Bibliography

[1] M. Bridson and A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, (1999).

[2] F.E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, Bull. Amer. Math. Soc. 74(1968) 660-665.

[3] K. S. Brown, Buildings, Springer-Verlag, New York, 1989.

[4] R.E. Bruck, Properties of fixed-point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. 179 (1973)

[5] F. Bruhat and J. Tits, Groupes réductifs sur un corps local. I. Données radicielles valuées, Inst. Hautes Études Sci. Publ. Math. 41 (1972), 5-251.

[6] D. Burago, Y. Burago, and S. Ivanov, A Course in Metric Geometry, Graduate Studies in Math. vol. 33, Amer. Math. Soc., Providence, RI, (2001).

[7] S. Dhompongsa, A. Kaewcharoen, A. Kaewkhao, The Domínguez-Lorenzo condition and fixed point for multi-valued mappings, Nonlinear Analysis. 64(2006) 958-970.

[8] S. Dhompongsa, W. Inthakon and A. Kaewkhao, Edelsteins method and fixed point theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 350 (2009), pp. 1217.

[9] M. Edelstein, The construction of an asymptotic center with a fixed point property, Bull. Amer. Math. Soc. 78 (1972) 206-208.

[10] J. Garcia-Falset, E. Llorens-Fuster and S. Prus, The fixed point property for mappings admitting a center, Nonlinear Anal. 66 (2007), pp. 12571274.

[11] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Univ. Press, 1990.

[12] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, Inc., New York, (1984).

[13] M. Gromov, Metric Structures for Riemannian and non-Riemannian Spaces, Progress in Mathematics 152, Birkhäuser, Boston, 1999.

[14] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44(1974) 147-150.

[15] A. Kaewkhao and K. Sokhuma, Remarks on asymptotic centers and fixed points, Abstract and Applied Analysis, in Press.

[16] M. A. Khamsi, On metric spaces with uniform normal structure, Proc. Amer. Math. Soc., Vol. 106, No.3 (1989), pp. 723-726.

[17] W. A. Kirk, Geodesic geometry and fixed point theory. In Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), pp. 195-225, Colecc. Abierta, 64, Univ. Sevilla Secr. Publ., Seville, (2003).

[18] W. A. Kirk, Geodesic geometry and fixed point theory II. In International Conference on Fixed Point Theory and Applications, pp. 113-142, Yokohama Publ., Yokohama, (2004).

[19] W.A. Kirk, S. Massa, Remarks on asymptotic and Chebyshev centers, Houston J. Math. 16 (1990) 364-375.

[20] B. Panyanak, Mann and Ishikawa iterative processes for multi-valued mappings in Banach space, Computers and Mathematics with Applications. 54(2007) 872-877.

[21] K.P.R. Sastry and G.V.R. Babu, Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point, Czechoslovak Mathematical Journal. 55(2005) 817-826.

[22] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43(1991) 153-159.

[23] N. Shahzad and H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Analysis. 71(2009) 838-844.

[24] K. Sokhuma and A. Kaewkhao, Ishikawa iterative process for a pair of single valued and multi-valued nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications, in Press.

[25] Y. Song and H. Wang, Erratum to Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces [Comput. Math. Appl. 54(2007) 872-877], Computers and Mathematics with Applications. 55(2008) 2999-3002.

[26] Y. Song and H. Wang, Convergence of iterative algorithms for multi-valued mappings in Banach spaces, Nonlinear Analysis. 70(2009) 1547-1556.

Output

ได้รับการตอบรับให้ตีพิมพ์แล้ว 2 เรื่อง

1. K. Sokhuma and A. Kaewkhao, Ishikawa iterative process for a pair of single valued and multi-valued nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications, in Press. (Impact Factor 1.525)
2. A. Kaewkhao and K. Sokhuma, Remarks on asymptotic centers and fixed points, Abstract and Applied Analysis, in Press. (Impact Factor 2.221)

ภาคผนวก

1. K. Sokhuma and A. Kaewkhao, Ishikawa iterative process for a pair of single valued and multi-valued nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications, in Press. (Impact Factor 1.525)

Ishikawa iterative process for a pair of single valued and multi-valued nonexpansive mappings in Banach spaces

K. Sokhuma[a] and A. Kaewkhao[b] *

[a] Department of Mathematics, Faculty of Science Burapha University, Chonburi 20131, THAILAND

[b] Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, THAILAND †

Abstract

Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and $t : E \to E$ and $T : E \to KC(E)$ be a single valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume in addition that $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. We prove that the sequence of the modified Ishikawa iteration method generated from an arbitrary $x_0 \in E$ by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n t y_n \end{aligned}$$

where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying

$$0 < a \leq \alpha_n, \beta_n \leq b < 1,$$

converges strongly to a common fixed point of t and T, i.e., there exists $x \in E$ such that $x = tx \in Tx$.

Keywords: Nonexpansive mapping, Fixed point, Uniformly convex Banach space, Ishikawa iteration.

1 Introduction

Let X be a Banach space and E a nonempty subset of X. We shall denote by $FB(E)$ the family of nonempty bounded closed subsets of E and by $KC(E)$ the family of nonempty compact convex subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, i.e.,

*Corresponding author.

† *E-mail addresses*: k_sokhuma@yahoo.co.th (Kritsana Sokhuma), akaewkhao@yahoo.com (Attapol Kaewkhao)

$$H(A,B) = \max\{ \sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A) \}, \quad A,B \in FB(X),$$

where $\operatorname{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the subset B.

A mapping $t : E \to E$ is said to be *nonexpansive* if

$$\|tx - ty\| \leq \|x - y\| \quad \text{for all } x, y \in E.$$

A point x is called a fixed point of t if $tx = x$.

A multi-valued mapping $T : E \to FB(X)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\| \quad \text{for all } x, y \in E.$$

A point x is called a fixed point for a multi-valued mapping T if $x \in Tx$.

We use the notation $\operatorname{Fix}(T)$ stands for the set of fixed points of a mapping T and $\operatorname{Fix}(t) \cap \operatorname{Fix}(T)$ stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $x = tx \in Tx$.

In 2006, S. Dhompongsa et al. ([2]) proved a common fixed point theorem for two nonexpansive commuting mappings.

Theorem 1.1. *[2, Theorem 4.2] Let E be a nonempty bounded closed convex subset of a uniformly Banach space X, $t : E \to E$, and $T : E \to KC(E)$ a nonexpansive mapping and a multi-valued nonexpansive mapping respectively. Assume that t and T are commuting , i.e. if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds $tx \in Tty$. Then t and T have a common fixed point.*

In this paper, we introduce an iterative process in a new sense, called the modified Ishikawa iteration method with respect to a pair of single valued and multi-valued nonexpansive mappings. We also establish the strong convergence theorem of a sequence from such process in a nonempty compact convex subset of a uniformly convex Banach space.

2 Preliminaries

The important property of a uniformly convex Banach space we use is the following lemma proved by Schu ([6]) in 1991.

Lemma 2.1. *([6]) Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of*

X such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ *and* $\lim_{n\to\infty} \|u_n x_n + (1-u_n)y_n\| = a$ *for some* $a \geq 0$. *Then,* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

The following observation will be used in proving our results and the proof is a straight-forward.

Lemma 2.2. *Let X be a Banach space and E be a nonempty closed convex subset of X. Then,*

$$dist(y, Ty) \leq \|y - x\| + dist(x, Tx) + H(Tx, Ty),$$

where $x, y \in E$ *and T is a multi-valued nonexpansive mapping from E into* $FB(E)$.

A fundamental principle which plays a key role in ergodic theory is the demiclosedness principle. A mapping t defined on a subset E of a Banach space X is said to be demiclosed if any sequence $\{x_n\}$ in E the following implication holds: $x_n \rightharpoonup x$ and $tx_n \to y$ implies $tx = y$.

Theorem 2.3. *([1]) Let E be a nonempty closed convex subset of a uniformly convex Banach space X and* $t : E \to E$ *be a nonexpansive mapping. If a sequence* $\{x_n\}$ *in E converges weakly to p and* $\{x_n - tx_n\}$ *converges to 0 as* $n \to \infty$, *then* $p \in Fix(t)$.

In 1974 , Ishikawa introduced the following well-known iteration.

Definition 2.4. *([3]) Let X be a Banach space. E a closed convex subset of X and t a selfmap on E. For* $x_0 \in E$, *the sequence* $\{x_n\}$ *of Ishikawa iterates of t is defined by,*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n t x_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n t y_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences.*

A nonempty subset K of E is said to be proximinal if, for any $x \in E$, there exists an element $y \in K$ such that $\|x - y\| = \text{dist}(x, K)$. We shall denote $P(K)$ by the family of nonempty proximinal bounded subsets of K.

In 2005, Sastry and Babu ([5]) defined the Ishikawa iterative scheme for multi-valued mappings as follows:

Let E be a compact convex subset of a Hilbert space X and $T : E \to P(E)$ be a multi-valued mapping and fix $p \in \text{Fix}(T)$.

$$\begin{cases} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,\ 1]$ with $z_n \in Tx_n$ such that $\|z_n - p\| = \text{dist}(p, Tx_n)$ and $\|z'_n - p\| = \text{dist}(p, Ty_n)$.

They also proved the strong convergence of the above Ishikawa iterative scheme for a multi-valued nonexpansive mapping T with a fixed point p under some certain conditions in a Hilbert space.

Recently, Panyanak ([4]) extended the results of Sastry and Babu ([5]) to a uniformly convex Banach space, and also modified the above Ishikawa iterative scheme as follows:

Let E be a nonempty convex subset of a uniformly convex Banach space X and $T : E \to P(E)$ be a multi-valued mapping

$$\begin{cases} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,\ 1]$ with $z_n \in Tx_n$ and $u_n \in \text{Fix}(T)$ such that $\|z_n - u_n\| = \text{dist}(u_n, Tx_n)$ and $\|x_n - u_n\| = \text{dist}(x_n, \text{Fix}(T))$, respectively. Moreover, $z'_n \in Tx_n$ and $v_n \in \text{Fix}(T)$ such that $\|z'_n - v_n\| = \text{dist}(v_n, Tx_n)$ and $\|y_n - v_n\| = \text{dist}(y_n, \text{Fix}(T))$, respectively.

Very recently, Song and Wang ([8, 9]) noted that there was a gap in the proofs of ([4, Theorem 3.1]), and ([5, Theorem 5]).

Thus they solved/revised the gap by means of the following Ishikawa iterative scheme:

Let $T : E \to FB(E)$ be a multi-valued mapping, where $\alpha_n, \beta_n \in [0, 1)$. The Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$ such that $\|z_n - z'_n\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z'_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$, respectively. Moreover, $\gamma_n \in (0, +\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$.

At the same period, Shahzad and Zegeye ([7]) modified the Ishikawa iterative scheme $\{x_n\}$ and extended the result of ([8, Theorem 2]) to a multi-valued quasi-nonexpansive mapping as follows:

Let K be a nonempty convex subset of a Banach space X and $T : E \to FB(E)$ a multi-valued mapping, where $\alpha_n, \beta_n \in [0, 1]$. The Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{cases} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{cases}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$.

In this paper we introduce a new iteration method modifying the above ones and call it the modified Ishikawa iteration method.

Definition 2.5. *Let E be a nonempty closed bounded convex subset of a Banach space X, $t : E \to E$ be a single valued nonexpansive mapping, and $T : E \to FB(E)$ be a multi-valued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n t y_n \end{aligned} \tag{2.1}$$

where $x_0 \in E$, $z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

3 Main Results

We first prove the following lemmas, which play very important roles in this section.

Lemma 3.1. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X. $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.1). Then $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in Fix(t) \cap Fix(T)$.*

Proof. Let $x_0 \in E$ and $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)x_n + \alpha_n t((1-\beta_n)x_n + \beta_n z_n) - w\| \\ &= \|(1-\alpha_n)x_n + \alpha_n t((1-\beta_n)x_n + \beta_n z_n) - (1-\alpha_n)w - \alpha_n w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|t((1-\beta_n)x_n + \beta_n z_n) - w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|(1-\beta_n)x_n + \beta_n z_n - w\| \\ &= (1-\alpha_n)\|x_n - w\| + \alpha_n \|(1-\beta_n)x_n + \beta_n z_n - (1-\beta_n)w - \beta_n w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \|z_n - w\| \\ &= (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \mathrm{dist}(z_n, Tw) \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n H(Tx_n, Tw) \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned}$$

Since $\{\|x_n - w\|\}$ is a decreasing and bounded sequence, we can conclude that the limit of $\{\|x_n - w\|\}$ exists. □

We can see how Lemma 2.1 is useful via the following lemma.

Lemma 3.2. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.1). If $0 < a \leq \alpha_n \leq b < 1$ for some a, $b \in \mathbb{R}$, then $\lim_{n\to\infty} \|ty_n - x_n\| = 0$.*

Proof. Let $w \in \text{Fix}(t) \cap \text{Fix}(T)$. By Lemma 3.1, we put $\lim_{n\to\infty} \|x_n - w\| = c$ and consider

$$\begin{aligned}
\|ty_n - w\| &\leq \|y_n - w\| \\
&= \|(1-\beta_n)x_n + \beta_n z_n - w\| \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \|z_n - w\| \\
&= (1-\beta_n)\|x_n - w\| + \beta_n \text{dist}(z_n, Tw) \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw) \\
&\leq (1-\beta_n)\|x_n - w\| + \beta_n \|x_n - w\| \\
&= \|x_n - w\|.
\end{aligned}$$

Then we have

$$\limsup_{n\to\infty} \|ty_n - w\| \leq \limsup_{n\to\infty} \|y_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c. \tag{3.1}$$

Further, we have

$$\begin{aligned}
c &= \lim_{n\to\infty} \|x_{n+1} - w\| \\
&= \lim_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n ty_n - w\| \\
&= \lim_{n\to\infty} \|\alpha_n ty_n - \alpha_n w + x_n - \alpha_n x_n + \alpha_n w - w\| \\
&= \lim_{n\to\infty} \|\alpha_n(ty_n - w) + (1-\alpha_n)(x_n - w)\|.
\end{aligned}$$

By Lemma 2.1, we can conclude that $\lim_{n\to\infty} \|(ty_n - w) - (x_n - w)\| = \lim_{n\to\infty} \|ty_n - x_n\| = 0$. □

Lemma 3.3. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.1). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some a, $b \in \mathbb{R}$, then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Proof. Let $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. We put, as in Lemma 3.2, $\lim_{n\to\infty} \|x_n - w\| = c$. For $n \geq 0$, we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)x_n + \alpha_n t y_n - w\| \\ &= \|(1-\alpha_n)x_n + \alpha_n t y_n - (1-\alpha_n)w - \alpha_n w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|t y_n - w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n \|y_n - w\|, \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - w\| - \|x_n - w\| &\leq -\alpha_n \|x_n - w\| + \alpha_n \|y_n - w\| \\ \|x_{n+1} - w\| - \|x_n - w\| &\leq \alpha_n(\|y_n - w\| - \|x_n - w\|) \\ \frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} &\leq \|y_n - w\| - \|x_n - w\|. \end{aligned}$$

Therefore, since $0 < a \leq \alpha_n \leq b < 1$,

$$\left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n}\right) + \|x_n - w\| \leq \|y_n - w\|.$$

Thus,

$$\liminf_{n\to\infty} \left\{\left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n}\right) + \|x_n - w\|\right\} \leq \liminf_{n\to\infty} \|y_n - w\|.$$

It follows that

$$c \leq \liminf_{n\to\infty} \|y_n - w\|.$$

Since, from (3.1), $\limsup_{n\to\infty} \|y_n - w\| \leq c$, we have

$$\begin{aligned} c &= \lim_{n\to\infty} \|y_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n z_n - w\| \\ &= \lim_{n\to\infty} \|(1-\beta_n)(x_n - w) + \beta_n(z_n - w)\|. \end{aligned} \tag{3.2}$$

Recall that

$$\begin{aligned} \|z_n - w\| &= \mathrm{dist}(z_n, Tw) \\ &\leq H(Tx_n, Tw) \\ &\leq \|x_n - w\|. \end{aligned}$$

Hence we have

$$\limsup_{n\to\infty} \|z_n - w\| \leq \limsup_{n\to\infty} \|x_n - w\| = c.$$

Using the fact that $0 < a \le \beta_n \le b < 1$ and (3.2), we can conclude that $\lim_{n\to\infty} \|x_n - z_n\| = 0$.
□

The following lemma allows us to go on.

Lemma 3.4. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.1). If $0 < a \le \alpha_n, \beta_n \le b < 1$, then $\lim_{n\to\infty} \|tx_n - x_n\| = 0$.*

Proof. Consider

$$\begin{aligned}
\|tx_n - x_n\| &= \|tx_n - ty_n + ty_n - x_n\| \\
&\le \|tx_n - ty_n\| + \|ty_n - x_n\| \\
&\le \|x_n - y_n\| + \|ty_n - x_n\| \\
&= \|x_n - (1-\beta_n)x_n - \beta_n z_n\| + \|ty_n - x_n\| \\
&= \|x_n - x_n + \beta_n x_n - \beta_n z_n\| + \|ty_n - x_n\| \\
&= \beta_n \|x_n - z_n\| + \|ty_n - x_n\|.
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty} \|tx_n - x_n\| \le \lim_{n\to\infty} \beta_n \|x_n - z_n\| + \lim_{n\to\infty} \|ty_n - x_n\|.$$

Hence, by Lemma 3.2 and Lemma 3.3, $\lim_{n\to\infty} \|tx_n - x_n\| = 0$. □

We give the sufficient conditions which imply the existence of common fixed points for single valued mappings and multi-valued nonexpansive mappings, respectively, as follow:

Theorem 3.5. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.1). If $0 < a \le \alpha_n, \beta_n \le b < 1$, then $x_{n_i} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in Fix(t) \cap Fix(T)$.*

Proof. Assumed that $\lim_{n\to\infty} \|x_{n_i} - y\| = 0$. From Lemma 3.4, we have

$$0 = \lim_{n\to\infty} \|tx_{n_i} - x_{n_i}\| = \lim_{n\to\infty} \|(I - t)(x_{n_i})\|.$$

Since $I - t$ is demiclosed at 0, we have $(I - t)(y) = 0$ and hence $y = ty$, i.e., $y \in \mathrm{Fix}(t)$. By Lemma 2.2 and by Lemma 3.4, we have

$$\begin{aligned}
\mathrm{dist}(y, Ty) &\le \|y - x_{n_i}\| + \mathrm{dist}(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty) \\
&\le \|y - x_{n_i}\| + \|x_{n_i} - z_{n_i}\| + \|x_{n_i} - y\| \to 0 \qquad \text{as } i \to \infty.
\end{aligned}$$

It follows that $y \in \mathrm{Fix}(T)$. Therefore $y \in \mathrm{Fix}(t) \cap Fix(T)$ as desired. □

Hereafter, we arrive at the convergence theorem of the sequence of the modified Ishikawa iteration. We conclude this paper with the following theorem.

Theorem 3.6. *Let E be a nonempty compact convex subset of a uniformly convex Banach space X, $t : E \to E$ and $T : E \to FB(E)$ a single valued and a multi-valued nonexpansive mapping, respectively, and $Fix(t) \cap Fix(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in Fix(t) \cap Fix(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.1) with $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of t and T.*

Proof. Since $\{x_n\}$ is contained in E which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in E$, i.e., $\lim_{i\to\infty} \|x_{n_i} - y\| = 0$. By Theorem 3.5, we have $y \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ and by Lemma 3.1, we have that $\lim_{n\to\infty} \|x_n - y\|$ exists. It must be the case that $\lim_{n\to\infty} \|x_n - y\| = \lim_{i\to\infty} \|x_{n_i} - y\| = 0$. Therefor $\{x_n\}$ converges strongly to a common fixed point y of t and T. □

4 Acknowledgements

The first author would like to thank the Office of the Higher Education Commission, Thailand for supporting by grant fund under the program Strategic Scholarships for Frontier Research Network for the Ph.D. Program Thai Doctoral degree for this research. This work was also completed with the support of the Commission on Higher Education and The Thailand Research Fund under grant MRG5180213. Moreover, we would like to express my deep gratitude to Prof. Dr. Sompong Dhompongsa whose guidance and support were valuable for the completion of the paper.

References

[1] F.E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, Bull. Amer. Math. Soc. 74(1968) 660-665.

[2] S. Dhompongsa, A. Kaewcharoen, A. Kaewkhao, The Domínguez-Lorenzo condition and fixed point for multi-valued mappings, Nonlinear Analysis. 64(2006) 958-970.

[3] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44(1974) 147-150.

[4] B. Panyanak, Mann and Ishikawa iterative processes for multi-valued mappings in Banach space, Computers and Mathematics with Applications. 54(2007) 872-877.

[5] K.P.R. Sastry and G.V.R. Babu, Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point, Czechoslovak Mathematical Journal. 55(2005) 817-826.

[6] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43(1991) 153-159.

[7] N. Shahzad and H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Analysis. 71(2009) 838-844.

[8] Y. Song and H. Wang, Erratum to *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces* [Comput. Math. Appl. 54(2007) 872-877], Computers and Mathematics with Applications. 55(2008) 2999-3002.

[9] Y. Song and H. Wang, Convergence of iterative algorithms for multi-valued mappings in Banach spaces, Nonlinear Analysis. 70(2009) 1547-1556.

2. A. Kaewkhao and K. Sokhuma, Remarks on asymptotic centers and fixed points, Abstract and Applied Analysis, in Press. (Impact Factor 2.221)

Remarks on asymptotic centers and fixed points

A. Kaewkhao[a] *and K. Sokhuma[b]

[a] Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, THAILAND

[b] Department of Mathematics, Faculty of Science, Burapha University, Chonburi 20131, THAILAND †

Dedicated to Professor Sompong Dhompongsa on the occation to his 60^{th} birthday

Abstract

We introduce a class of nonlinear continuous mappings defined on a bounded closed convex subset of a Banach space X. We characterize the Banach spaces in which every asymptotic center of each bounded sequence in any weakly compact convex subset is compact as those spaces having the weak fixed point property for this type of mappings.

1 Introduction

A mapping T on a subset E of a Banach space X is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. Although nonexpansive mappings are widely studied, there are many nonlinear mappings which are more general. The study of the existence of fixed points for those mappings is very useful in solving in the problems of equations in science and applied science.

The technique of employing the asymptotic centers and their Chebyshev radii in fixed point theory was first discovered by Edelstein [3] and the compactness assumption given on asymptotic centers was introduced by Kirk and Massa [6]. Recently, Dhompongsa et al. proved in [2] a theorem of existence of fixed points for some generalized nonexpansive mappings on a bounded closed convex subset E of a Banach space with assumption that every asymptotic center of a bounded sequence relative to E is nonempty and compact. However, spaces or sets in which asymptotic centers are compact have not been completely characterized, but partial results are known (see [5, pp. 93]).

In this paper, we introduce a class of nonlinear continuous mappings in Banach spaces which allows us to characterize the Banach spaces in which every asymptotic center of each bounded sequence in any weakly compact convex subset is compact as those spaces having the weak fixed point property for this type of mappings.

*Corresponding author.

†*E-mail addresses*: akaewkhao@yahoo.com (Attapol Kaewkhao), k_sokhuma@yahoo.co.th (Kritsana Sokhuma)

2 Preliminaries

Let E be a nonempty closed and convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

Let

$$r \equiv r(E, \{x_n\}) := \inf\{r(x, \{x_n\}) : x \in E\}$$

and

$$A \equiv A(E, \{x_n\}) := \{x \in E : r(x, \{x_n\}) = r\}.$$

The number r and the set A are, respectively, called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to E. It is known that $A(E, \{x_n\})$ is nonempty, weakly compact and convex as E is [5, pp. 90].

Let $T : E \to E$ be a nonexpansive and $z \in E$. Then for $\alpha \in (0, 1)$ the mapping $T_\alpha : E \to E$ defined by setting

$$T_\alpha x = (1 - \alpha)z + \alpha T x$$

is a contraction mapping. As we have known, Banach contraction mapping theorem assures the existence of a unique fixed point $x_\alpha \in E$. Since

$$\lim_{\alpha\to 1^-} \|x_\alpha - Tx_\alpha\| = \lim_{\alpha\to 1^-} (1 - \alpha)\|z - Tx_\alpha\| = 0,$$

we have the following.

Lemma 2.1. *If E is a bounded closed and convex subset of a Banach space and if $T : E \to E$ is nonexpansive, then there exists a sequence $\{x_n\} \subset E$ such that*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

3 Main Results

Definition 3.1. *Let E be a bounded closed convex subset of a Banach space X. We say that a sequence $\{x_n\}$ in X is an asymptotic center sequence for a mapping $T : E \to X$ if, for each $x \in E$,*

$$\limsup_{n\to\infty} \|x_n - Tx\| \le \limsup_{n\to\infty} \|x_n - x\|.$$

We say that $T : E \to X$ is a *D-type mapping* whenever it is continuous and there is an asymptotic center sequence for T.

The following observation shows that the concept of D-type mappings is a generalization of nonexpansiveness.

Proposition 3.2. *Let $T : E \to E$ be a nonexpansive mapping. Then T is a D-type mapping.*

Proof. It is easy to see that T is continuous. By Lemma 2.1 there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

For $x \in E$,

$$\|x_n - Tx\| \leq \|x_n - Tx_n\| + \|Tx_n - Tx\| \leq \|x_n - Tx_n\| + \|x_n - x\|.$$

Then

$$\limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

This implies that $\{x_n\}$ is an asymptotic center sequence for T. Thus T is a D-type mapping. □

Definition 3.3. *We say that a Banach space $(X, \|\cdot\|)$ has the weak fixed point property for D-type mappings if every D-type self-mapping on every weakly compact convex subset of X has a fixed point.*

Now we are in the position to prove our main theorem.

Theorem 3.4. *Let X be a Banach space. Then X has the weak fixed point property for D-type mappings if and only if the asymptotic center relative to each nonempty weakly compact convex subset of each bounded sequence of X is compact.*

Proof. Suppose the asymptotic center of any bounded sequence of X relative to any nonempty weakly compact convex subset of X is compact. Let E be a weakly compact convex subset of X and $T : E \to E$ be a D-type mapping having $\{x_n\}$ as an asymptotic center sequence. Let r and A, respectively, be the asymptotic radius and the asymptotic center of $\{x_n\}$ relative to E. Since E is weakly compact and convex, A is nonempty weakly compact and convex. For every $x \in A$, since $\{x_n\}$ is an asymptotic center sequence for T, we have

$$r \leq \limsup_{n\to\infty} \|x_n - Tx\| \leq \limsup_{n\to\infty} \|x_n - x\| = r.$$

Hence $T(x) \in A$, which implies that A is $T-$invariant. By the assumption, A is a compact set. By using Schauder's fixed point theorem we can conclude that T has a fixed point in A and hence T has a fixed point in E.

Now suppose X has the weak fixed point property for D-type mappings, and suppose there exists a weakly compact convex subset K of X and a bounded sequence $\{x_n\}$ in X whose asymptotic center A relative to K is not compact. By Klee's theorem (see [5, pp. 203]), there exists a continuous, fixed point free mapping $T : A \to A$. We see that $\{x_n\}$ is an asymptotic center sequence for T. Indeed, since $Tx \in A$ for each $x \in A$, we have

$$\limsup_{n\to\infty} \|x_n - Tx\| = r = \limsup_{n\to\infty} \|x_n - x\|.$$

Then T is a D-type mapping. Thus T should have a fixed point which is a contradiction. □

In 2007, Garcia-Falset et al. [4] introduced another concept of centers of mappings.

Definition 3.5. *Let E be a bounded closed convex subset of a Banach space X. A point $x_0 \in X$ is said to be a center for a mapping $T : E \to X$ if, for each $x \in E$,*

$$\|Tx - x_0\| \leq \|x - x_0\|.$$

A mapping $T : E \to X$ is said to be a J-type mapping whenever it is continuous and it has some center $x_0 \in X$.

Definition 3.6. *We say that a Banach space X has the J-weak fixed point property if every J-type self-mapping of every weakly compact subset E of X has a fixed point.*

Employing the above definitions, the authors proved a characterization of the geometrical property (C) of the Banach spaces introduced in 1973 by R. E. Bruck [1] : A Banach space X has property (C) whenever the weakly compact convex subsets of its unit sphere are compact sets.

Theorem 3.7. *[4, Theorem 16] Let X be a Banach space. Then X has property (C) if and only if X has the J-weak fixed point property.*

It is easy to see that a center $x_0 \in X$ of a mapping $T : E \to X$ is can be seen as an asymptotic center sequence $\{x_n\}$ for the mapping T by setting $x_n \equiv x_0$ for all $n \in \mathbb{N}$. This leads to the following conclusion.

Proposition 3.8. *Let $T : E \to X$ be a J-type mapping. Then T is a D-type mapping.*

Consequently, we have

Proposition 3.9. *Let X be a Banach space. If X has the weak fixed point property for D-type mappings, then X has the J-weak fixed point property.*

From Theorem 3.4, Theorem 3.7, and Proposition 3.9 we can conclude this paper by the following result:

Theorem 3.10. *Let X be a Banach space. If the asymptotic center relative to every nonempty weakly compact convex subset of each bounded sequence of X is compact, then X has property (C).*

4 Acknowledgements

This work was completed with the support of the Commission on Higher Education and The Thailand Research Fund under grant MRG5180213. The author would like to express my deep gratitude to Prof. Dr. Sompong Dhompongsa whose guidance and support were valuable for the completion of the paper.

References

[1] R.E. Bruck, Properties of fixed-point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. 179 (1973)

[2] S. Dhompongsa, W. Inthakon and A. Kaewkhao, Edelsteins method and fixed point theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 350 (2009), pp. 1217.

[3] M. Edelstein, The construction of an asymptotic center with a fixed point property, Bull. Amer. Math. Soc. 78 (1972) 206-208.

[4] J. Garcia-Falset, E. Llorens-Fuster and S. Prus, The fixed point property for mappings admitting a center, Nonlinear Anal. 66 (2007), pp. 12571274.

[5] K. Goebel, W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Univ. Press, 1990.

[6] W.A. Kirk, S. Massa, Remarks on asymptotic and Chebyshev centers, Houston J. Math. 16 (1990) 364-375.