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Geometric Constants and Metric Fixed Point Theory

1. The Jordan von Neumann constant

1.1. Introduction

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $\mathbb{R}-$trees (see [1]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [12]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry see Bridson and Haefliger [1]. Burago, et al. [6] contains a somewhat more elementary treatment, and Gromov [13] a deeper study. Fixed point theory in a CAT(0) space was first studied by Kirk (see [17] and [18]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and much papers have appeared.

If x, y_1, y_2 are points in a CAT(0) space and if $y_0 = m[y_1, y_0]$, then the CAT(0) inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \qquad \text{(CN)}$$

This is the (CN) inequality of Bruhat and Tits [5]. In fact (cf. [1], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies (CN).

1.2. Results

Let (X,d) be a geodesic metric space with $card(X) \geq 2$. We define the Jordan von Neumann constant of (X,d) by

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{d(y,z)^2 + 4d(x,m[y,z])^2}{2(d(x,y)^2 + d(y,z)^2)} : x,y,z \in M, d(x,y) + d(y,z) \neq (0,0) \right\},$$

where $m[y,z]$ is the (geodesic) midpoint of y and z.

Proposition 1. We have

$$1 \leq C_{\mathrm{NJ}}(X) \leq 2.$$

It follows from the use of the (CN) inequality of Bruhat and Tits that :

Theorem 2. Let X be a complete geodesic metric space. Then X is a CAT(0) space if and only if the Jordan von Neumann constant $C_{\mathrm{NJ}}(X) = 1$.

Theorem 3. Let X be a complete geodesic metric space. If $C_{\mathrm{NJ}}(X) < \frac{5}{4}$, then X has uniform normal structure.

Since our work uses the fixed point property, let us give the following definition.

Definition 4. A mapping $T : X \to X$ is said to be nonexpansive if

$$d(Tx, Ty) \le d(x, y)$$

for all $x, y \in X$. We say that X has the fixed point property if any nonexpansive mapping defined on X has a fixed point.

The analogue of Kirk's fixed point theorem in metric spaces can be stated by the following.

Theorem 5. [16] Let X be a complete bounded metric space. Asuume that X has uniform normal structure. Then M has the fixed point property.

Using Theorem 3 and Theorem 5 together, we can conclude that :

Theorem 6. Let X be a complete bounded metric space. Asuume that $C_{\mathrm{NJ}}(X) < \frac{5}{4}$. Then X has the fixed point property.

Using Theorem 3 and Theorem 6 together, we obtain the known result of Kirk [17] and [18].

Theorem 7. Let X be a complete bounded CAT(0) space. Then X has the fixed point property.

2. Ishikawa iterative process for a pair of single valued and multi-valued nonexpansive mappings in Banach spaces

2.1 Introduction

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

$$H(A, B) = \max\{ \sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A) \}, \quad A, B \in FB(X),$$

where $\operatorname{dist}(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point a to the subset B.

A mapping $t : E \to E$ is said to be *nonexpansive* if

$$\|tx - ty\| \le \|x - y\| \quad \text{for all } x, y \in E.$$

A point x is called a fixed point of t if $tx = x$.

A multi-valued mapping $T : E \longrightarrow FB(X)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\| \quad \text{for all } x, y \in E.$$

A point x is called a fixed point for a multi-valued mapping T if $x \in Tx$.

We use the notation Fix(T) stands for the set of fixed points of a mapping T and Fix$(t) \cap$ Fix(T) stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $x = tx \in Tx$.

In 2006, S. Dhompongsa et al. ([7]) proved a common fixed point theorem for two nonexpansive commuting mappings.

Theorem 8. *[7, Theorem 4.2] Let E be a nonempty bounded closed convex subset of a uniformly Banach space X, $t : E \longrightarrow E$, and $T : E \longrightarrow KC(E)$ a nonexpansive mapping and a multi-valued nonexpansive mapping respectively. Assume that t and T are commuting , i.e. if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds $tx \in Tty$. Then t and T have a common fixed point.*

In this project, we introduce an iterative process in a new sense, called the modified Ishikawa iteration method with respect to a pair of single valued and multi-valued nonexpansive mappings. We also establish the strong convergence theorem of a sequence from such process in a nonempty compact convex subset of a uniformly convex Banach space.

The important property of a uniformly convex Banach space we use is the following lemma proved by Schu ([22]) in 1991.

Lemma 9. *([22]) Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|u_n x_n + (1 - u_n)y_n\| = a$ for some $a \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

The following observation will be used in proving our results and the proof is a straightforward.

Lemma 10. *Let X be a Banach space and E be a nonempty closed convex subset of X. Then,*

$$dist(y, Ty) \leq \|y - x\| + dist(x, Tx) + H(Tx, Ty),$$

where $x, y \in E$ and T is a multi-valued nonexpansive mapping from E into $FB(E)$.

A fundamental principle which plays a key role in ergodic theory is the demiclosedness principle. A mapping t defined on a subset E of a Banach space X is said to be demiclosed if any sequence $\{x_n\}$ in E the following implication holds: $x_n \rightharpoonup x$ and $tx_n \longrightarrow y$ implies $tx = y$.

Theorem 11. *([2]) Let E be a nonempty closed convex subset of a uniformly convex Banach space X and $t : E \longrightarrow E$ be a nonexpansive mapping. If a sequence $\{x_n\}$ in E converges weakly to p and $\{x_n - tx_n\}$ converges to 0 as $n \longrightarrow \infty$, then $p \in Fix(t)$.*

In 1974 , Ishikawa introduced the following well-known iteration.

Definition 12. ([14]) Let X be a Banach space, E a closed convex subset of X and t a selfmap on E. For $x_0 \in E$, the sequence $\{x_n\}$ of Ishikawa iterates of t is defined by,

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n t x_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n t y_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences.

A nonempty subset K of E is said to be proximinal if, for any $x \in E$, there exists an element $y \in K$ such that $\|x - y\| = \text{dist}(x, K)$. We shall denote $P(K)$ by the family of nonempty proximinal bounded subsets of K.

In 2005, Sastry and Babu ([21]) defined the Ishikawa iterative scheme for multi-valued mappings as follows:

Let E be a compact convex subset of a Hilbert space X and $T : E \to P(E)$ be a multi-valued mapping and fix $p \in \text{Fix}(T)$.

$$\begin{cases} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ with $z_n \in Tx_n$ such that $\|z_n - p\| = \text{dist}(p, Tx_n)$ and $\|z'_n - p\| = \text{dist}(p, Ty_n)$.

They also proved the strong convergence of the above Ishikawa iterative scheme for a multi-valued nonexpansive mapping T with a fixed point p under some certain conditions in a Hilbert space.

Recently, Panyanak ([20]) extended the results of Sastry and Babu ([21]) to a uniformly convex Banach space, and also modified the above Ishikawa iterative scheme as follows:

Let E be a nonempty convex subset of a uniformly convex Banach space X and $T : E \to P(E)$ be a multi-valued mapping

$$\begin{cases} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ with $z_n \in Tx_n$ and $u_n \in \text{Fix}(T)$ such that $\|z_n - u_n\| = \text{dist}(u_n, Tx_n)$ and $\|x_n - u_n\| = \text{dist}(x_n, \text{Fix}(T))$, respectively. Moreover, $z'_n \in Tx_n$ and $v_n \in \text{Fix}(T)$ such that $\|z'_n - v_n\| = \text{dist}(v_n, Tx_n)$ and $\|y_n - v_n\| = \text{dist}(y_n, \text{Fix}(T))$, respectively.

Very recently, Song and Wang ([25, 26]) noted that there was a gap in the proofs of ([20, Theorem 3.1]), and ([21, Theorem 5]). Thus they solved/revised the gap by means of the following Ishikawa iterative scheme:

Let $T : E \to FB(E)$ be a multi-valued mapping, where $\alpha_n, \beta_n \in [0, 1)$. The Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$ such that $\|z_n - z'_n\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z'_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$, respectively. Moreover, $\gamma_n \in (0, +\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$.

At the same period, Shahzad and Zegeye ([23]) modified the Ishikawa iterative scheme $\{x_n\}$ and extended the result of ([25, Theorem 2]) to a multi-valued quasi-nonexpansive mapping as follows:

Let K be a nonempty convex subset of a Banach space X and $T : E \to FB(E)$ a multi-valued mapping, where $\alpha_n, \beta_n \in [0, 1]$. The Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in E, \\ y_n = (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$.

2.2 Results

In this project we introduce a new iteration method modifying the above ones and call it the modified Ishikawa iteration method.

Definition 13. Let E be a nonempty closed bounded convex subset of a Banach space X, $t : E \to E$ be a single valued nonexpansive mapping, and $T : E \to FB(E)$ be a multi-valued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n z_n \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n t y_n \end{aligned} \tag{1}$$

where $x_0 \in E, z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

We first prove the following lemmas, which play very important roles in this section.

Lemma 14. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). Then $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in Fix(t) \cap Fix(T)$.*

We can see how Lemma 9 is useful via the following lemma.

Lemma 15. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n \leq b < 1$ for some a, $b \in \mathbb{R}$, then $\lim_{n\to\infty} \|ty_n - x_n\| = 0$.*

Lemma 16. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some a, $b \in \mathbb{R}$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

The following lemma allows us to go on.

Lemma 17. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $\lim_{n\to\infty} \|tx_n - x_n\| = 0$.*

We give the sufficient conditions which imply the existence of common fixed points for single valued mappings and multi-valued nonexpansive mappings, respectively, as follow:

Theorem 18. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$, then $x_{n_i} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in Fix(t) \cap Fix(T)$.*

Hereafter, we arrive at the convergence theorem of the sequence of the modified Ishikawa iteration. We conclude this project with the following theorem.

Theorem 19. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (1) with $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

3. Asymptotic centers and fixed points

3.1 Introduction

A mapping T on a subset E of a Banach space X is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. Although nonexpansive mappings are widely studied, there are many nonlinear mappings which are more general. The study of the existence of fixed points for those mappings is very useful in studying in the problem of equations in science and applied science.

The technique of employing the asymptotic centers and their Chebyshev radii in fixed point theory was first discovered by Edelstein [9] and the compactness assumption given on asymptotic centers was introduced by Kirk and Massa [19]. Recently, Dhompongsa et al. proved in [8] a theorem of existence of fixed points for some generalized nonexpansive mappings on a bounded closed convex subset E of a Banach space with assumption that every asymptotic center of a bounded sequence relative to E is nonempty and compact. However, spaces or sets in which asymptotic centers are compact have not been completely characterized, but partial results are known (see [11, pp. 93]). In this project, we introduce a class of nonlinear continuous mappings in Banach spaces which allows us to characterize the Banach spaces with compact asymptotic centers of bounded sequence relative to their weakly compact convex subsets as those that have the weak fixed point property for this type of mappings.

Let E be a nonempty closed and convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n \to \infty} \|x_n - x\|.$$

Let

$$r \equiv r(E, \{x_n\}) := \inf\{r(x, \{x_n\}) : x \in E\}$$

and

$$A \equiv A(E, \{x_n\}) := \{x \in E : r(x, \{x_n\}) = r\}.$$

The number r and the set A are, respectively, called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to E. It is known that $A(E, \{x_n\})$ is nonempty, weakly compact and convex as E is [11, pp. 90].

Let $T : E \to E$ be a nonexpansive and $z \in E$. Then for $\alpha \in (0, 1)$ the mapping $T_\alpha : E \to E$ defined by setting

$$T_\alpha x = (1 - \alpha)z + \alpha T x$$

is a contraction mapping. As we have known, Banach contraction mapping theorem assures the existence of a unique fixed point $x_\alpha \in E$. Since

$$\lim_{\alpha \to 1^-} \|x_\alpha - T x_\alpha\| = \lim_{\alpha \to 1^-} (1 - \alpha)\|z - T x_\alpha\| = 0,$$

we have the following.

Lemma 20. *If E is a bounded closed and convex subset of a Banach space and if $T : E \to E$ is nonexpansive, then there exists a sequence $\{x_n\} \subset E$ such that*

$$\lim_{n \to \infty} \|x_n - T x_n\| = 0.$$

3.2 Results

Definition 21. Let E be a bounded closed convex subset of a Banach space X. We say that a sequence $\{x_n\}$ in X is an asymptotic center sequence for the mapping $T : E \to X$ if, for each $x \in E$,

$$\limsup_{n \to \infty} \|x_n - T x\| \le \limsup_{n \to \infty} \|x_n - x\|.$$

We say that $T : E \to X$ is a *D-type mapping* whenever it is continuous and there is an asymptotic center sequence for T.

The following observation shows that the concept of D-type mappings is a generalization of nonexpansiveness.

Proposition 22. *Let $T : E \to E$ be a nonexpansive mapping. Then T is a D-type mapping.*

Definition 23. We say that a Banach space $(X, \|\cdot\|)$ has the weak fixed point property for D-type mappings if every D-type self-mapping of every weakly compact convex subset of X has a fixed point.

Now we are in the position to prove our main theorem.

Theorem 24. *Let X be a Banach space. Then X has the weak fixed point property for D-type mappings if and only if the asymptotic center relative to every nonempty weakly compact convex subset of each bounded sequence of X is compact.*

In 2007, Garcia-Falset et al. [10] introduced another concept of a center of mappings.

Definition 25. Let E be a bounded closed convex subset of a Banach space X. A point $x_0 \in X$ is said to be a center for a mapping $T : E \to X$ if, for each $x \in E$,

$$\|Tx - x_0\| \le \|x - x_0\|.$$

A mapping $T : E \to X$ is said to be a J-type mapping whenever it is continuous and it has some center $x_0 \in X$.

Definition 26. We say that a Banach space X has the J-weak fixed point property if every J-type self-mapping of every weakly compact subset E of X has a fixed point.

Employing the above definitions, the authors proved a characterization of the geometrical property (C) of the Banach spaces introduced in 1973 by R. E. Bruck [4] : A Banach space X has property (C) whenever the weakly compact convex subsets of its unit sphere are compact sets.

Theorem 27. *[10, Theorem 16] Let X be a Banach space. Then X has property (C) if and only if X has the J-weak fixed point property.*

It is easy to see that a center $x_0 \in X$ of a mapping $T : E \to X$ is can be seen as an asymptotic center sequence $\{x_n\}$ for the mapping T by setting $x_n \equiv x_0$ for all $n \in \mathbb{N}$. This leads to the following conclusion.

Proposition 28. *Let $T : E \to X$ be a J-type mapping. Then T is a D-type mapping.*

Consequently, we have

Proposition 29. *Let X be a Banach space. If X has the weak fixed point property for D-type mappings, then X has the J-weak fixed point property.*

From Theorem 24, Theorem 16 of [10], and Proposition 29 we can conclude this project by the following.

Theorem 30. *Let X be a Banach space. If the asymptotic center relative to every nonempty weakly compact convex subset of each bounded sequence of X is compact, then X has property (C).*