 K_1 \leq u + c_0) \cap (K_1 = 1)\right)
 \end{aligned} \tag{3.2.10}$$

By Proposition (3.2.1), we obtain

$$\varphi_1(u) = P(K_1 = 0) + P\left((X_1 \leq u + c_0) \cap (K_1 = 1)\right)$$

Since X_1 and K_1 are independent, we have

$$\begin{aligned}
 \varphi_1(u) &= P(K_1 = 0) + P(X_1 \leq u + c_0) \cdot P(K_1 = 1) \\
 &= (1 - p) + p \int_{-\infty}^{u+c_0} dF_{X_1}(x).
 \end{aligned} \tag{3.2.11}$$

Since $u + c_0 - x \geq 0$ for all $x \leq u + c_0$, we have

$$\varphi_0(u + c_0 - x) = 1 \quad \text{for all } x \leq u + c_0.$$

Therefore,

$$\varphi_1(u) = (1 - p) + p \int_{-\infty}^{u+c_0} \varphi_0(u + c_0 - x) dF_{X_1}(x).$$

This proves for $N = 1$. Now assume that (3.2.9) holds for $N = k \geq 2$. Then

$$\begin{aligned}
\varphi_{k+1}(u) &= \mathbb{P}(\Lambda_1 \leq u + c_0, \Lambda_2 \leq u + 2c_0, \dots, \Lambda_{k+1} \leq u + (k+1)c_0) \\
&= \mathbb{P}\left(\bigcap_{i=1}^{k+1} (\Lambda_i \leq u + ic_0)\right) \\
&= \mathbb{P}\left(\bigcap_{i=1}^{k+1} (\Lambda_i \leq u + ic_0) \cap [(K_1 = 0) \cup (K_1 = 1)]\right) \\
&= \mathbb{P}\left(\left[\bigcap_{i=1}^{k+1} (\Lambda_i \leq u + ic_0) \cap (K_1 = 0)\right] \cup \right. \\
&\quad \left. \left[\bigcap_{i=1}^{k+1} (\Lambda_i \leq u + ic_0) \cap (K_1 = 1)\right]\right) \\
&= \mathbb{P}\left(\left[\bigcap_{i=2}^{k+1} (\Lambda_i - X_1 K_1 \leq u + ic_0) \cap (K_1 = 0)\right] \cup \right. \\
&\quad \left. \left[\left\{\bigcap_{i=2}^{k+1} (\Lambda_i - X_1 K_1 \leq u + ic_0 - X_1) \cap (X_1 \leq u + c)\right\} \cap (K_1 = 1)\right]\right) \\
&= \mathbb{P}\left(\left[\bigcap_{i=2}^{k+1} (\Lambda_i - X_1 K_1 \leq u + ic_0) \cap (K_1 = 0)\right] + \right. \\
&\quad \left. \mathbb{P}\left[\left\{\bigcap_{i=2}^{k+1} (\Lambda_i - X_1 K_1 \leq u + ic_0 - X_1) \cap (X_1 \leq u + c)\right\} \cap (K_1 = 1)\right]\right).
\end{aligned} \tag{3.2.12}$$

Firstly, we consider the first term of right-hand side of (3.2.12). Since $X_i K_i$ and $X_j K_j$ have the same distribution for all $i, j \geq 1$ and $i \neq j$, we have

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{i=2}^{k+1} (\Lambda_i - X_1 K_1 \leq u + ic_0) \cap (K_1 = 0)\right] \\
&= \mathbb{P}\left[\bigcap_{i=1}^k (\Lambda_i \leq u + (i+1)c_0) \cap (K_1 = 0)\right] \\
&= \mathbb{P}(K_1 = 0) \cdot \mathbb{P}\left(\bigcap_{i=1}^k (\Lambda_i \leq u + (i+1)c_0)\right) \\
&= (1-p) \cdot \varphi_k(u+c).
\end{aligned} \tag{3.2.13}$$

Secondary, we consider the second term of right-hand side of (3.2.12), then

$$\begin{aligned}
& \mathbb{P}\left[\left\{\bigcap_{i=2}^{k+1} (\Lambda_i - X_1 K_1 \leq u + ic_0 - X_1) \cap (X_1 \leq u + c)\right\} \cap (K_1 = 1)\right] \\
&= \mathbb{P}\left[\bigcap_{i=1}^k (\Lambda_i \leq u + (i+1)c_0 - X_1) \cap (X_1 \leq u + c) \cap (K_1 = 1)\right] \\
&= \mathbb{P}(K_1 = 1) \cdot \mathbb{P}\left(\bigcap_{i=1}^k (\Lambda_i \leq u + (i+1)c_0 - X_1) \cap (X_1 \leq u + c)\right) \\
&= p \cdot \int_{-\infty}^{u+c_0} \varphi_k(u + c - x) dF_{X_1}(x). \tag{3.2.14}
\end{aligned}$$

Combining equation (3.2.14) with equation (3.2.13), we obtain

$$\varphi_{k+1}(u) = (1 - p) \cdot \varphi_k(u + c) + p \cdot \int_{-\infty}^{u+c_0} \varphi_k(u + c - x) dF_{X_1}(x) \tag{3.2.15}$$

which proves (3.2.9) for $N = k + 1$ and concludes the proof. \square

Theorem 3.2.3 *Under Assumption 1, let $c_0 > 0, u \geq 0$ and $N \in \mathbb{N}$ be given. If $\{X_n K_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process and $\mathbb{P}(K_1 = 1) = p = 1 - \mathbb{P}(K_1 = 0)$, then the ruin probability at one of the times $1, 2, 3, \dots, N$ satisfies the following equation*

$$\Phi_N(u) = \Phi_1(u) + (1 - p)\Phi_{N-1}(u + c_0) + p \int_{-\infty}^{u+c_0} \Phi_{N-1}(u + c_0 - x) dF_{X_1}(x) \tag{3.2.16}$$

where $\Phi_0(u) = 0$.

Proof We consider the following cases:

Case 1. $N = 1$. Since $\Phi_0(u) = 0$ for all $u \geq 0$, we have

$$\Phi_1(u) = \Phi_1(u) + p \int_{-\infty}^{u+c_0} \Phi_0(u + c_0 - x) dF_{X_1}(x) = 0. \tag{3.2.17}$$

Case 2. $N \geq 2$. Since $\Phi_N(u) = 1 - \varphi_N(u)$ and by Lemma (3.2.2), we have

$$\begin{aligned}
\Phi_N(u) &= 1 - \varphi_N(u) \\
&= 1 - \left[(1-p)\varphi_{N-1}(u+c) + p \cdot \int_{-\infty}^{u+c_0} \varphi_{N-1}(u+c_0-x) dF_{X_1}(x) \right] \\
&= 1 - \left[(1-p)(1 - \Phi_{N-1}(u+c)) + p \cdot \int_{-\infty}^{u+c_0} (1 - \Phi_{N-1}(u+c_0-x)) dF_{X_1}(x) \right] \\
&= 1 - \left[(1 - \Phi_{N-1}(u+c) - p + p \cdot \Phi_{N-1}(u+c) + p \cdot \int_{-\infty}^{u+c_0} dF_{X_1}(x) \right. \\
&\quad \left. - p \cdot \int_{-\infty}^{u+c_0} \Phi_{N-1}(u+c_0-x) dF_{X_1}(x) \right] \\
&= \Phi_{N-1}(u+c) - p \cdot \Phi_{N-1}(u+c) + p - p \cdot \int_{-\infty}^{u+c_0} dF_{X_1}(x) \\
&\quad + p \cdot \int_{-\infty}^{u+c_0} \Phi_{N-1}(u+c_0-x) dF_{X_1}(x) \\
&= (1-p)\Phi_{N-1}(u+c) + p(1 - \int_{-\infty}^{u+c_0} dF_{X_1}(x)) \\
&\quad + p \cdot \int_{-\infty}^{u+c_0} \Phi_{N-1}(u+c_0-x) dF_{X_1}(x) \\
&= \Phi_1(u) + (1-p)\Phi_{N-1}(u+c_0) + p \int_{-\infty}^{u+c_0} \Phi_{N-1}(u+c_0-x) dF_{X_1}(x).
\end{aligned} \tag{3.2.18}$$

This completes the proof. □

3.2.2 Exponential Claims

We consider the discrete-time surplus process (3.1.2) with $\{X_n, n \geq 1\}$ is an i.i.d. random variable exponential claim size distribution and $\{K_n, n \geq 1\}$ is a sequence of i.i.d. random variable with Bernoulli distribution.

Corollary 3.2.4 *Under Assumption 1, let $c_0 > 0, u \geq 0$ and $N \in \mathbb{N}$ be given. Assume that X_1 has exponential distributed with intensity $\lambda > 0$, i.e., X_1 has the probability density function*

$$f_{X_1}(x) = \lambda e^{-\lambda x}$$

and $P(K_1 = 1) = p = 1 - P(K_1 = 0)$, then the obtain ruin probability is in the following form

$$\Phi_N(u) = \Delta_{N-1}(u) \cdot p e^{-\lambda(u+Nc_0)} \quad (3.2.19)$$

where

$$\Delta_0(u) = 1, \quad \Delta_N(u) = e^{N\lambda c_0} + (1-p)\Delta_{N-1} + \lambda p \int_{-\infty}^{u+c_0} \Delta_{N-1}(x) dx$$

and premium rate $c_0 > E[X_1] = 1/\lambda$.

proof We proceed to show the expression (3.2.19) by induction. We start with $N = 1$. Since $\Delta_0(u) = 1$ for all $u \geq 0$, we have

$$\Delta_{N-1}(u) \cdot p e^{-\lambda(u+Nc_0)} = p e^{-\lambda(u+c_0)}.$$

Combining equation (3.2.19) with equation (3.2.11), we obtain

$$\Phi_1(u) = p \left(1 - \int_{-\infty}^{u+c_0} f(x) dx \right) = p \left(1 - [1 - e^{-\lambda(u+c_0)}] \right) = p e^{-\lambda(u+c_0)}. \quad (3.2.20)$$

This proves (3.2.19) for $N = 1$. Now, we assume that (3.2.19) holds for $N = k \geq 2$, then we need to show that the formula is also true for case of $(k+1)$, that is,

$$\Phi_{k+1}(u) = \Delta_k(u) \cdot p e^{-\lambda(u+(k+1)c_0)}. \quad (3.2.21)$$

By Theorem (3.2.16) and the induction assumption that the case of k is true, we have

$$\begin{aligned}
\Phi_{k+1}(u) &= \Phi_1(u) + (1-p)\Phi_k(u+c_0) + p \int_{-\infty}^{u+c_0} \Phi_k(u+c_0-x) dF_{X_1}(x) \\
&= \Phi_1(u) + (1-p)\Phi_k(u+c_0) + p \int_{-\infty}^{u+c_0} \Phi_k(x) f(u+c-x) dx \\
&= pe^{-\lambda(u+c_0)} + (1-p) \left[\Delta_{k-1}(u+c_0) (pe^{-\lambda(u+(k+1)c_0)}) \right] \\
&\quad + \lambda p \int_{-\infty}^{u+c_0} \Delta_{k-1}(x) pe^{-\lambda(u+(k+1)c_0)} dx \\
&= (e^{k\lambda c}) pe^{-\lambda(u+(k+1)c_0)} + (1-p) \left[\Delta_{k-1}(u+c_0) (pe^{-\lambda(u+(k+1)c_0)}) \right] \\
&\quad + \lambda p \int_{-\infty}^{u+c_0} \Delta_{k-1}(x) dx \left(pe^{-\lambda(u+(k+1)c_0)} \right) \\
&= \left[e^{k\lambda c} + (1-p)\Delta_{k-1}(u+c_0) + \lambda p \int_{-\infty}^{u+c_0} \Delta_{k-1}(x) dx \right] pe^{-\lambda(u+(k+1)c_0)} \\
&= \Delta_k(u) \cdot pe^{-\lambda(u+(k+1)c_0)}. \tag{3.2.22}
\end{aligned}$$

This completes the proof. □

Remark 3.2.5 We observe that $\Delta_N(u)$ is a polynomial function of degree N whereas λ, c_0 and p are parameter combinations, for example:

$$\begin{aligned}
\Delta_0(u) &= 1 \\
\Delta_1(u) &= e^{\lambda c_0} + (1-p) + \lambda p(u+c_0) \\
\Delta_2(u) &= e^{2\lambda c_0} + (1-p) \left[e^{\lambda c_0} + (1-p) + \lambda p(u+2c_0) \right] \\
&\quad + \lambda p \left[e^{\lambda c_0}(u+c_0) + (1-p)(u+c_0) + \frac{\lambda p(u+2c_0)^2}{2} \right].
\end{aligned}$$

3.3 Numerical Results

We provide numerical illustrations of the main results. We calculate the finite-time ruin probabilities of the discrete-time surplus process (3.1.2) in the case of $\{X_n, n \in \mathbb{N}\}$, a sequence of i.i.d. exponential distribution by using equation(3.2.19) in Corollary(3.2.4). The chosen model parameter combinations are $\lambda = 1.0; p = 0.5; \theta = 0.1$ and 0.25 (i.e., $c = 1.1$ and 1.25 , respectively); and $u = 0$ and 5 .

Table 3.1 Ruin Probabilities ($\Phi_N(u)$) in the Discrete-Time Risk Model with exponential Claims

N	$u = 0$		$u = 5$	
	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$
5	0.27869	0.22771	0.00560	0.00393
10	0.29672	0.23767	0.00799	0.00502
15	0.30058	0.23917	0.00879	0.00527
20	0.30167	0.23947	0.00906	0.00533
25	0.30201	0.23954	0.00916	0.00534
30	0.30213	0.23956	0.00920	0.00535
35	0.30217	0.23956	0.00922	0.00535
40	0.30219	0.23956	0.00922	0.00535
45	0.30219	0.23957	0.00922	0.00535
50	0.30219	0.23957	0.00923	0.00535
60	0.30220	0.23957	0.00923	0.00535
70	0.30220	0.23957	0.00923	0.00535
80	0.30220	0.23957	0.00923	0.00535

Table 3.1 shows values of $\Phi_N(u)$ to five decimal places for various model parameters which is computed from equations (3.2.19).

From Table 3.1, if we choose $p = 1$, the result is the same as in Chan and Zhang (2006)[5].

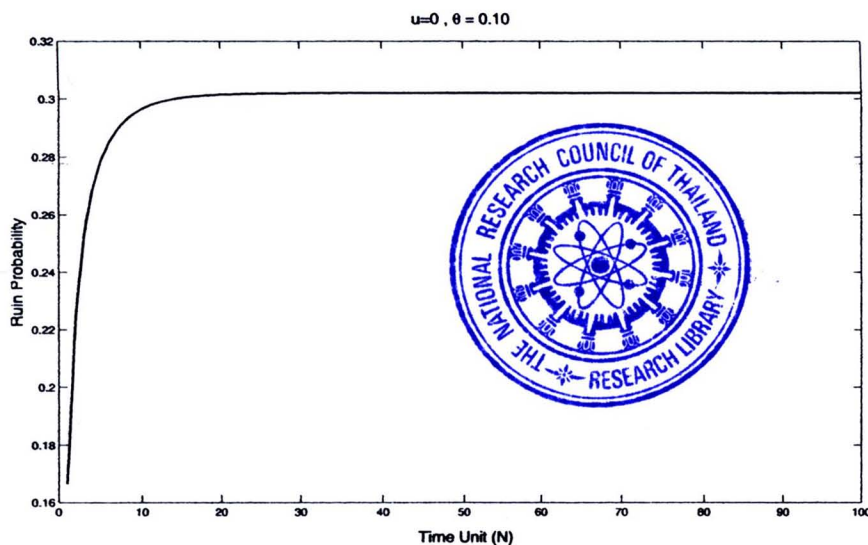


Figure 3.1 Ruin Probabilities in the Discrete-Time Risk Model with Exponential Claims ($\lambda = 1, p = 0.5, u = 0$ and $\theta = 0.10$)

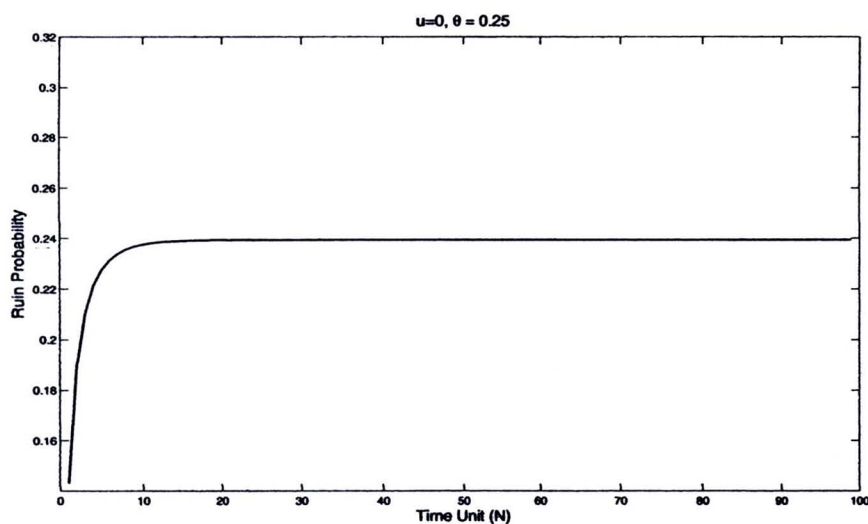


Figure 3.2 Ruin Probabilities in the Discrete-Time Risk Model with Exponential Claims ($\lambda = 1, p = 0.5, u = 0$ and $\theta = 0.25$)

Figure 3.1 and 3.2 show the approximation of ruin probabilities for the various values of unit time N from 1 to 82 as mentioned in Corollary (3.2.4). Here we choose $\lambda = 1.0, p = 0.5$ and $u = 0$. The chosen model parameter combinations are $\theta = 0.10$ and $\theta = 0.25$.

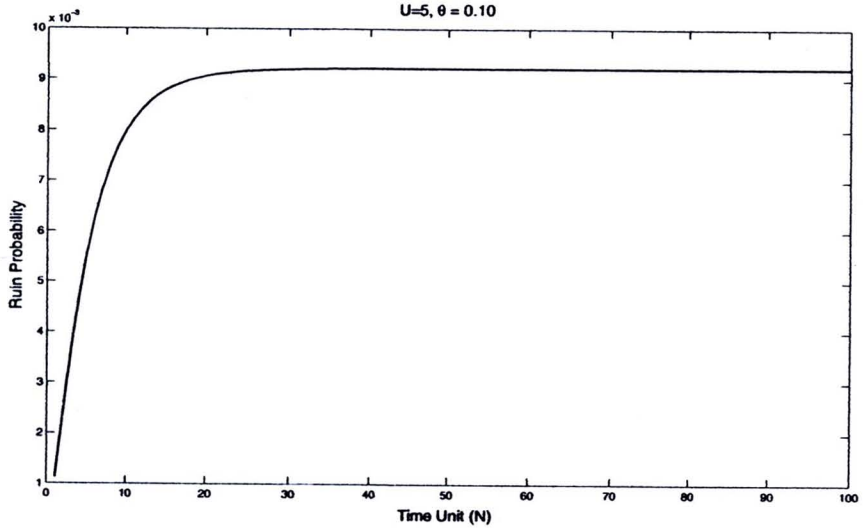


Figure 3.3 Ruin Probabilities in the Discrete-Time Risk Model with Exponential Claims ($\lambda = 1$, $p = 0.5$, $u = 5$ and $\theta = 0.10$)

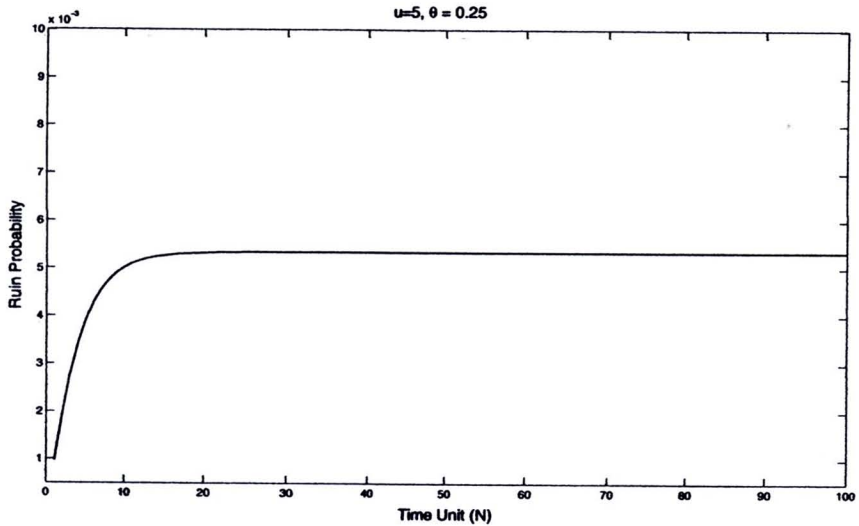


Figure 3.4 Ruin Probabilities in the Discrete-Time Risk Model with Exponential Claims ($\lambda = 1$, $p = 0.5$, $u = 5$ and $\theta = 0.25$)

Figure 3.3 and 3.4 show the approximation of ruin probabilities for the various values of unit time N from 1 to 82 as mentioned in Corollary (3.2.4). Here we choose $\lambda = 1.0$, $p = 0.5$ and $u = 5$. The chosen model parameter combinations are $\theta = 0.10$ and $\theta = 0.25$.

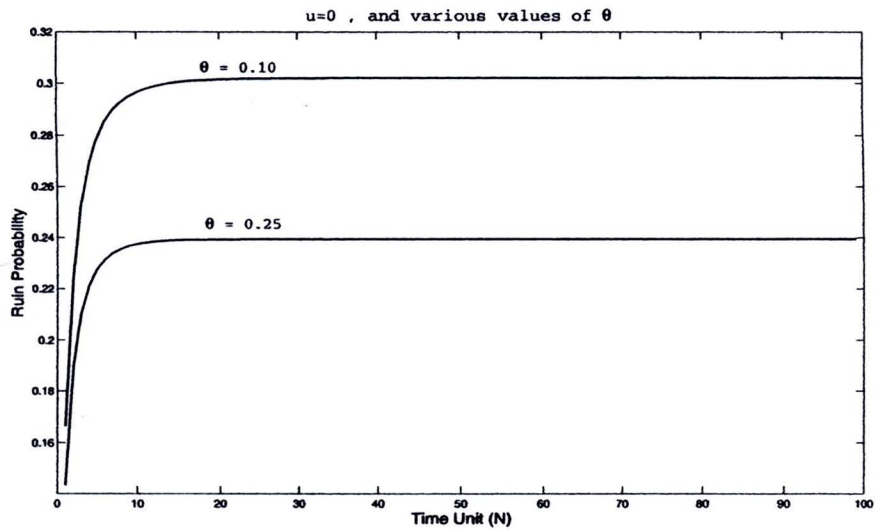


Figure 3.5 Ruin Probabilities in the Discrete-Time Risk Model with Exponential Claims ($\lambda = 1$, $p = 0.5$, $u = 0$ and various values of θ)

Figure 3.5 shows the comparison of ruin probabilities for various values of θ with $u = 0$ as mentioned in Corollary (3.2.4). Here we choose $\lambda = 0.10$ and $p = 0.5$. To compare the shape of the ruin probability functions, vertical and horizon axes of the graphs are rescaled linearly. It is clear that if ruin occurs, It is likely to occur sooner when $\theta = 0.25$ than when $\theta = 0.10$.

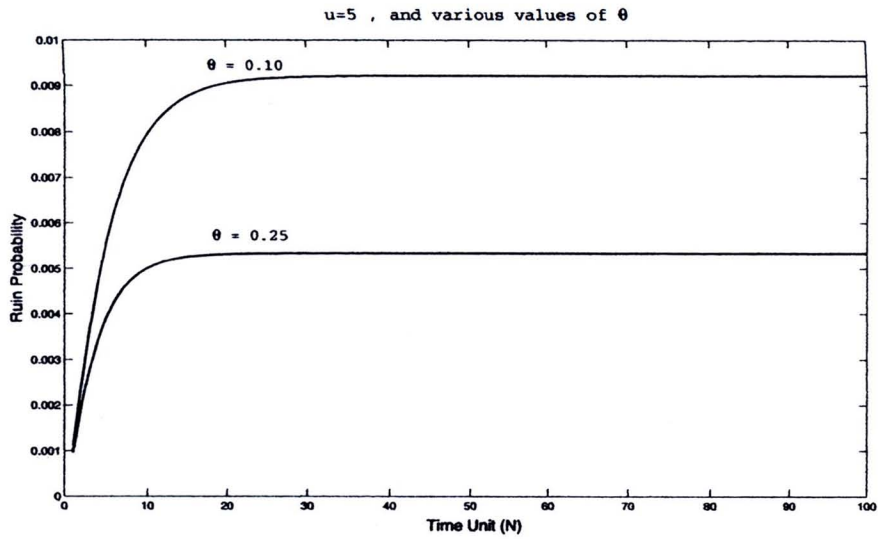


Figure 3.6 Ruin Probabilities in the Discrete-Time Risk Model with Exponential Claims ($\lambda = 1$, $p = 0.5$, $u = 5$ and various values of θ)

Figure 3.6 shows the comparison of ruin probabilities for various values of θ with $u = 5$ as mentioned in Corollary (3.2.4). Here we choose $\lambda = 0.10$ and $p = 0.5$. Again, for comparison purposes, the shape of the ruin probability functions, vertical and horizon axes of the graphs are rescaled linearly. The result is similarly to the case of $u = 0$.

From figure 3.5 and 3.6, we observe that, as n increase, values of $\Phi_N(u)$ converge to some constant. And, if the initial capital u is small, the values of the ruin probability function at one of the times to ruin will be greater than the larger initial capital comparing at the same time.

3.4 Geometric Interarrival Time Process

In this section we consider the discrete-time surplus process

$$V_0 = u, \quad V_n = V_{n-1} + c_0 Z_n - X_n \quad (3.4.1)$$

where $\{X_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process and $\{Z_n, n \in \mathbb{N}\}$ is an i.i.d. inter-arrival time process taking values in $N = \{0, 1, 2, \dots\}$, and the common probability function for Z is $\{p_i, i = 0, 1, 2, \dots\}$, where $p_i = P(Z = i)$. Let $\{T_n, n \in \mathbb{N}\}$ be an i.i.d. arrivaltime process, i.e.,

$$0 = T_0 \leq T_1 \leq T_2 \leq \dots$$

Thus, the interarrival time is defined by

$$Z_1 = T_1, \quad Z_2 = T_2 - T_1, \quad \dots, \quad Z_n = T_n - T_{n-1}.$$

The time of ruin is defined by

$$\tau(u) = \min\{n \geq 1 : V_n < 0\}. \quad (3.4.2)$$

Let $V_0 = u \geq 0$ be an initial capital. For each $n \in \mathbb{N}$, we let

$$\tilde{\varphi}_n(u) := P(V_1 \geq 0, V_2 \geq 0, V_3 \geq 0, \dots, V_n \geq 0 | V_0 = u) \quad (3.4.3)$$

denote the *survival probability* at times n . Thus, the *ruin probability* at one of times $1, 2, 3, \dots, n$ is denoted by

$$\begin{aligned} \tilde{\Phi}_n(u) &= 1 - \tilde{\varphi}_n(u) \\ &= P(V_i < 0 \text{ for some } i \in \{1, 2, 3, \dots, n\} | V_0 = u). \end{aligned} \quad (3.4.4)$$

Recall the discrete-time surplus process (3.1.2)

$$U_0 = u, \quad U_n = u + c_0 n - \sum_{i=1}^n X_i K_i.$$

Then the arrival time process $\{\tilde{T}_n, n \in \mathbb{N}\}$ can be defined as follow:

$$\tilde{T}_n = \min\{m \geq 1 \mid \sum_{i=1}^m K_i = n, K_m = 1\}$$

And the interarrival time is defined by

$$\tilde{Z}_n = \tilde{T}_n - \tilde{T}_{n-1}; \quad \tilde{T}_0 = 0.$$

Remark 3.4.1 The interarrival time process $\{\tilde{Z}_n, n \in \mathbb{N}\}$ has a geometric distribution.

Proof Let $i \in \{0, 1, 2, \dots\}$ and $\gamma \in \mathbb{N}$ be given, we consider the following cases:

Case 1. $n=1$, we have

$$\begin{aligned}
 P(\tilde{Z}_1 = i) &= P(\tilde{T}_1 = i) \\
 &= P(K_1 = 0, K_2 = 0, \dots, K_{i-1} = 0, K_i = 1) \\
 &= P(K_1 = 0)P(K_2 = 0) \cdots P(K_{i-1} = 0)P(K_i = 1) \\
 &= (1 - p)^{i-1}p.
 \end{aligned} \tag{3.4.5}$$

Case 2. $n \geq 2$, we have

$$\begin{aligned}
 P(\tilde{Z}_n = i) &= P(\tilde{T}_n - \tilde{T}_{n-1} = i) \\
 &= P(\tilde{T}_n - \tilde{T}_{n-1} = i \mid T_{n-1} = \gamma) \\
 &= \frac{P(K_{\gamma+1} = 0, K_{\gamma+2} = 0, \dots, K_{\gamma+(i-1)} = 0, K_{\gamma+i} = 1, \tilde{T}_{n-1} = \gamma)}{P(\tilde{T}_{n-1} = \gamma)}
 \end{aligned} \tag{3.4.6}$$

Since $\{K_n, n \in \mathbb{N}\}$ is an i.i.d. random variable, then

$$\begin{aligned}
 P(\tilde{Z}_n = i) &= \frac{P(K_{\gamma+1} = 0)P(K_{\gamma+2} = 0) \cdots P(K_{\gamma+(i-1)} = 0)P(K_{\gamma+i} = 1)P(\tilde{T}_{n-1} = \gamma)}{P(\tilde{T}_{n-1} = \gamma)} \\
 &= P(K_1 = 0)P(K_2 = 0) \cdots P(K_{i-1} = 0)P(K_i = 1) \\
 &= (1 - p)^{i-1}p.
 \end{aligned} \tag{3.4.7}$$

This completes the proof.

□

With a little effort, we have the following result.

Remark 3.4.2 We consider the first claim time N of the claim times defined by

$$N = \min\{n \geq 1 | K_n = 1\}.$$

Then the ruin probability of the discrete-time surplus process (3.4.1) at the first claim time with exponential claim size distribution is in the following form

$$\tilde{\Phi}_1(u) = pe^{-\lambda(u+c_0)} \cdot \frac{1}{1 - (1-p)e^{-\lambda c_0}}.$$

Proof Since $\{X_n K_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process, we consider the ruin probability at the first claim time

$$\begin{aligned} & P(u + c_0 Z_1 - X_1 < 0) \\ &= P((u + c_0 Z_1 - X_1 < 0) \cap [(Z_1 = 1) \cup (Z_1 = 2) \cup (Z_1 = 3) \cup \dots]) \\ &= P\left(\bigcup_{i=1}^{\infty} (u + c_0 Z_1 - X_1 < 0) \cap (Z_1 = i)\right) \\ &= \sum_{i=1}^{\infty} P((u + c_0 Z_1 - X_1 < 0) \cap (Z_1 = i)) \\ &= \sum_{i=1}^{\infty} P((u + c_0 i - X_1 < 0) \cap (Z_1 = i)) \\ &= \sum_{i=1}^{\infty} P(u + c_0 i - X_1 < 0) \cdot P(Z_1 = i) \\ &= \sum_{i=1}^{\infty} P(X_1 > u + c_0 i) \cdot P(K_1 = 0, K_2 = 0, \dots, K_{i-1} = 0, K_i = 1) \\ &= \sum_{i=1}^{\infty} P(X_1 > u + (i-1)c_0 + c_0) \cdot p(1-p)^{i-1} \end{aligned} \tag{3.4.8}$$

From Corollary (3.2.4), we have $\Phi_1(u) = P(X_1 > u + c_0) = p \cdot e^{-\lambda(u+c_0)}$. Then we obtain

$$\begin{aligned}
P(u + c_0 Z_1 - X_1 < 0) &= \sum_{i=1}^{\infty} \Phi_1(u + (i+1)c_0) \cdot p(1-p)^{i-1} \\
&= \sum_{i=1}^{\infty} p \cdot e^{-\lambda(u+(i-1)c_0+c_0)} \cdot p(1-p)^{i-1} \\
&= p \sum_{i=1}^{\infty} e^{-\lambda(u+ic_0)} \cdot p(1-p)^{i-1} \\
&= pe^{-\lambda u} \sum_{i=1}^{\infty} e^{-\lambda ic_0} \cdot p(1-p)^{i-1} \\
&= pe^{-\lambda u} \cdot M_{Z_1}(-\lambda c_0) \\
&= pe^{-\lambda u} \cdot \frac{pe^{-\lambda c_0}}{1 - (1-p)e^{-\lambda c_0}} \\
&= pe^{-\lambda(u+c_0)} \cdot \frac{1}{1 - (1-p)e^{-\lambda c_0}}
\end{aligned} \tag{3.4.9}$$

where M_{Z_1} is a moment generating function of Z_1 . □

We observe that, if we choose $P(K_1 = 1) = 1$ in the surplus process (3.1.2) and $Z_n = 1$ for all $n \in \mathbb{N}$ in the surplus process (3.4.1) (i.e., the claim occurs every unit time n), then the surplus process (3.1.2) and (3.4.1) are equivalent, and we have the ruin probability at the first claim time is in the form

$$\Phi_1(u) = \tilde{\Phi}_1(u) = e^{-\lambda(u+c_0)}.$$

And the recursive and explicit formula of $\Phi_n(u)$ is in the form

$$\begin{aligned}
\Phi_n(u) &= \Phi_{n-1}(u) + \frac{[\lambda(u+nc)]^{n-1}}{(n-1)!} e^{-\lambda(u+nc)} \frac{(u+c)}{(u+nc)} \\
&= \sum_{k=1}^n \frac{[\lambda(u+kc)]^{k-1}}{(k-1)!} e^{-\lambda(u+kc)} \frac{(u+c)}{(u+kc)}.
\end{aligned} \tag{3.4.10}$$

This result is the same as in Chan and Zhang (2006)[5].