CHAPTER 2 THEORY

2.1 Shallow Water Equations

The shallow water equations can be used to describe a thin layer of constant density fluid or the atmosphere in which their thickness or depth is much less than their horizontal extent. Assume also that there is no vertical wind shear, the vertical advection terms do not appear in the shallow water equations. Assume that the fluid is incompressible (constant density) in response to gravitational and rotational accelerations, inviscid shallow fluid layer in rotating frame of reference may be written as the momentum equations (Equation 2.1 and Equation 2.2), hydrostatic equation (Equation 2.3) and continuity equation (Equation 2.4) (Holton, 2004).

$$\frac{du}{dt} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2.1}$$

$$\frac{dv}{dt} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \tag{2.2}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.4}$$

where u, v and w are the x, y and z components, respectively, of the wind vector.

f is the Coriolis parameter, given by $f = 2\Omega \sin\theta$, (Ω is the angular velocity of the earth, θ is the latitude).

ρ is the density of the fluid.

p is the pressure.

g is the acceleration due to gravity.

The total time derivative is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Consider a thin layer of fluid above a flat surface (that is neglected bottom mountains) in Figure 2.1.

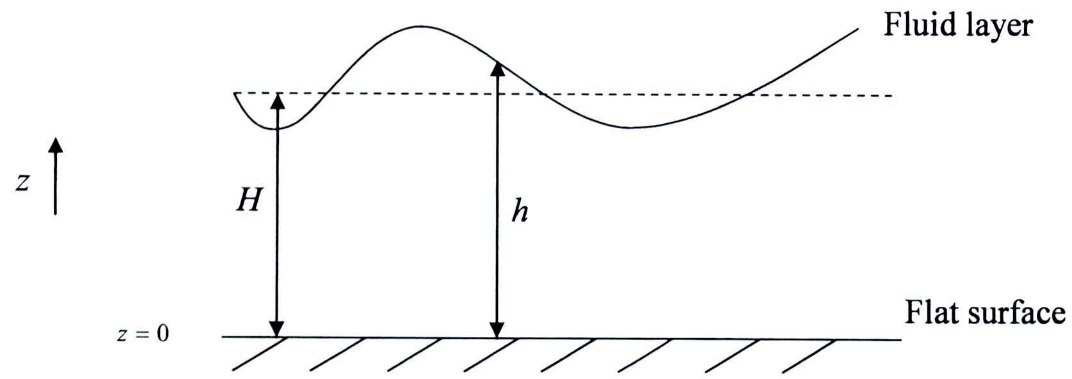

Figure 2.1 A single-level shallow water system. $h = h(x, y)$ is the depth of the water column, H its mean depth (Holton, 2004).

Assume that the pressure at the top of the fluid layer is constant p_0. Integrating the hydrostatic equation (Equation 2.3) between the limits z and h to get

$$\frac{\partial p}{\partial z} = -\rho g$$

$$p = p_0 + \int_z^h \rho g dz$$

$$p = \rho g(h - z) + p_0 \tag{2.5}$$

Assume that the pressure at a point is given by the weight of fluid above it (plus p_0). It implies that the horizontal pressure gradient at a depth z is given by

$$\frac{\partial p}{\partial x} = \frac{\partial[\rho g(h - z) + p_0]}{\partial x}$$

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial h}{\partial x}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\frac{\partial h}{\partial x} \tag{2.6}$$

$$\frac{\partial p}{\partial y} = \frac{\partial[\rho g(h - z) + p_0]}{\partial y}$$

$$\frac{\partial p}{\partial y} = \rho g \frac{\partial h}{\partial y}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = g\frac{\partial h}{\partial y} \tag{2.7}$$

The expressions on the right hand sides are independent of z.

Substituting Equation (2.6) into Equation (2.1);

$$\frac{du}{dt} - fv + g\frac{\partial h}{\partial x} = 0 \tag{2.8}$$

and substituting Equation (2.7) into Equation (2.2),

$$\frac{dv}{dt} + fu + g\frac{\partial h}{\partial y} = 0 \tag{2.9}$$

Assume that the horizontal velocity (u, v) is constant throughout the fluid layer. Since the bottom is flat, the vertical velocity must vanish there. Moreover, the vertical velocity of a fluid particle at the top surface is given by $w(h) = \frac{dh}{dt}$. Integrating the continuity equation (Equation 2.4) through the full depth of the fluid.

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dz = 0$$

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz + \int_0^h \frac{\partial w}{\partial z}dz = 0$$

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + w(h) - w(0) = 0$$

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{dh}{dt} = 0$$

or

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{2.10}$$

Assume that the depth at the sea level is $h = Z$. Here $Z = \Phi(z)/g$ is the geopotential height. , where $g = 9.81\ m/s^2$ is the global average of gravity at mean sea level and $\Phi(z)$ is the potential function of z and $\frac{d\Phi}{dz} = g$.

If the value of geopotential is set to zero at mean sea level, the geopotential $\Phi(z)$ at height z is just the work required to raise a unit mass to height z from mean sea level, $\Phi = \int_0^z gdz$ (Holton, 2004).

Therefore Equations (2.8), (2.9), and (2.10) comprise the set of shallow water equations as

$$\frac{du}{dt} - fv + g\frac{\partial Z}{\partial x} = 0 \tag{2.11}$$

$$\frac{dv}{dt} + fu + g\frac{\partial Z}{\partial y} = 0 \tag{2.12}$$

$$\frac{dZ}{dt} + Z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{2.13}$$

This is the set of three equations (u, v, Z).

2.2 Closed Domain Invariants of the Shallow Water Equation

A domain invariant is a property of a class of mathematical objects that remains unchanged when transformations of a certain type are applied to the objects on the domain. Invariants are used on the domain of SILEPE such as mean potential vorticity, mean geopotential height, and mean total energy. The relative equations are as follow.

2.2.1 Mean Potential Vorticity

From the vorticity equation obtained from Equation (2.11) and Equation (2.12),

$$\frac{\partial(\zeta + f)}{\partial t} = -V_H \cdot \nabla(\zeta + f) - (\zeta + f)\nabla \cdot V_H$$

where $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ is the relative vorticity.

$$\frac{\partial(\zeta + f)}{\partial t} + V_H \cdot \nabla(\zeta + f) + (\zeta + f)\nabla \cdot V_H = 0$$

$$\frac{\partial(\zeta+f)}{\partial t}+u\frac{\partial(\zeta+f)}{\partial x}+v\frac{\partial(\zeta+f)}{\partial y}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \qquad (2.14)$$

From the continuity Equation (2.13),

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\frac{1}{Z}\left(\frac{\partial Z}{\partial t}+u\frac{\partial Z}{\partial x}+v\frac{\partial Z}{\partial y}\right) \qquad (2.15)$$

Substituting Equation (2.15) in Equation (2.14),

$$\frac{\partial(\zeta+f)}{\partial t}+u\frac{\partial(\zeta+f)}{\partial x}+v\frac{\partial(\zeta+f)}{\partial y}-\left(\frac{\zeta+f}{Z}\right)\left(\frac{\partial Z}{\partial t}+u\frac{\partial Z}{\partial x}+v\frac{\partial Z}{\partial y}\right)=0$$

$$\frac{\partial(\zeta+f)}{\partial t}+u\frac{\partial(\zeta+f)}{\partial x}+v\frac{\partial(\zeta+f)}{\partial y}-\left(\frac{\zeta+f}{Z}\right)\frac{\partial Z}{\partial t}-u\left(\frac{\zeta+f}{Z}\right)\frac{\partial Z}{\partial x}-v\left(\frac{\zeta+f}{Z}\right)\frac{\partial Z}{\partial y}=0 \qquad (2.16)$$

Multiplying Equation (2.16) by $\frac{1}{Z}$,

$$\frac{1}{Z}\frac{\partial(\zeta+f)}{\partial t}+\frac{u}{Z}\frac{\partial(\zeta+f)}{\partial x}+\frac{v}{Z}\frac{\partial(\zeta+f)}{\partial y}-\left(\frac{\zeta+f}{Z^2}\right)\frac{\partial Z}{\partial t}-u\left(\frac{\zeta+f}{Z^2}\right)\frac{\partial Z}{\partial x}-v\left(\frac{\zeta+f}{Z^2}\right)\frac{\partial Z}{\partial y}=0$$

$$\left[\frac{Z}{Z^2}\frac{\partial(\zeta+f)}{\partial t}-\left(\frac{\zeta+f}{Z^2}\right)\frac{\partial Z}{\partial t}\right]+u\left[\frac{Z}{Z^2}\frac{\partial(\zeta+f)}{\partial x}-\left(\frac{\zeta+f}{Z^2}\right)\frac{\partial Z}{\partial x}\right]$$
$$+v\left[\frac{Z}{Z^2}\frac{\partial(\zeta+f)}{\partial y}-\left(\frac{\zeta+f}{Z^2}\right)\frac{\partial Z}{\partial y}\right]=0$$

$$\frac{\partial}{\partial t}\left(\frac{\zeta+f}{Z}\right)+u\frac{\partial}{\partial x}\left(\frac{\zeta+f}{Z}\right)+v\frac{\partial}{\partial y}\left(\frac{\zeta+f}{Z}\right)=0$$

$$\frac{d}{dt}\left(\frac{\zeta+f}{Z}\right)=0 \qquad (2.17)$$

Multiplying Equation (18) by $n\left(\frac{\zeta+f}{Z}\right)^{n-1}$;

$$n\left(\frac{\zeta+f}{Z}\right)^{n-1}\frac{d}{dt}\left(\frac{\zeta+f}{Z}\right)=0$$

$$\frac{d}{dt}\left(\frac{\zeta+f}{Z}\right)^n=0 \qquad (2.18)$$

Equation (2.18) states that all powers n of the potential vorticity of a parcel following the motion are conserved.

$$\frac{\partial}{\partial t}\left(\frac{\zeta+f}{Z}\right)^n + u\frac{\partial}{\partial x}\left(\frac{\zeta+f}{Z}\right)^n + v\frac{\partial}{\partial y}\left(\frac{\zeta+f}{Z}\right)^n = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\zeta+f}{Z}\right)^n + V_H.\nabla\left(\frac{\zeta+f}{Z}\right)^n = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\zeta+f}{Z}\right)^n = -V_H.\nabla\left(\frac{\zeta+f}{Z}\right)^n$$

This implies that over a close domain, the domain mean potential vorticity and all its powers are conserved.

$$\frac{\partial}{\partial t}\int_x\int_y\left(\frac{\zeta+f}{Z}\right)^n dxdy = 0 \tag{2.19}$$

2.2.2 Mean Geopotential Height

From the continuity equation (2.13),

$$\frac{\partial Z}{\partial t} + u\frac{\partial Z}{\partial x} + v\frac{\partial Z}{\partial y} = -Z\frac{\partial u}{\partial x} - Z\frac{\partial v}{\partial y}$$

$$\frac{\partial Z}{\partial t} + u\frac{\partial Z}{\partial x} + v\frac{\partial Z}{\partial y} + Z\frac{\partial u}{\partial x} + Z\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial Z}{\partial t} + \left(Z\frac{\partial u}{\partial x} + u\frac{\partial Z}{\partial x}\right) + \left(Z\frac{\partial v}{\partial y} + v\frac{\partial Z}{\partial y}\right) = 0$$

$$\frac{\partial Z}{\partial t} + \frac{\partial Zu}{\partial x} + \frac{\partial Zv}{\partial y} = 0 \tag{2.20}$$

It can be written in the following form

$$\frac{\partial Z}{\partial t} + \nabla \cdot ZV_H = 0$$

where $V_H = u\vec{i} + v\vec{j}$

$$\frac{\partial Z}{\partial t} = -\nabla \cdot ZV_H \tag{2.21}$$

On integration over a closed domain,

$$\frac{\partial}{\partial t}\int_y\int_x Zdxdy = 0 \tag{2.22}$$

Equation (2.22) states that over a closed domain, the domain mean geopotential height is an invariant of SILEPE.

2.2.3 Mean Total Energy

From the equations of motion (2.11) and (2.12),

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - g\frac{\partial Z}{\partial x} + fv \tag{2.23}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - g\frac{\partial Z}{\partial y} - fu \tag{2.24}$$

Multiplying Equation (2.23) by u,

$$u\frac{\partial u}{\partial t} = -u^2\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} - ug\frac{\partial Z}{\partial x} + fuv \tag{2.25}$$

Multiplying Equation (2.24) by v,

$$v\frac{\partial v}{\partial t} = -uv\frac{\partial v}{\partial x} - v^2\frac{\partial v}{\partial y} - vg\frac{\partial Z}{\partial y} - fuv \tag{2.26}$$

Adding Equation (2.25) and Equation (2.26),

$$u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} = -\left(u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uv\frac{\partial u}{\partial y} + v^2\frac{\partial v}{\partial y}\right) - \left(ug\frac{\partial Z}{\partial x} + vg\frac{\partial Z}{\partial y}\right) \tag{2.27}$$

$$\frac{\partial k}{\partial t} = -V_H \cdot \nabla k - V_H \cdot \nabla gZ \tag{2.28}$$

where $k = \frac{1}{2}(u^2 + v^2)$ is the kinetic energy.

Equation (2.28) is the time rate of change of kinetic energy in SILEPE.

From the continuity Equation (2.13), multiplying this equation by $k + gZ$;

$$k\frac{\partial Z}{\partial t}+gZ\frac{\partial Z}{\partial t}+ku\frac{\partial Z}{\partial x}+gZu\frac{\partial Z}{\partial x}+kv\frac{\partial Z}{\partial y}+gZv\frac{\partial Z}{\partial y}+kZ\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+ZgZ\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

(2.29)

$$k\frac{\partial Z}{\partial t}+gZ\frac{\partial Z}{\partial t}+\left(ku\frac{\partial Z}{\partial x}+kv\frac{\partial Z}{\partial y}\right)+\left(gZu\frac{\partial Z}{\partial x}+gZv\frac{\partial Z}{\partial y}\right)+kZ\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+ZgZ\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$k\frac{\partial Z}{\partial t}+gZ\frac{\partial Z}{\partial t}+kV_H\cdot\nabla Z+gZV_H\cdot\nabla Z+kZ\nabla\cdot V_H+ZgZ\nabla\cdot V_H=0 \qquad (2.30)$$

From Equation (2.28), multiplying this equation by Z ;

$$Z\frac{\partial k}{\partial t}+ZV_H\cdot\nabla k+ZV_H\cdot\nabla gZ=0 \qquad (2.31)$$

Adding Equation (2.31) and Equation (2.30),

$$\left(Z\frac{\partial k}{\partial t}+k\frac{\partial Z}{\partial t}+gZ\frac{\partial Z}{\partial t}\right)+\left(ZV_H\cdot\nabla k+kV_H\cdot\nabla Z\right)+kZ\nabla\cdot V_H+gZV_H\cdot\nabla Z$$
$$+\left(ZgZ\nabla\cdot V_H+ZV_H\cdot\nabla gZ\right)=0$$

$$\frac{\partial}{\partial t}Z(k+\frac{gZ}{2})+V_H\cdot\nabla kZ+kZ\nabla\cdot V_H+gZV_H\cdot\nabla Z+Z\left(gZ\nabla\cdot V_H+V_H\cdot\nabla gZ\right)=0$$

$$\frac{\partial}{\partial t}Z(k+\frac{gZ}{2})+\nabla\cdot kZV_H+gZV_H\cdot\nabla Z+Z\nabla\cdot gZV_H=0$$

$$\frac{\partial}{\partial t}Z(k+\frac{gZ}{2})+\nabla\cdot kZV_H+\nabla\cdot Z(gZV_H)=0$$

$$\frac{\partial}{\partial t}Z(k+\frac{gZ}{2})=-\nabla\cdot kZV_H-\nabla\cdot Z(gZV_H) \qquad (2.32)$$

Integration of the Equation (2.32) over a closed domain leads to

$$\frac{\partial}{\partial t}\int_y\int_x Z(k+\frac{gZ}{2})dxdy=0 \qquad (2.33)$$

This integral states that the domain mean total energy parameter $Z(k+\frac{gZ}{2})$ is invariant.

2.3 Specification of Initialization

The u and v component winds are specified at model grid points through objective analysis of the observations. The geopotential height field is then deduced from the horizontal motion field by a heirachy of models of increasing complexity. This process is called static initialization (Krishnamurti, 1986).

2.3.1 Static Initialization

The single-level static initialization techniques are based on the non-linear balance equation and on simplified versions of it. The following laws are relevant to the static initialization problem.

Geostrophic law

$$\nabla^2 gZ = f_0 \nabla^2 \psi$$

Linear balance law

$$\nabla^2 gZ = \nabla \cdot f\nabla \psi$$

Non linear balance law

$$\nabla^2 gZ = \nabla \cdot f\nabla \psi + 2J\left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right)$$

where f_0 is the Coriolis constant.

ψ is the stream function. From the computed ψ field u and v are obtained by using the relations $u = -\dfrac{\partial \psi}{\partial y}$; $v = \dfrac{\partial \psi}{\partial x}$.

∇ and ∇^2 are the gradient and Laplacian operators, respectively, and J is Jacobean notation.

2.3.2 Dynamic Initialization

To improve further on the geopotential height-wind field balance that has been deduced through static initialization, a process called dynamic initialization is performed. The dynamic initialization procedure utilizes the shallow water equation as its basis. The procedure entails a forward-backward integration of the single level primitive equations of the u, v and Z fields obtained from static initialization.

The primitive equation are integrated forward and backward utilizing small time steps of a few minutes, the purpose being to let the motion and pressure fields adjust to an equilibrium that may depart from the so called balance laws. In the process of this forward and backward integration, inertial-gravity oscillations are excited and the final state of iterative initialization is one which varies very slowly on forward integration. The adjusted motion fields vary very slightly from the initial field, the large change being in the adjustment of the pressure field in the low latitudes.

The boundary condition specified for the model during dynamic initialization consists of cyclic continuity in the zonal direction and open boundary conditions in the southern and northern boundaries.

2.4 Ensemble Forecasting Techniques

Ensemble forecasts are designed to capture the probabilities for weather events and the range of uncertainty inherent in each forecast situation. Ensemble forecasts can be assessed by the rate of growth in spread of ensemble member forecasts. Ensemble size and adequate initial condition perturbation are important in obtaining an adequate ensemble spread and a measure of atmospheric predictability. Figure 2.2 shows the essential components of an ensemble: a control forecast started from the analysis, forecasts started from two perturbations to the analysis (in this case equal and opposite), the ensemble average, and the "truth", or forecast verification, which becomes available later. The first schematic shows an example of a "good ensemble" in which "truth" looks like a member of the ensemble. In this case, the ensemble spread is related to the forecast error. The second schematic is an example of a "bad ensemble": due to poor initial perturbations and/or model deficiencies, the forecasts are not able to track the verifying truth, and remain relatively close to each other. In this case the ensemble is not helpful to the forecasters at all, since the lack of ensemble spread would give them unjustified confidence in the erroneous forecast (Kalnay, 2002).

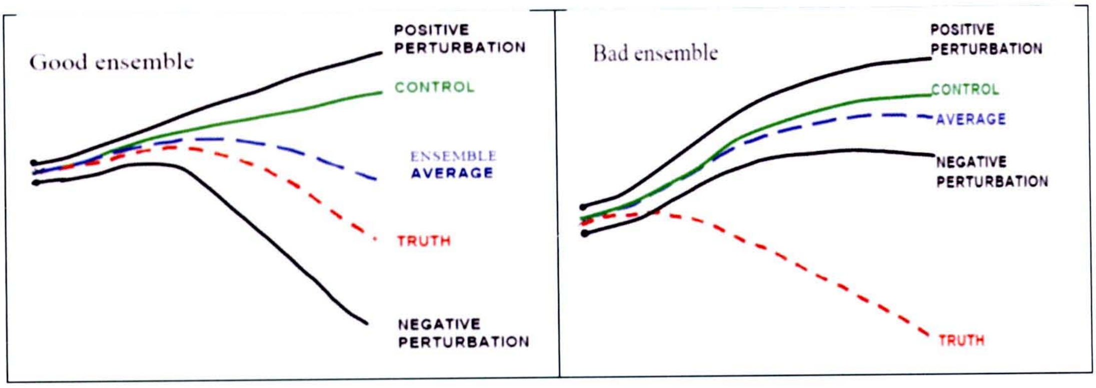

Figure 2.2 Schematic of the essential components of an ensemble of forecasts (Kalnay, 2002).

The two essential problems in the design of an ensemble forecasting system are how to create effective initial perturbations, and how to handle model deficiencies which make the ensemble forecast spread smaller than the forecast error. In order to improve these problems, in the recent years many ensemble forecast techniques have been developed. Some of the interesting ensemble forecasting techniques are described below.

2.4.1 Singular Vector Method

The singular vector analysis was first used by Lorenz (1965) for atmospheric study to compute the largest error growth rates to estimate atmospheric predictability of an idealized model atmosphere. The most successful application of singular vector analysis has been at the European Centre for Medium-Range Weather Forecasts (ECMWF) for generating the initial perturbation state for ensemble forecasts. A brief outline of this method is described below (Holton, 2004).

A set of n nonlinear evolution equations of an atmospheric model using a spectral expansion leading to n degrees of freedom can be written as

$$\frac{dx}{dt} = A(x) \tag{2.34}$$

Here, x is the state vector consisting of spherical harmonic components of atmospheric variables such as vorticity, divergence, temperature, humidity, surface pressure, etc. First, find the Tangent Linear Model (TLM), which is defined such that;

$$L = \frac{\partial M}{\partial x} \tag{2.35}$$

where M is the time integration of the numerical scheme from the initial condition to time t.

L is a tangent linear model of M .

Second, find the adjoint tangent linear model, which is the transpose of the tangent linear model. It is defined with respect to the inner product of two arbitrary vectors.

$$\langle Lu, v \rangle = \langle u, L^T v \rangle \tag{2.36}$$

where u and v are arbitrary vectors and L^T is the adjoint TLM.

Finally, find the singular vector decomposition by consider any $(n \times n)$ real matrix L which can be written as the product of an $(n \times n)$ orthogonal matrix U, an $(n \times n)$ diagonal matrix S, and the transpose of an $(n \times n)$ orthogonal matrix V as

$$L = USV^T \tag{2.37}$$

As U and V are orthogonal matrices

$$UU^T = I \quad \text{and} \quad VV^T = I \tag{2.38}$$

Alternatively, pre-multiplying Equation (2.37) by U^T and post-multiplying it by V gives

$$U^T LV = S = \begin{bmatrix} \sigma_1 & 0 & 0 & .... & 0 \\ 0 & \sigma_2 & 0 & .... & 0 \\ . & . & . & .... & . \\ . & . & . & \sigma_{n-1} & . \\ . & . & . & . & \sigma_n \end{bmatrix} \tag{2.39}$$

and U, V may be written as

$$U = [u_1, u_2, ..., u_n], \qquad V = [v_1, v_2, ..., v_n] \tag{2.40}$$

Elements of Equation (2.39) satisfy the relation

$$\sigma_1 \geq \sigma_2 ... \geq \sigma_n \geq 0. \tag{2.41}$$

Multiplying the left-hand side of Equation (2.39) by U gives

$$LV = US, \text{ i.e.,} \quad Lv_i = \sigma_i u_i, \tag{2.42}$$

where v_i are the right singular vectors or initial singular vectors of L. Multiplying the right-hand side of Equation (2.39) by V^T gives

$$U^T L = SV^T, \tag{2.43}$$

which on transposing gives

$$L^T U = VS, \text{ i.e.,} \quad L^T u_i = \sigma_i v_i, \tag{2.44}$$

where u_i are the left singular vectors or final singular vectors of L. From Equation (2.42) and Equation (2.44)

$$L^T Lv_i = \sigma L^T u_i = \sigma_i^2 v_i \tag{2.45}$$

Therefore, the initial singular vectors, v_i, can be obtained as eigenvectors of $L^T L$, a normal matrix whose eigenvalues are squares of the singular values.

2.4.2 Monte Carlo Method

The idea of the Monte Carlo method is to perturb all data simultaneously with random numbers of a realistic magnitude. The resulting forecast will differ almost throughout the forecast domain. Repeating the experiment many times with different sets of numbers, one can get the idea of the forecast errors that are due to the uncertainty of the observations and analyses. The Monte Carlo method for ensemble forecasting was first applied by Leith (1974) in a perfect model environment. He generates sets of perturbation which are normally distributed with a zero mean and the perturbation sets are orthogonal to each other. If X (0) is the initial analysis and E_i is the *i*th set of random errors (random numbers), then the *i*th perturbed initial state $X(0)$ is given by

$$X_i^E(0) = X(0) + E_i, \tag{2.46}$$

where the multivariable vector E_i satisfies the conditions

$$\langle E_i \rangle = 0 \tag{2.47}$$

and

$$\langle E_i E_j^* \rangle = \sigma\delta_{ij} \tag{2.48}$$

where σ is the variance of random errors.

If $X_i^E(t)$ is the forecast of the *i*th ensemble member after time *t*, then

$$\overline{X}(t) = \frac{1}{N}\sum_{i=1}^{N} X_i^E(t) \tag{2.49}$$

is the ensemble mean forecast and

$$\sigma_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(X_i(t) - \overline{X}(t))^2} \tag{2.50}$$

is the forecast variance, a measure of the spread of the ensemble forecast.

2.4.3 Breeding Technique

Breeding (Toth and Kalnay, 1993, 1997, Cai et al., 2002) was developed as a technique to generate initial perturbations for ensemble forecasting at NCEP. The method involves simply running the nonlinear model used for the control a second time, periodically subtracting the control from the perturbed solution, and rescaling the difference so that it has the same size as the original perturbation. The rescaled difference (a bred vector) is added to the control run and the process repeated (Figure 2.3). Their growth rate is a measure of the local instability of the flow.

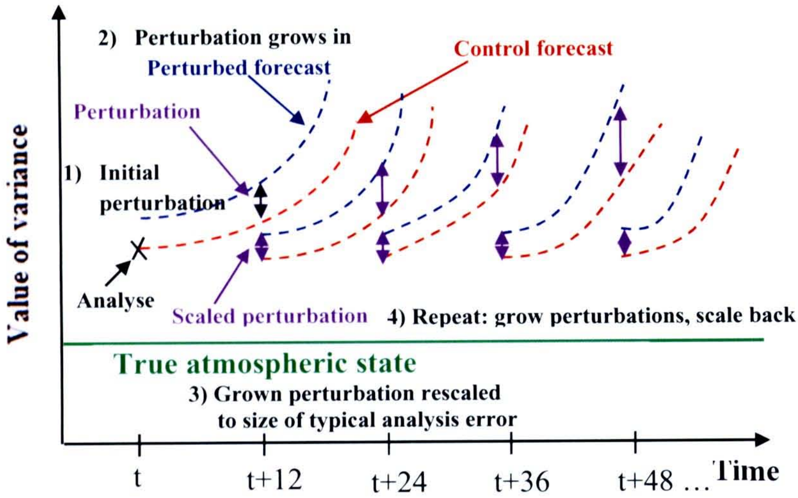

Figure 2.3 Schematic of the method to generate bred vectors (Stephen, 2002).

In Figure 2.3, the breeding method procedure consists of the following steps:

1. Add a small arbitrary perturbation to the atmospheric analysis.

2. Integrate the model for a short period (e.g. 12 hours) from both the unperturbed (control) and the perturbed initial conditions.

3. Subtract the control forecast from the perturbed forecast. Scale down the difference field so that it has the same size as the initial perturbation.

4. Repeat steps 2 and 3 through the end of breeding experiments.

2.4.4 Kalman Filter Method

The Kalman filter (Dan, 2006) is an estimator for what is called the linear-quadratic-Gaussian problem, which is the problem of estimating the instantaneous "state" of a linear dynamic system perturbed by Gaussian white noise by using measurements linearly related to the state but corrupted by the noise. The resulting estimator is statistically optimal with respect to any quadratic function of estimation error. Assume that the equations that describe the behavior of a dynamical system are known, and that there are some noisy observations of the system, and the current state of the system or predict the state of the system in the future is required.

Consider a discrete time dynamical system governed by the equation

$$x_k = Ax_{k-1} + w_{k-1} \tag{2.51}$$

$$z_k = Hx_k + v_k \tag{2.52}$$

Here x_k, x_{k-1}, w_{k-1}, z_k and v_k are vectors and the subscripts refer to the time steps rather than indexing elements of the vectors.

Let x_k is the state of the system at time k

x_{k-1} is the state of the system at time $k-1$

w_{k-1} is the random noise affecting the system at time $k-1$

z_k is the vector of measurements at time k

v_k is the random noise in the observation z_k at time k

Assume that

w_{k-1} has a multivariate normal distribution with mean 0 and covariance matrix Q

v_k is normally distributed with mean 0 and covariance matrix R.

The matrices A, H, Q, and R are all assumed to be known.

For now the goal is to estimate x_k and predict $x_{k+1}, x_{k+2}, \ldots$ as accurately as possible given $z_1, z_2, \ldots, z_k$. Pick x_k so as to minimize $\|z_k - Hx_k\|$.

Begin the estimation process with an initial guess for the state of the system at time 0. Since it is required to keep track of the uncertainty in the estimates, the uncertainty in the initial guess has to be specified. This can be described by using a multivariate normal distribution.

$$x_0 \sim N(\hat{x}_0, \hat{P}_0) \tag{2.53}$$

In the prediction step, an estimate $\hat{x}_{k-1}$ of the state of the system at time $k-1$ is given, with associated covariance matrix $\hat{P}_{k-1}$.

$$\hat{x}_k^- = A\hat{x}_{k-1} + w_{k-1} \tag{2.54}$$

Define $\hat{x}_k^- \in R^n$ to be a priori state estimate at step *k* given knowledge of the process prior to step *k*, and $\hat{x}_k \in R^n$ to be a posteriori state estimate at step *k* given measurement z_k.

The covariance of the new estimate is

$$\hat{P}_k^- = Cov(\hat{x}_k^-) \tag{2.55}$$

$$\hat{P}_k^- = Cov(A\hat{x}_{k-1} + w_{k-1}) \tag{2.56}$$

$$\hat{P}_k^- = Cov(A\hat{x}_{k-1}) + Cov(w_{k-1}) \tag{2.57}$$

The covariance of w_{k-1} is Q. The covariance of $A\hat{x}_{k-1}$ is $ACov(\hat{x}_{k-1})A^T$. Thus

$$\hat{P}_k^- = ACov(A\hat{x}_{k-1}) + Q \tag{2.58}$$

$$\hat{P}_k^- = A\hat{P}_{k-1}A^T + Q \tag{2.59}$$

Repeat this process for $x_1, x_2,...$. If no observations of the system are available, that would be an appropriate way to estimate the system state.

In the update step, modify the prediction the prediction estimate to include the observation.

$$\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-) \tag{2.60}$$

$$\hat{x}_k = \hat{x}_k^- + K_k z_k - K_k H\hat{x}_k^- \tag{2.61}$$

$$\hat{x}_k = (I - K_k H)\hat{x}_k^- + K_k z_k \tag{2.62}$$

Here the factor K_k is called the Kalman gain. It adjusts the relative influence of z_k and x_k^-.

It can be shown that

$$K_k = \hat{P}_k^- H^T(H\hat{P}_k^- H^T + R)^{-1} \tag{2.63}$$

is optimal in the sense that it minimizes the trace of $\hat{P}_k$.

The covariance of the updated estimate is

$$\hat{P}_k = Cov(\hat{x}_k). \tag{2.64}$$

From Equation (2.62);

$$\hat{P}_k = Cov((I - K_k H)\hat{x}_k^- + K_k z_k) \tag{2.65}$$

$$\hat{P}_k = Cov((I - K_k H)\hat{x}_k^-) + Cov(K_k z_k) \qquad (2.66)$$

$$\hat{P}_k = (I - K_k H)Cov(\hat{x}_k^-)(I - K_k H)^T + K_k Cov(z_k) K_k^T \qquad (2.67)$$

Since $Cov(\hat{x}_k^-) = \hat{P}_k^-$ and $Cov(z_k) = R$,

$$\hat{P}_k = (I - K_k H)\hat{P}_k^-(I - K_k H)^T + K_k R K_k^T \qquad (2.68)$$

This simplifies to

$$\hat{P}_k = (I - K_k H)\hat{P}_k^-(I^T - (K_k H)^T) + K_k R K_k^T \qquad (2.69)$$

$$\hat{P}_k = (I - K_k H)\hat{P}_k^-(I - H^T K_k^T) + K_k R K_k^T \qquad (2.70)$$

$$\hat{P}_k = (\hat{P}_k^- - K_k H\hat{P}_k^-)(I - H^T K_k^T) + K_k R K_k^T \qquad (2.71)$$

$$\hat{P}_k = \hat{P}_k^- - K_k H\hat{P}_k^- - \hat{P}_k^- H^T K_k^T + K_k H\hat{P}_k^- H^T K_k^T + K_k R K_k^T \qquad (2.72)$$

$$\hat{P}_k = \hat{P}_k^- - K_k H\hat{P}_k^- - \hat{P}_k^- H^T K_k^T + K_k (H\hat{P}_k^- H^T + R) K_k^T \qquad (2.73)$$

It is required to minimize the trace of $\hat{P}_k$.

Consider

$$tr(\hat{P}_k) = tr(\hat{P}_k^- - K_k H\hat{P}_k^- - \hat{P}_k^- H^T K_k^T + K_k (H\hat{P}_k^- H^T + R) K_k^T) \qquad (2.74)$$

$$tr(\hat{P}_k) = tr(\hat{P}_k^-) - tr(K_k H\hat{P}_k^-) - tr(\hat{P}_k^- H^T K_k^T) + tr(K_k (H\hat{P}_k^- H^T + R) K_k^T) \qquad (2.75)$$

Since $tr(\hat{P}_k^- H^T K_k^T) = tr(\hat{P}_k^- H^T K_k^T)^T = tr(K_k H\hat{P}_k^-)$

$$tr(\hat{P}_k) = tr(\hat{P}_k^-) - 2tr(K_k H\hat{P}_k^-) + tr(K_k (H\hat{P}_k^- H^T + R) K_k^T) \qquad (2.76)$$

Consider

$$\frac{dtr(\hat{P}_k)}{dK_k} = 0 \qquad (2.77)$$

$$\frac{dtr(\hat{P}_k^-)}{dK_k} - 2\frac{dtr(K_k H\hat{P}_k^-)}{dK_k} + \frac{dtr(K_k (H\hat{P}_k^- H^T + R) K_k^T)}{dK_k} = 0 \qquad (2.78)$$

Since $\frac{dtr(K_k A)}{dK_k} = A^T$

$$0 - 2(H\hat{P}_k^-)^T + \frac{dtr(H\hat{P}_k^- H^T + R)K_k^2)}{dK_k} = 0 \tag{2.79}$$

$$-2(H\hat{P}_k^-)^T + 2K_k(H\hat{P}_k^- H^T + R) = 0 \tag{2.80}$$

Therefore (same as Equation (2.63))

$$K_k = \hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1} \tag{2.81}$$

Using this optimal Kalman gain, $\hat{P}_k$ simplifies further.

From Equation (2.73),

$$\hat{P}_k = \hat{P}_k^- - K_k H\hat{P}_k^- - \hat{P}_k^- H^T K_k^T + K_k(H\hat{P}_k^- H^T + R)K_k^T \tag{2.82}$$

Substituting $K_k = \hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1}$ in Equation (2.82)

$$\begin{aligned}\hat{P}_k &= \hat{P}_k^- - (\hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1}) H\hat{P}_k^- - \hat{P}_k^- H^T (\hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1})^T \\ &\quad + (\hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1})(H\hat{P}_k^- H^T + R)(\hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1})^T\end{aligned} \tag{2.83}$$

$$\begin{aligned}\hat{P}_k &= \hat{P}_k^- - (\hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1}) H\hat{P}_k^- - \hat{P}_k^- H^T (((H\hat{P}_k^- H^T + R)^{-1})^T) H\hat{P}_k^- \\ &\quad + (\hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1})(H\hat{P}_k^- H^T + R)(((H\hat{P}_k^- H^T + R)^{-1})^T) H\hat{P}_k^-\end{aligned} \tag{2.84}$$

$$\begin{aligned}\hat{P}_k &= \hat{P}_k^- - \hat{P}_k^- H^T [(H\hat{P}_k^- H^T + R)^{-1} - ((H\hat{P}_k^- H^T + R)^{-1})^T \\ &\quad + (H\hat{P}_k^- H^T + R)^{-1}(H\hat{P}_k^- H^T + R)((H\hat{P}_k^- H^T + R)^{-1})^T] H\hat{P}_k^-\end{aligned} \tag{2.85}$$

$$\hat{P}_k = \hat{P}_k^- - \hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1} H\hat{P}_k^- \tag{2.86}$$

Since $K_k = \hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1}$

$$\hat{P}_k = \hat{P}_k^- - K_k H\hat{P}_k^- \tag{2.87}$$

$$\hat{P}_k = (I - K_k H)\hat{P}_k^- \tag{2.88}$$

Therefore Equations (2.54), (2.59), (2.60), (2.63), and (2.88) comprise the "Kalman filter equation" as follows.

Prediction Step:

The state $\hat{x}_k^- = A\hat{x}_{k-1} + w_{k-1}$

The error covariance $\hat{P}_k^- = A\hat{P}_{k-1}A^T + Q$

Update Step:

Kalman gain $K_k = \hat{P}_k^- H^T (H\hat{P}_k^- H^T + R)^{-1}$

Update with new measurement $\hat{x}_k = \hat{x}_k^- + K_k(z_k - H\hat{x}_k^-)$

Update with new error covariance $\hat{P}_k = (I - K_k H)\hat{P}_k^-$

The sequence of computational steps for the Kalman filter estimator is shown in Figure 2.4.

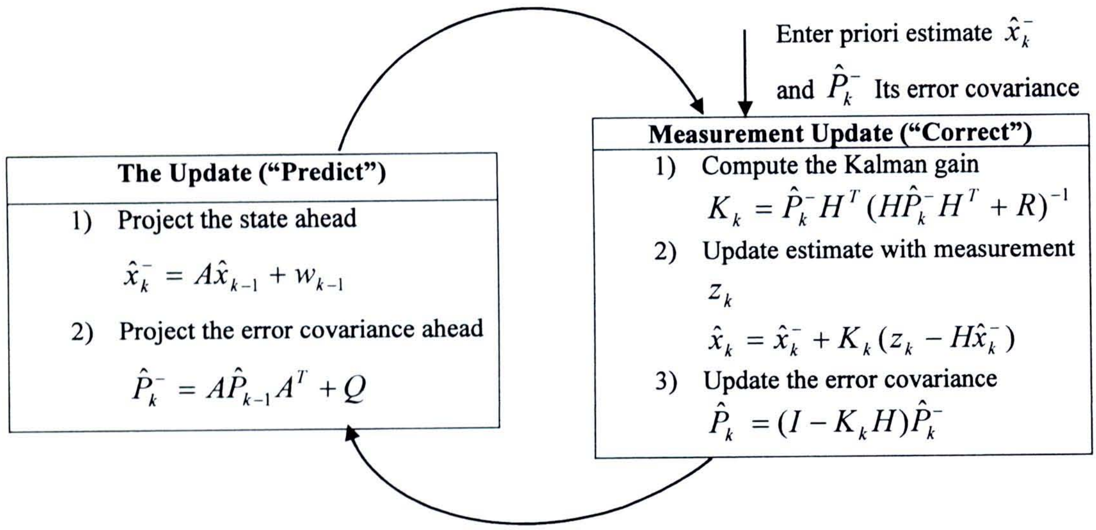

Figure 2.4 The sequence of computational steps for the Kalman filter estimator (Greg and Gary, 2006).

2.4.5 Extended Kalman Filter Method

Kalman filter method is an effective tool for estimation, but it is limited to linear systems. Most real world systems are nonlinear, in which case Kalman filters do not directly apply. In the real world, nonlinear filters are used more often than linear filters. Therefore the extended Kalman filter is developed for estimation of nonlinear system (Dan, 2006).

Consider a discrete-time nonlinear dynamical system

$$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1} \tag{2.89}$$

with the measurement equation

$$y_k = h(x_k) + v_k \tag{2.90}$$

where $f(\bullet)$ and $h(\bullet)$ are nonlinear functions vectors.

Perform a first-order Taylor series expansion of the state equation around $x_{k-1} = \hat{x}_{k-1}$ to obtain the following:

$$\begin{aligned} x_k &= f_{k-1}(\hat{x}_{k-1}, u_{k-1}) + \left.\frac{\partial f_{k-1}}{\partial x_{k-1}}\right|_{\hat{x}_{k-1}} (x_{k-1} - \hat{x}_{k-1}) + w_{k-1} \\ &= f_{k-1}(\hat{x}_{k-1}, u_{k-1}) + F_{k-1}(x_{k-1} - \hat{x}_{k-1}) + w_{k-1} \\ &= F_{k-1}x_{k-1} + [f_{k-1}(\hat{x}_{k-1}, u_{k-1}) - F_{k-1}\hat{x}_{k-1}] + w_{k-1} \\ &= F_{k-1}x_{k-1} + \widetilde{u}_{k-1} + w_{k-1} \end{aligned} \tag{2.91}$$

where

$$F_{k-1} = \left.\frac{\partial f_{k-1}}{\partial x_{k-1}}\right|_{\hat{x}_{k-1}} \tag{2.92}$$

$$\widetilde{u}_{k-1} = f_{k-1}(\hat{x}_{k-1}, u_{k-1}) - F_{k-1}\hat{x}_{k-1} \tag{2.93}$$

Linearize the measurement equation by a first-order Taylor series expansion around $x_k = \hat{x}_k^-$ to obtain,

$$\begin{aligned} y_k &= h_k(\hat{x}_k^-) + \left.\frac{\partial h_k}{\partial x_k}\right|_{\hat{x}_{k-1}} (x_k - \hat{x}_k^-) + v_k \\ &= h_k(\hat{x}_k^-) + G_k(x_k - \hat{x}_k^-) + v_k \\ &= G_k x_k + [h_k(\hat{x}_k^-) - G_k\hat{x}_k^-] + v_k \\ &= G_k x_k + z_k + v_k \end{aligned} \tag{2.94}$$

where

$$G_k = \left.\frac{\partial h_k}{\partial x_k}\right|_{\hat{x}_{k-1}} \tag{2.95}$$

$$z_k = h_k(\hat{x}_k^-) - G_k\hat{x}_k^- \tag{2.96}$$

Equation (2.91) is a linear state space system and Equation (2.94) is a linear measurement equation. That is the Kalman filter can be used to estimate the state. This result in the following equations for the extended Kalman filter.

Prediction Step:

The state $\hat{x}_k^- = f_{k-1}(\hat{x}_{k-1}, u_{k-1}) + w_{k-1}$ (2.97)

The error covariance $\hat{P}_k^- = F_{k-1}\hat{P}_{k-1}F_{k-1}^T + Q_{k-1}$ (2.98)

Update Step:

Kalman gain $K_k = \hat{P}_k^- G_k^T (G_k \hat{P}_k^- G_k^T + R_k)^{-1}$ (2.99)

Update with new state estimate $\hat{x}_k = \hat{x}_k^- + K_k(y_k - G_k\hat{x}_k^- - z_k)$ (2.100)

Update with new error covariance $\hat{P}_k = (I - K_k G_k)\hat{P}_k^-$ (2.101)

The extended Kalman filter can be summarized as follows.

1. The system and measurement equations are given as

$$x_k = f(x_{k-1}, u_{k-1}) + w_{k-1} \tag{2.102}$$

$$y_k = h(x_k) + v_k \tag{2.103}$$

$$w_k \sim (0, Q_k) \tag{2.104}$$

$$v_k \sim (0, R_k) \tag{2.105}$$

2. Initialize the filter as follows:

$$\hat{x}_0 = E(x_0) \tag{2.106}$$

$$\hat{P}_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \tag{2.107}$$

3. For $k = 1, 2, 3, \ldots$ perform the following.

a) Compute the following partial derivative matrices:

$$F_{k-1} = \left.\frac{\partial f_{k-1}}{\partial x_{k-1}}\right|_{\hat{x}_{k-1}} \tag{2.108}$$

b) Perform the time update of the state estimate and estimation-error covariance as

$$\hat{x}_k^- = f_{k-1}(\hat{x}_{k-1}, u_{k-1}) + w_{k-1} \tag{2.109}$$

$$\hat{P}_k^- = F_{k-1}\hat{P}_{k-1}F_{k-1}^T + Q_{k-1} \tag{2.110}$$

c) Compute the following partial derivative matrices:

$$G_k = \left.\frac{\partial h_k}{\partial x_k}\right|_{\hat{x}_{k-1}} \tag{2.111}$$

d) Perform the measurement update of the state estimate and estimation-error covariance:

$$K_k = \hat{P}_k^- G_k^T (G_k \hat{P}_k^- G_k^T + R_k)^{-1} \quad (2.112)$$

$$\hat{x}_k = \hat{x}_k^- + K_k (y_k - G_k \hat{x}_k^- - z_k) \quad (2.113)$$

$$\hat{P}_k = (I - K_k G_k) \hat{P}_k^- \quad (2.114)$$

Several studies show that the breeding method which is based on a nonlinear model when applied with finite perturbation amplitudes, it naturally accounts for all nonlinear system. It is computational efficient and success in operational atmospheric ensemble forecasts. The breeding method is a good candidate for generating ensemble perturbations for seasonal to multiseasonal climate predictions and not complex for implementation. Therefore, in this research, a breeding method is used to generate initial perturbations for ensemble forecasts of cold surge over Southeast Asia under the influence of global warming.