CHAPTER 3 TWO TYPES OF DP TRIANGULAR PATHCES

In this chapter, two new models of triangular patches are introduced by using the notions of DP univariate polynomial basis [11]. This basis is interesting the following reasons:

1. Linear complexity: Its evaluation algorithm has linear complexity.
2. Normalized Totally Positive Property(NTP): Its blending functions always have the normalized totally positive property. Consequently, it is a shape preserving representation [14].
3. Coefficient calculation avoidance: There is no need to calculate for the coefficients of the polynomials.

Employing this univariate basis for DP curves, we can derive a new bivariate basis. As a result, we expect to obtain a new set of blending functions that can form a new basis of bivariate polynomials as well as providing proper DP form characteristics.
In this result, the key requirements for new bivariate polynomials of the surface model adapted from these characteristics of DP blending functions, will be:

1. An algorithm that can calculate a point on this triangular surface within a quadratic time.
2. The proposed set of polynomials are defined almost with no coefficient values in order to avoid calculation time. Hence, only some coefficients will be calculated.

An easy way to explain and get a point on a DP curve is to repeatedly interpolate two consecutive points together. For DP curve, there are two main types of interpolations, (1) with parameter t and (2) with exactly $\frac{1}{2}$.
Interpolated with t, the new point will be $q = (1-t)p_1 + tp_2$, while interpolated with $\frac{1}{2}$, the result is $q = \frac{1}{2}(p_1 + p_2)$, as illustrated as follows.

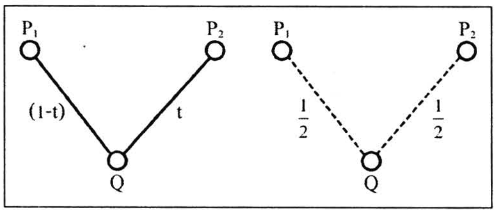

Figure 3.1 Two Types of Interpolations for DP Curve

Thus, a point on DP curve can be recursively computed by the interpolation concept as shown in Figure 3.2. In practice, we will use an iterative method to calculate a point on DP curve instead of this recursive one.

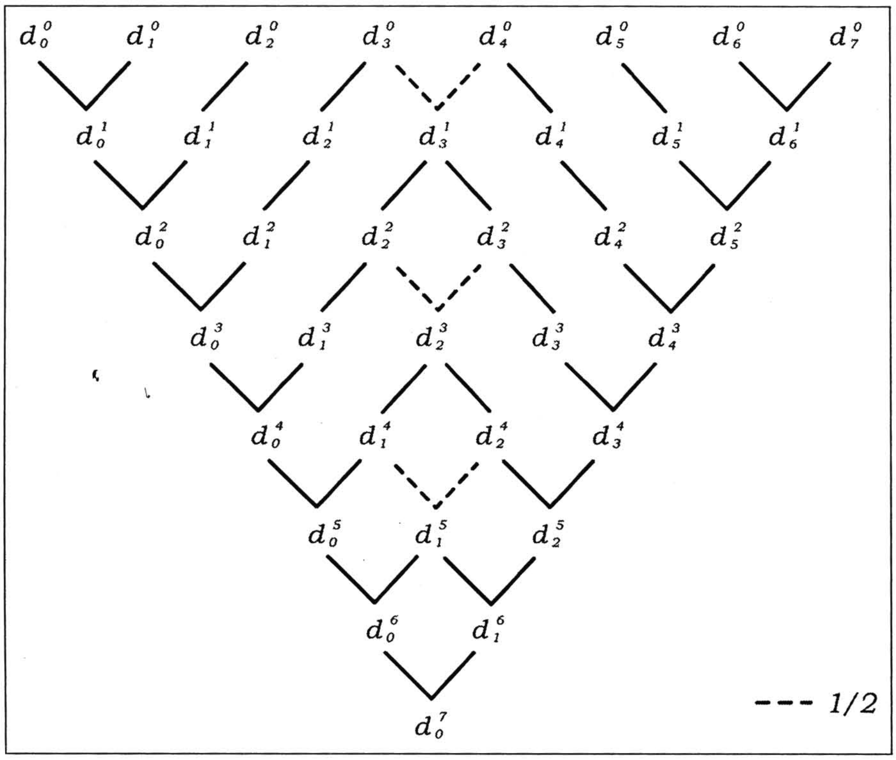

Figure 3.2 The Triangular Schema of DP Curve

Similarly, there are two main types of interpolations in our model. Given three points, P_1, P_2 and P_3 respectively, the triangular interpolations can be explicitly expressed as following.

(1) $Q = uP_1 + vP_2 + wP_3$.

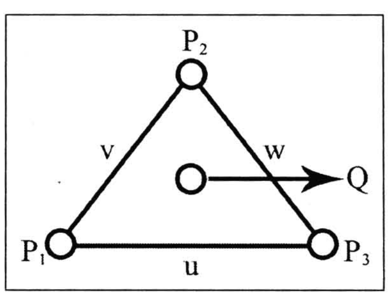

Figure 3.3 First Type of Interpolations for Our Model

(2) $Q = \frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$.

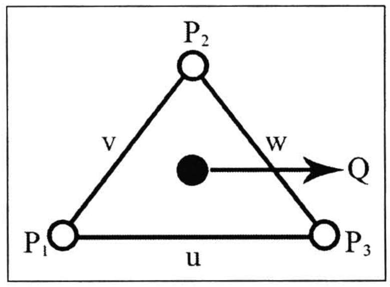

Figure 3.4 Second Type of Interpolations for Our Model

Futhermore, there are two kinds of anti-interpolations used to derive the affine-combination, i.e., Partition of unity ($\sum_{i+j+k=n} \mathcal{D}_{i,j,k}(u, v, w) = 1$)

The technique is to duplicate or generate two or three points at the same location, in order to obtain a set of new control points with the true topology of control points in the next level of interpolations as shown in Figures 3.5 and 3.6

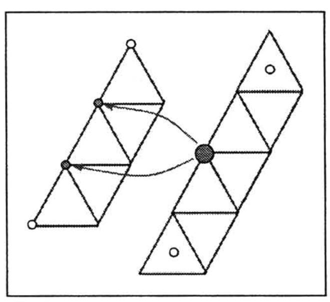

Figure 3.5 Two Points at the Same Location

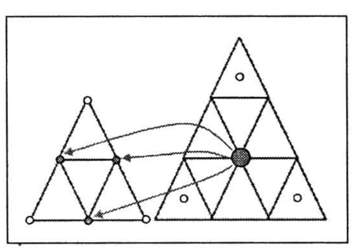

Figure 3.6 Three Points at the Same Location

3.1 Our First Model of Triangular DP Surfaces

In this section, the first model of triangular DP surfaces is presented with recursive algorithm, recurrence algorithm and quadratic computational complexity, $O(n^2)$.

3.1.1 Bivariate Basis Function

Our first bivariate DP basis functions can be constructed on a triangular domain by a given set of control nets. The control points are denoted by $\mathbf{d}_{i,j,k}$, where $i + j + k$ is equal to the degree of surface. The DP basis, denoted by $\mathcal{D}^n_{i,j,k}(u, v, w)$, will then be used for defining the new model as follows:

$$\mathcal{D}^n(u, v, w) = \sum_{i+j+k=n} \mathbf{d}_{i,j,k} \cdot \mathcal{D}^n_{i,j,k}(u, v, w), \tag{3.1}$$

where for each $u, v, w \geq 0$ and $u + v + w = 1$.

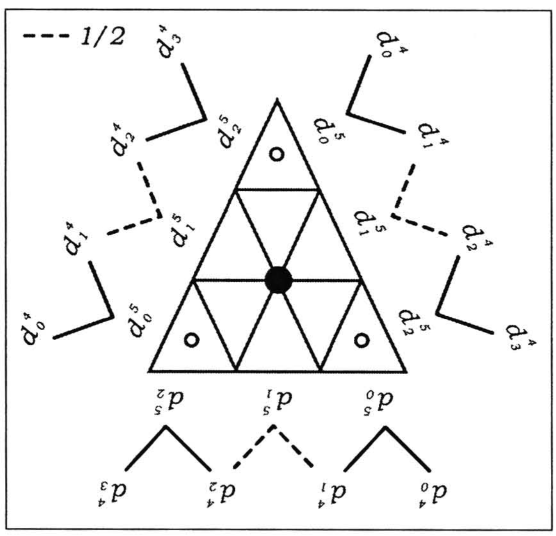

Figure 3.7 How to Interpolate the Control Points for Our First Triangular DP Surface

We can construct our first triangular DP surfaces by using the interpolation technique from the DP curve. We can apply the characteristics of the DP curve to all sides of each triangle as shown in Figure 3.7. At the middle of each side, we will press the point into the center of the triangular. We will interpolate with $\frac{1}{3}$ at the center of the triangle and corners of inner triangles.

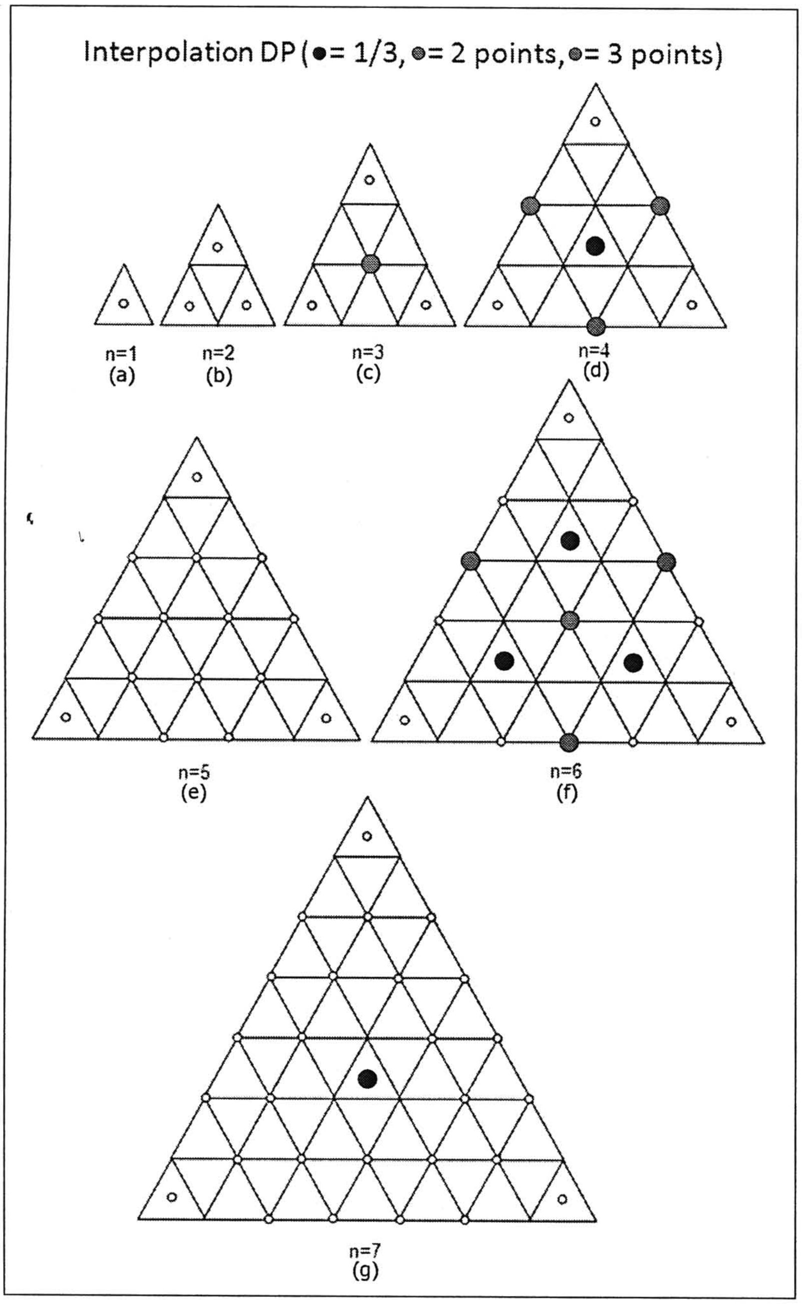

Figure 3.8 Our First Proposed Schema for Triangular DP Surface

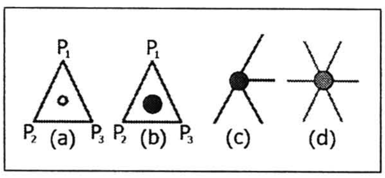

Figure 3.9 The Concepts of Our First DP Interpolations

Figure 3.8 shows some examples of how to interpolate the control points for degree 3, 4, 5, 6 and 7 respectively. The concepts of our first DP interpolations can be explained as follows:

1. From Figure 3.9(a), any points located at each corner must be interpolated with u, v, and w. The 3-point interpolation result can be computed by $uP_1 + vP_2 + wP_3$.
2. From Figure 3.9(b), a point at the center of the triangle and any points at the corner of inner triangles will be assigned to interpolate with $\frac{1}{3}$. The interpolation result can be computed by $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$.
3. From Figure 3.9(c), this point will be used to generate two duplicate points (two points with the same coordinates).
4. From Figure 3.9(d), this point will be used to generate three duplicate points (three points with the same coordinates).
5. The remaining points remain the same.

Definition 3.1

The DP basis functions of our first triangular DP surface are defined as follows:

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} (u+v)w^- + (u+w)v^- + (v+w)u^- & , \text{ if } i = j = k, \\ u^i v + uv^j & , \text{ if } i = j \text{ and } j > k, \\ u^i w + uw^j & , \text{ if } k = i \text{ and } i > j, \\ v^i w + vw^j & , \text{ if } j = k \text{ and } k > i, \\ u^i v & , \text{ if } k = 0 \text{ and } i > \lfloor \frac{n}{2} \rfloor, \\ v^j u & , \text{ if } k = 0 \text{ and } j > \lfloor \frac{n}{2} \rfloor, \\ u^i w & , \text{ if } j = 0 \text{ and } i > \lfloor \frac{n}{2} \rfloor, \\ w^k u & , \text{ if } j = 0 \text{ and } k > \lfloor \frac{n}{2} \rfloor, \\ v^j w & , \text{ if } i = 0 \text{ and } j > \lfloor \frac{n}{2} \rfloor, \\ w^k v & , \text{ if } i = 0 \text{ and } k > \lfloor \frac{n}{2} \rfloor. \end{cases} \tag{3.2}$$

if $i = n$ or $j = n$ or $k = n$

$$\mathcal{D}^n_{i,j,k}(u,v,w) = u^i v^j w^k. \tag{3.3}$$

if $i = 1$ or $j = 1$ or $k = 1$

$$\mathcal{D}^n_{i,j,k}(u,v,w) = (\frac{u}{3^{\lfloor - \rfloor}} + \frac{v}{3^{\lfloor - \rfloor}})w + (\frac{u}{3^{\lfloor - \rfloor}} + \frac{w}{3^{\lfloor - \rfloor}})v + (\frac{v}{3^{\lfloor - \rfloor}} + \frac{w}{3^{\lfloor - \rfloor}})u. \tag{3.4}$$

otherwise

$$\mathcal{D}^n_{i,j,k}(u,v,w) = (\frac{u}{3} + \frac{v}{3})w^{\lfloor - \rfloor} + (\frac{u}{3} + \frac{w}{3})v^{\lfloor - \rfloor} + (\frac{v}{3} + \frac{w}{3})u^{\lfloor - \rfloor}. \tag{3.5}$$

Some examples of the DP polynomials are given below.

Example 3.1

For degree n=3,

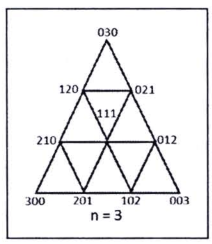

Figure 3.10 Schema for Triangular DP Surface of Degree 3

$\mathcal{D}^3_{3,0,0}(u,v,w) = u^3, \mathcal{D}^3_{2,1,0}(u,v,w) = u^2v, \mathcal{D}^3_{2,0,1}(u,v,w) = u^2w, \mathcal{D}^3_{1,2,0}(u,v,w) = uv^2,$
$\mathcal{D}^3_{1,1,1}(u,v,w) = (u+v)w + v(u+w) + u(v+w), \mathcal{D}^3_{1,0,2}(u,v,w) = uw^2,$
$\mathcal{D}^3_{0,3,0}(u,v,w) = v^3, \mathcal{D}^3_{0,2,1}(u,v,w) = v^2w, \mathcal{D}^3_{0,1,2}(u,v,w) = vw^2, \mathcal{D}^3_{0,0,3}(u,v,w) = w^3.$

For degree n=4,

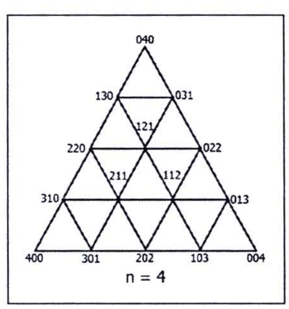

Figure 3.11 Schema for Triangular DP Surface of Degree 4

$\mathcal{D}^4_{4,0,0}(u,v,w) = u^4, \mathcal{D}^4_{3,1,0}(u,v,w) = u^3v, \mathcal{D}^4_{3,0,1}(u,v,w) = u^3w,$
$\mathcal{D}^4_{2,2,0}(u,v,w) = u^2v + uv^2, \mathcal{D}^4_{2,1,1}(u,v,w) = v(\frac{u}{3} + \frac{w}{3}) + u(\frac{v}{3} + \frac{w}{3}) + (\frac{u}{3} + \frac{v}{3})w,$
$\mathcal{D}^4_{2,0,2}(u,v,w) = u^2w + uw^2, \mathcal{D}^4_{1,3,0}(u,v,w) = uv^3,$
$\mathcal{D}^4_{1,2,1}(u,v,w) = v(\frac{u}{3} + \frac{w}{3}) + u(\frac{v}{3} + \frac{w}{3}) + (\frac{u}{3} + \frac{v}{3})w,$
$\mathcal{D}^4_{1,1,2}(u,v,w) = v(\frac{u}{3} + \frac{w}{3}) + u(\frac{v}{3} + \frac{w}{3}) + (\frac{u}{3} + \frac{v}{3})w, \mathcal{D}^4_{1,0,3}(u,v,w) = uw^3,$
$\mathcal{D}^4_{0,4,0}(u,v,w) = v^4, \mathcal{D}^4_{0,3,1}(u,v,w) = v^3w, \mathcal{D}^4_{0,2,2}(u,v,w) = v^2w + vw^2,$
$\mathcal{D}^4_{0,1,3}(u,v,w) = vw^3, \mathcal{D}^4_{0,0,4}(u,v,w) = w^4.$

For degree n=5,

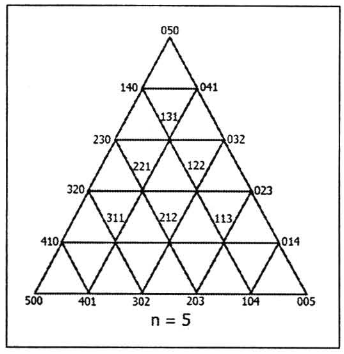

Figure 3.12 Schema for Triangular DP Surface of Degree 5

$$\mathcal{D}^5_{5,0,0}(u,v,w) = u^5, \mathcal{D}^5_{4,1,0}(u,v,w) = u^4v, \mathcal{D}^5_{4,0,1}(u,v,w) = u^4w,$$
$$\mathcal{D}^5_{3,2,0}(u,v,w) = uv^3, \mathcal{D}^5_{3,1,1}(u,v,w) = v(\tfrac{u}{3} + \tfrac{w}{3}) + u(\tfrac{v}{3} + \tfrac{w}{3}) + (\tfrac{u}{3} + \tfrac{v}{3})w,$$
$$\mathcal{D}^5_{3,0,2}(u,v,w) = u^3w, \mathcal{D}^5_{2,3,0}(u,v,w)uv^3, \mathcal{D}^5_{2,2,1}(u,v,w) = u^2v + uv^2,$$
$$\mathcal{D}^5_{2,1,2}(u,v,w) = u^2w + uw^2, \mathcal{D}^5_{2,0,3}(u,v,w) = uw^3, \mathcal{D}^5_{1,4,0}(u,v,w) = uv^4,$$
$$\mathcal{D}^5_{1,3,1}(u,v,w) = v(\tfrac{u}{3} + \tfrac{w}{3}) + u(\tfrac{v}{3} + \tfrac{w}{3}) + (\tfrac{u}{3} + \tfrac{v}{3})w, \mathcal{D}^5_{1,2,2}(u,v,w) = v^2w + vw^2,$$
$$\mathcal{D}^5_{1,1,3}(u,v,w) = v(\tfrac{u}{3} + \tfrac{w}{3}) + u(\tfrac{v}{3} + \tfrac{w}{3}) + (\tfrac{u}{3} + \tfrac{v}{3})w, \mathcal{D}^5_{1,0,4}(u,v,w) = uw^4,$$
$$\mathcal{D}^5_{0,5,0}(u,v,w) = v^5, \mathcal{D}^5_{0,4,1}(u,v,w) = v^4w, \mathcal{D}^5_{0,3,2}(u,v,w) = v^3w,$$
$$\mathcal{D}^5_{0,2,3}(u,v,w) = vw^3, \mathcal{D}^5_{0,1,4}(u,v,w) = vw^4, \mathcal{D}^5_{0,0,5}(u,v,w) = w^5.$$

For degree n=6,

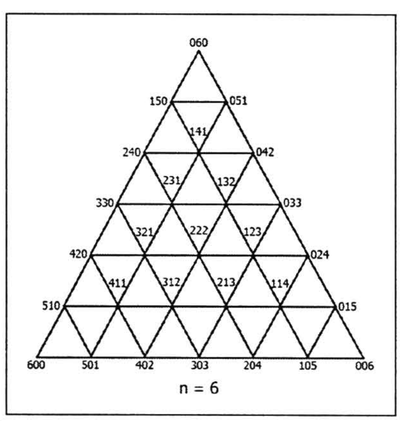

Figure 3.13 Schema for Triangular DP Surface of Degree 6

$$\mathcal{D}^6_{6,0,0}(u,v,w) = u^6, \mathcal{D}^6_{5,1,0}(u,v,w) = u^5v, \mathcal{D}^6_{5,0,1}(u,v,w) = u^5w,$$
$$\mathcal{D}^6_{4,2,0}(u,v,w) = u^4v, \mathcal{D}^6_{4,1,1}(u,v,w) = v(\tfrac{u}{9} + \tfrac{w}{9}) + u(\tfrac{v}{9} + \tfrac{w}{9}) + (\tfrac{u}{9} + \tfrac{v}{9})w,$$
$$\mathcal{D}^6_{4,0,2}(u,v,w) = u^4w, \mathcal{D}^6_{3,3,0}(u,v,w) = u^3v + uv^3,$$
$$\mathcal{D}^6_{3,2,1}(u,v,w) = v(\tfrac{u}{9} + \tfrac{w}{9}) + u(\tfrac{v}{9} + \tfrac{w}{9}) + (\tfrac{u}{9} + \tfrac{v}{9})w,$$
$$\mathcal{D}^6_{3,1,2}(u,v,w) = v(\tfrac{u}{9} + \tfrac{w}{9}) + u(\tfrac{v}{9} + \tfrac{w}{9}) + (\tfrac{u}{9} + \tfrac{v}{9})w,$$

$\mathcal{D}^6_{3,0,3}(u,v,w) = u^3w + uw^3, \mathcal{D}^6_{2,4,0}(u,v,w) = uv^4,$
$\mathcal{D}^6_{2,3,1}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^6_{2,2,2}(u,v,w) = v^2(u+w)u^2(v+w) + (u+v)w^2,$
$\mathcal{D}^6_{2,1,3}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^6_{2,0,4}(u,v,w) = uw^4, \mathcal{D}^6_{1,5,0}(u,v,w) = uv^5,$
$\mathcal{D}^6_{1,4,1}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^6_{1,3,2}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^6_{1,2,3}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^6_{1,1,4}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^6_{1,0,5}(u,v,w) = uw^5, \mathcal{D}^6_{0,6,0}(u,v,w) = v^6, \mathcal{D}^6_{0,5,1}(u,v,w) = v^5w,$
$\mathcal{D}^6_{0,4,2}(u,v,w) = v^4w, \mathcal{D}^6_{0,3,3}(u,v,w) = v^3w + vw^3, \mathcal{D}^6_{0,2,4}(u,v,w) = vw^4,$
$\mathcal{D}^6_{0,1,5}(u,v,w) = vw^5, \mathcal{D}^6_{0,0,6}(u,v,w) = w^6.$

For degree n=7,

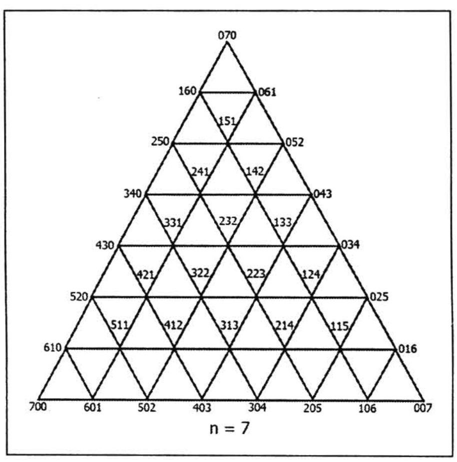

Figure 3.14 Schema for Triangular DP Surface of Degree 7

$\mathcal{D}^7_{7,0,0}(u,v,w) = u^7, \mathcal{D}^7_{6,1,0}(u,v,w) = u^6v, \mathcal{D}^7_{6,0,1}(u,v,w) = u^6w,$
$\mathcal{D}^7_{5,2,0}(u,v,w) = u^5v, \mathcal{D}^7_{5,1,2}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^7_{5,0,2}(u,v,w) = u^5w, \mathcal{D}^7_{4,3,0}(u,v,w) = u^4v,$
$\mathcal{D}^7_{4,2,1}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^7_{4,1,2}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w, \mathcal{D}^7_{4,0,3}(u,v,w) = u^4w,$
$\mathcal{D}^7_{3,4,0}(u,v,w) = uv^4, \mathcal{D}^7_{3,3,1}(u,v,w) = u^3v + uv3,$
$\mathcal{D}^7_{3,2,2}(u,v,w) = v^2(\frac{u}{3} + \frac{w}{3}) + u^2(\frac{v}{3} + \frac{w}{3}) + (\frac{u}{3} + \frac{v}{3})w^2,$
$\mathcal{D}^7_{3,1,3}(u,v,w) = u^3w + uw3, \mathcal{D}^7_{3,0,4}(u,v,w) = uw^4, \mathcal{D}^7_{2,5,0}(u,v,w) = uv^5,$
$\mathcal{D}^7_{2,4,1}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^7_{2,3,2}(u,v,w) = v^2(\frac{u}{3} + \frac{w}{3}) + u^2(\frac{v}{3} + \frac{w}{3}) + (\frac{u}{3} + \frac{v}{3})w^2,$
$\mathcal{D}^7_{2,2,3}(u,v,w) = v^2(\frac{u}{3} + \frac{w}{3}) + u^2(\frac{v}{3} + \frac{w}{3}) + (\frac{u}{3} + \frac{v}{3})w^2,$
$\mathcal{D}^7_{2,1,4}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$

$\mathcal{D}^7_{2,0,5}(u,v,w) = uw^5, \mathcal{D}^7_{1,6,0}(u,v,w) = uv^6,$
$\mathcal{D}^7_{1,5,1}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^7_{1,4,2}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^7_{1,3,3}(u,v,w) = v^3w + vw^3, \mathcal{D}^7_{1,2,4}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^7_{1,1,5}(u,v,w) = v(\frac{u}{9} + \frac{w}{9}) + u(\frac{v}{9} + \frac{w}{9}) + (\frac{u}{9} + \frac{v}{9})w,$
$\mathcal{D}^7_{1,0,6}(u,v,w) = uw^6, \mathcal{D}^7_{0,7,0}(u,v,w) = v^7, \mathcal{D}^7_{0,6,1}(u,v,w) = v^6w,$
$\mathcal{D}^7_{0,5,2}(u,v,w) = v^5w, \mathcal{D}^7_{0,4,3}(u,v,w) = v^4w, \mathcal{D}^7_{0,3,4}(u,v,w) = vw^5,$
$\mathcal{D}^7_{0,2,5}(u,v,w) = vw^5, \mathcal{D}^7_{0,1,6}(u,v,w) = vw^6, \mathcal{D}^7_{0,0,7}(u,v,w) = w^7.$

3.1.2 Recursive Formula

Definition 3.2

For $n = 2$, DP blending function are defined as same as for the Bézier-Bernstein basis. In addition, for the trivial case $n = 1$, $\mathbf{d}^1_{1,0,0}(u, v, w) = u$, $\mathbf{d}^1_{0,1,0}(u, v, w) = v$ and $\mathbf{d}^1_{0,0,1}(u, v, w) = w$.

For $n \geq 3$, the DP blending function of degree n can be recursively defined as follows:

$$
\mathbf{d}^r_{i,j,k} = \begin{cases}
u\mathbf{d}^{r-1}_{i+1,j,k} + v\mathbf{d}^{r-1}_{i,j+1,k} + w\mathbf{d}^{r-1}_{i,j,k+1} & , \quad i = n \text{ or } j = n \text{ or } k = n, \\
\frac{1}{3}\mathbf{d}^{r-1}_{i+1,j,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j+1,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j,k+1} & , \quad i \geq j = k \text{ and } n \bmod 2 = 0 \text{ and } j \bmod 2 = 0, \\
\frac{1}{3}\mathbf{d}^{r-1}_{i+1,j,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j+1,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j,k+1} & , \quad j \geq i = k \text{ and } n \bmod 2 = 0 \text{ and } k \bmod 2 = 0, \\
\frac{1}{3}\mathbf{d}^{r-1}_{i+1,j,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j+1,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j,k+1} & , \quad k \geq i = j \text{ and } n \bmod 2 = 0 \text{ and } i \bmod 2 = 0, \\
\frac{1}{3}\mathbf{d}^{r-1}_{i+1,j,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j+1,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j,k+1} & , \quad i \geq j = k \text{ and } n \bmod 2 \neq 0 \text{ and } j \bmod 2 \neq 0, \\
\frac{1}{3}\mathbf{d}^{r-1}_{i+1,j,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j+1,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j,k+1} & , \quad j \geq i = k \text{ and } n \bmod 2 \neq 0 \text{ and } k \bmod 2 \neq 0, \\
\frac{1}{3}\mathbf{d}^{r-1}_{i+1,j,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j+1,k} + \frac{1}{3}\mathbf{d}^{r-1}_{i,j,k+1} & , \quad k \geq i = j \text{ and } n \bmod 2 \neq 0 \text{ and } i \bmod 2 \neq 0, \\
u\mathbf{d}^{r-1}_{i+1,j,k} & , \quad j = k \text{ and } k > i, \\
v\mathbf{d}^{r-1}_{i,j+1,k} & , \quad k = i \text{ and } i > j, \\
w\mathbf{d}^{r-1}_{i,j,k+1} & , \quad i = j \text{ and } j > i, \\
u\mathbf{d}^{r-1}_{i+1,j,k} & , \quad k > i > j \text{ and } n \bmod 2 = 0, \\
v\mathbf{d}^{r-1}_{i,j+1,k} & , \quad i > j > k \text{ and } n \bmod 2 = 0, \\
w\mathbf{d}^{r-1}_{i,j,k+1} & , \quad j > k > i \text{ and } n \bmod 2 = 0, \\
u\mathbf{d}^{r-1}_{i+1,j,k} & , \quad k < i < j \text{ and } k \bmod 2 = 0, \\
v\mathbf{d}^{r-1}_{i,j+1,k} & , \quad i < j < k \text{ and } i \bmod 2 = 0, \\
w\mathbf{d}^{r-1}_{i,j,k+1} & , \quad j < k < i \text{ and } j \bmod 2 = 0, \\
u\mathbf{d}^{r-1}_{i+1,j,k} & , \quad i \geq \lfloor \frac{n}{2} \rfloor, \\
v\mathbf{d}^{r-1}_{i,j+1,k} & , \quad j \geq \lfloor \frac{n}{2} \rfloor, \\
w\mathbf{d}^{r-1}_{i,j,k+1} & , \quad k \geq \lfloor \frac{n}{2} \rfloor.
\end{cases} \tag{3.6}
$$

where $\mathbf{d}^0_{i,j,k} = \mathbf{d}_{i,j,k}$, $r = 1, ..., n$ and $i, j, k = 0, ..., n - r$. Hence, a point on this surface can be computed from $\mathbf{d}^n_{0,0,0}$ since $\delta^n(u, v, w) = \mathbf{d}^n_{0,0,0}$.

3.1.3 Computational Complexity

We can find the number of interpolations of our first triangular DP surfaces by counting the points needed to interpolate from the schema or we can figure it out by recursive algorithms as well. The relationships between the number of interpolations for each surface and the degree from 1 to 13 are shown in Table 3.1

Table 3.1 The number of interpolations for evaluating a point on our first proposed triangular DP surface compared to the existing triangular surfaces

Degree (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Bézier	1	4	10	20	35	56	84	120	165	220	286	364	455
Said-Ball	1	4	7	13	23	36	54	77	105	140	181	230	287
Hu's Wang-Ball	1	4	8	14	22	32	45	60	78	100	125	154	187
Our First DP	1	4	7	11	14	20	24	30	36	43	49	58	65

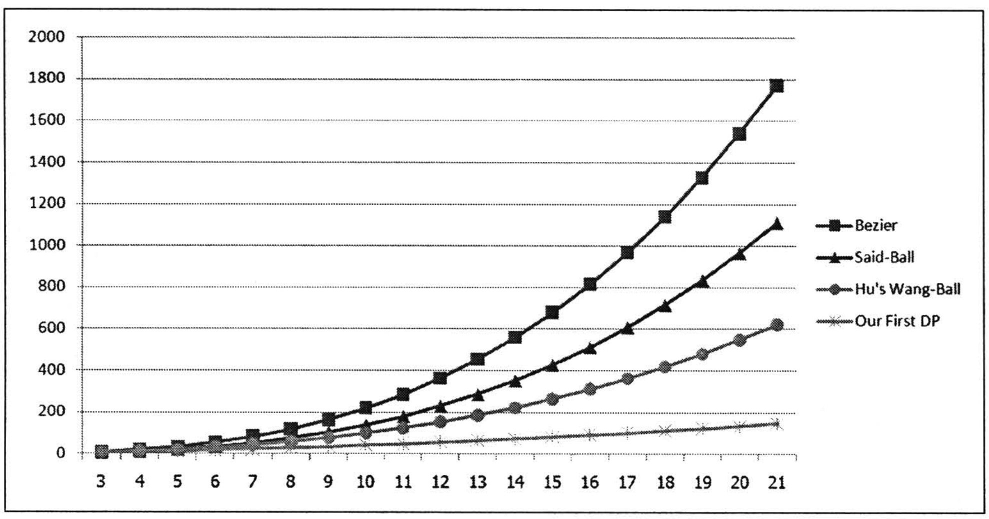

Figure 3.15 The Number of Interpolations for Evaluating a Point on Our First Proposed Triangular DP Surface Compared to The Existing Triangular Surfaces

The computation time of our first triangular DP surface is very fast compared to the existing algorithms, that is, triangular Bézier surface, triangular Said-Ball surface and triangular Hu's Wang-Ball surface. Our first algorithms require about quarter of the computational time for triangular Bézier surfaces. From Table 3.1, we can find the number of interpolations according to the following equations:

if $n\ mod\ 3 = 0$, then

$$\mathcal{A}_n = \frac{1}{6}(36 + 14n - 9\lceil\frac{1}{2} - \frac{n}{6}\rceil + 9\lceil\frac{1}{2} - \frac{n}{6}\rceil^2 + 54\lfloor\frac{1}{6}(n-9)\rfloor + 18\lfloor\frac{1}{6}(n-9)\rfloor^2 + 9\lfloor\frac{n}{6}\rfloor + 27\lfloor\frac{n}{6}\rfloor^2). \tag{3.7}$$

if $n\ mod\ 3 = 1$, then

$$\mathcal{A}_n = \frac{1}{6}(28 + 14n - 9\lceil\frac{2}{3} - \frac{n}{6}\rceil + 9\lceil\frac{2}{3} - \frac{n}{6}\rceil^2 + 18\lceil\frac{1}{6}(n-1)\rceil + 54\lfloor\frac{1}{6}(n-10)\rfloor + 18\lfloor\frac{1}{6}(n-10)\rfloor + 9\lfloor\frac{1}{6}(n-1)\rfloor + 27\lfloor\frac{1}{6}(n-1)\rfloor^2). \tag{3.8}$$

if $n\ mod\ 3 = 2$, then

$$\mathcal{A}_n = \frac{1}{6}(32 + 14n + 9\lceil\frac{5}{6} - \frac{n}{6}\rceil + 9\lceil\frac{5}{6} - \frac{n}{6}\rceil^2 + 18\lceil\frac{1}{6}(n-2)\rceil + 54\lfloor\frac{1}{6}(n-11)\rfloor + 18\lfloor\frac{n}{6}(n-11)\rfloor^2 + 9\lfloor\frac{1}{6}(n-2)\rfloor + 27\lfloor\frac{1}{6}(n-2)\rfloor^2 + 18\lfloor\frac{n}{6}\rfloor). \tag{3.9}$$

Each of the above equations, has a maximum degree of 2. It is obvious that the evaluation of a point on this proposed triangular DP surface can be computed in the quadratic time, $O(n^2)$.

However, it was found later that this algorithm is not linearly independent from degree 5. Since the determinant of monomial matrix of degree 5 is equal to 0. Consequently, the determinant of monomial matrix beginning with degree 6 is also equal to 0. This property is one of the most important characteristics for controlling convexity in geometric modeling. Moreover, a new system of these blending functions does not form a basis since the linear independence is not satisfied. Thus, the interpolation schema should be modified in order to accomplish all the necessary properties of surface model construction.

3.2 Our Second Model of Triangular DP Surfaces

In this section, an alternative method for triangular DP surface modeling is proposed, which possesses many significant geometric properties for surface modeling, for example, convexity, partition of unity and linear independence. Moreover, the properties of this algorithm are also presented with a basis recurrence formulae and a recursive algorithm with quadratic complexity, denoted by $O(n^2)$.

3.2.1 Bivariate Basis Function

Our second bivariate DP basis functions can be constructed on a triangular domain by a given set of control nets. The control points are denoted by $\mathbf{d}_{i,j,k}$, where $i+j+k$ is equal to the degree of surface. DP polynomial, denoted by $\mathcal{D}^n_{i,j,k}(u,v,w)$, can be defined by:

$$\mathcal{D}^n(u,v,w) = \sum_{i+j+k=n} \mathbf{d}_{i,j,k} \cdot \mathcal{D}^n_{i,j,k}(u,v,w), \tag{3.10}$$

where for each $u, v, w \geq 0$ and $u + v + w = 1$.

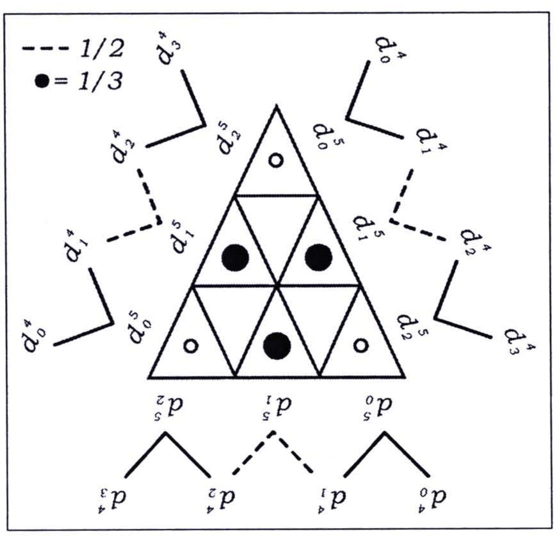

Figure 3.16 How to Interpolate the Control Points for a New Triangular DP Surface

We can construct a new triangular DP surfaces by slightly changing the first schema. Two points on each side of triangle, will be interpolated with the weight $\frac{1}{3}$. At the center of the triangle, we will interpolate with u, v, and w. Moreover, this interpolation scheme creates linear dependence from the surfaces of degree 5 and so on. A subtle adjustment at degree 3 has been done and coincidentally appears to be the same of an interpolation of cubic Bézier curve. The interpolations of our new triangular DP surfaces are shown in Figure 3.16

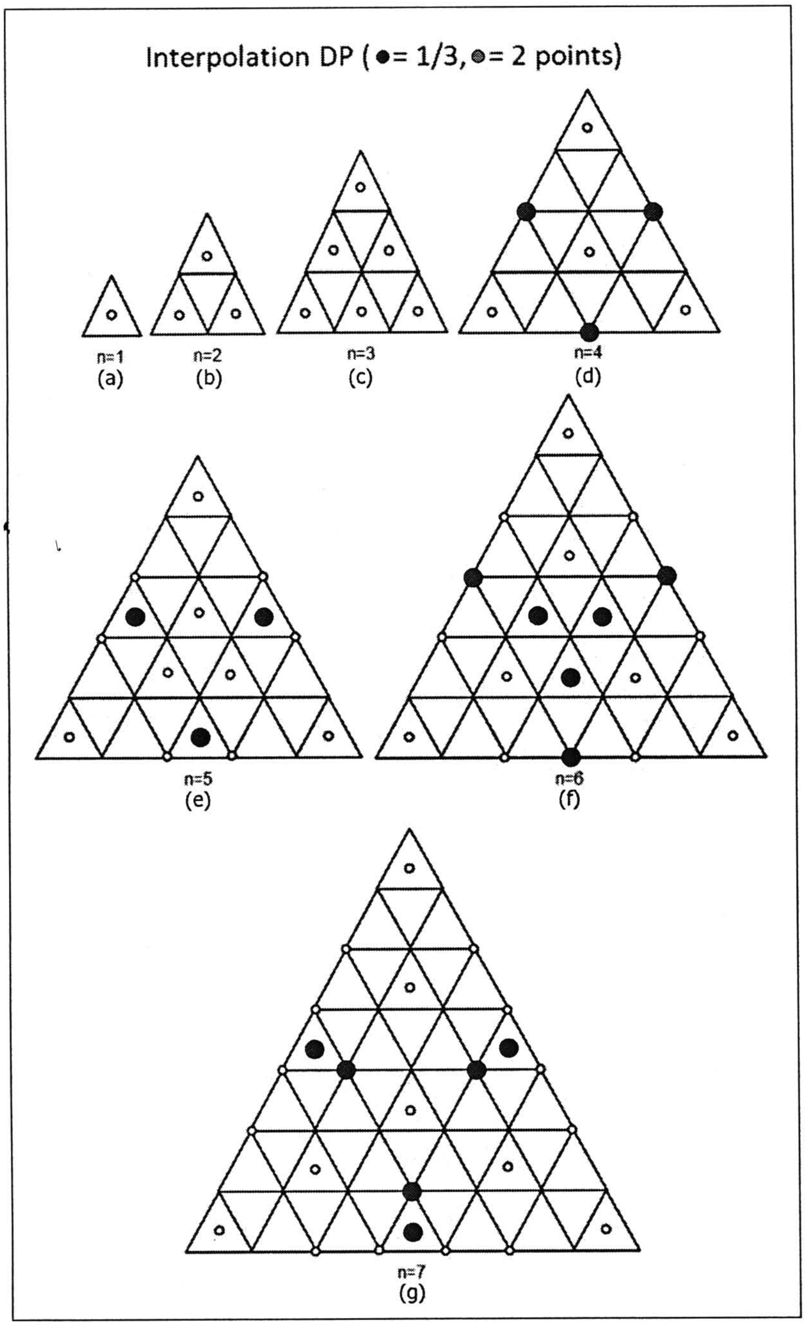

Figure 3.17 Our Second Proposed Schema for Triangular DP Surface

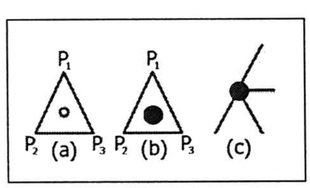

Figure 3.18 The Concepts of Our Second DP Interpolations

Figure 3.17 shows some examples of how to interpolate the control points for degree 4, 5, 6, and 7, respectively. The concepts of our second DP interpolations can be explained in details as follows:

1. From Figure 3.18(a), the point located at each corner must be interpolated with u, v, and w. The 3-point interpolation result can be computed by $uP_1 + vP_2 + wP_3$.
2. From Figure 3.18(b), the point at the center along each side of the triangle will be assigned to interpolate with $\frac{1}{3}$. The interpolation result can be computed by $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$.
3. From Figure 3.18(c), this point will be used to generate two duplicate points (two points of the same coordinates).
4. Other points remain the same.

3.2.2 Recurrence Formula

Definition 3.3

For $n = 2$ and 3, a DP blending function is defined to be the same as that for the Bézier-Bernstein basis. In addition, for a trivial case $n = 1$, $\mathcal{D}^1_{1,0,0}(u,v,w) = u$, $\mathcal{D}^1_{0,1,0}(u,v,w) = v$, $\mathcal{D}^1_{0,0,1}(u,v,w) = w$ and $\triangle = \mathcal{D}^n_{i,j,k}(u,v,w)$.
For $n \geq 4$, the DP blending function of degree n can be recursively defined as follows:

$$\triangle = \begin{cases} u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad i > j \text{ and } j = k, \\ v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad j > i \text{ and } i = k, \\ w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad k > i \text{ and } i = j. \end{cases} \tag{3.11}$$

$$\triangle = \begin{cases} \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad i = j \text{ and } k = 0, \\ \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad j = k \text{ and } i = 0, \\ \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) + \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad k = i \text{ and } j = 0. \end{cases} \tag{3.12}$$

$$\triangle = \begin{cases} \frac{1}{3}\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad j = k = i+1, \\ u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \frac{1}{3}\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad k = i = j+1, \\ u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \frac{1}{3}\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad i = j = k+1. \end{cases} \tag{3.13}$$

$$\triangle = \begin{cases} \frac{1}{3}\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad j = k > i, \\ \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \frac{1}{3}\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad k = i > j, \\ \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \frac{1}{3}\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad i = j > k, \\ \frac{1}{3}(\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w)) & , \quad i = j = k. \end{cases} \tag{3.14}$$

$$\triangle = \begin{cases} u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad j > i = k+1 \text{ or } k > i \text{ and } i = j+1, \\ v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad i > j = k+1 \text{ or } k > j \text{ and } j = i+1, \\ w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad i > k = j+1 \text{ or } j > k \text{ and } k = i+1. \end{cases} \tag{3.15}$$

$$\triangle = \begin{cases} \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad k > i > j \text{ or } k < i < j, \\ \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad i > j > k \text{ or } i < j < k, \\ \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad j > k > i \text{ or } j < k < i. \end{cases} \tag{3.16}$$

$$\triangle = \begin{cases} \frac{1}{3}u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad i = j+1 = k+2, \\ u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \frac{1}{3}v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad j = i+1 = k+2, \\ \frac{1}{3}v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad j = k+1 = i+2, \\ v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \frac{1}{3}w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad k = j+1 = i+2, \\ \frac{1}{3}w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) + u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad i = k+1 = j+2, \\ w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) + \frac{1}{3}u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad k = i+1 = j+2. \end{cases} \tag{3.17}$$

$$\triangle = \begin{cases} \frac{1}{3}\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad i = j+1 \text{ and } i > k, \\ \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \frac{1}{3}\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad j = i+1 \text{ and } j > k, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad j = k+1 \text{ and } j > i, \\ \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \frac{1}{3}\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad k = j+1 \text{ and } k > i, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) + \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad i = k+1 \text{ and } i > j, \\ \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) + \frac{1}{3}\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad k = i+1 \text{ and } k > j. \end{cases} \tag{3.18}$$

From $u+v+w=1$ and $u,v,w \geq 0$

Since the triangular DP basis functions $\sum_{i+j+k=n} \mathcal{D}^n_{i,j,k}(u,v,w) \cdot \mathbf{d}_{i,j,k}$, denoted by $\mathcal{D}_{i,j,k}(u,v,w)$, where $i+j+k=n$, it satisfies

- Convexity: Since $0 \leq u,v,w \leq 1$, $\mathcal{D}_{i,j,k}(u,v,w) \geq 0$.

- Partition of Unity: Since $\sum_{i+j+k=n} \mathcal{D}_{i,j,k}(u,v,w) = 1$ where $0 \le u,v,w \le 1$.
- Linear Independence: Since the determinant of monomial matrix is not equal to 0, $\mathcal{D}_{i,j,k}(u,v,w)$ forms a bivariate polynomial basis.

Example 3.2

Some examples of the bivariate DP polynomials are listed below.
For degree n=4,
$\mathcal{D}^4_{4,0,0}(u,v,w) = u^4, \mathcal{D}^4_{3,1,0}(u,v,w) = u^3v, \mathcal{D}^4_{3,0,1}(u,v,w) = u^3w,$
$\mathcal{D}^4_{2,2,0}(u,v,w) = 3uv(u+v), \mathcal{D}^4_{2,1,1}(u,v,w) = 6u^2vw, \mathcal{D}^4_{2,0,2}(u,v,w) = 3uw(u+w),$
$\mathcal{D}^4_{1,3,0}(u,v,w) = uv^3, \mathcal{D}^4_{1,2,1}(u,v,w) = 6uv^2w, \mathcal{D}^4_{1,1,2}(u,v,w) = 6uvw^2,$
$\mathcal{D}^4_{1,0,3}(u,v,w) = uw^3, \mathcal{D}^4_{0,4,0}(u,v,w) = v^4, \mathcal{D}^4_{0,3,1}(u,v,w) = v^3w,$
$\mathcal{D}^4_{0,2,2}(u,v,w) = 3vw(v+w), \mathcal{D}^4_{0,1,3}(u,v,w) = vw^3, \mathcal{D}^4_{0,0,4}(u,v,w) = w^4.$

For degree n=5,
$\mathcal{D}^5_{5,0,0}(u,v,w) = u^5, \mathcal{D}^5_{4,1,0}(u,v,w) = u^4v, \mathcal{D}^5_{4,0,1}(u,v,w) = u^4w,$
$\mathcal{D}^5_{3,2,0}(u,v,w) = uv(u+u^2+v), \mathcal{D}^5_{3,1,1}(u,v,w) = 6u^3vw,$
$\mathcal{D}^5_{3,0,2}(u,v,w) = uw(u+u^2+w), \mathcal{D}^5_{2,3,0}(u,v,w) = uv(u+v+v^2),$
$\mathcal{D}^5_{2,2,1}(u,v,w) = uv(u+v+12uvw), \mathcal{D}^5_{2,1,2}(u,v,w) = uw(u+w+12uvw),$
$\mathcal{D}^5_{2,0,3}(u,v,w) = uw(u+w+w^2), \mathcal{D}^5_{1,4,0}(u,v,w) = uv^4, \mathcal{D}^5_{1,3,1}(u,v,w) = 6uv^3w,$
$\mathcal{D}^5_{1,2,2}(u,v,w) = vw(v+w+12uvw), \mathcal{D}^5_{1,1,3}(u,v,w) = 6uvw^3, \mathcal{D}^5_{1,0,4}(u,v,w) = uw^4,$
$\mathcal{D}^5_{0,5,0}(u,v,w) = v^5, \mathcal{D}^5_{0,4,1}(u,v,w) = v^4w, \mathcal{D}^5_{0,3,2}(u,v,w) = vw(v+v^2+w),$
$\mathcal{D}^5_{0,2,3}(u,v,w) = vw(v+w+w^2), \mathcal{D}^5_{0,1,4}(u,v,w) = vw^4, \mathcal{D}^5_{0,0,5}(u,v,w) = w^5.$

For degree n=6,
$\mathcal{D}^6_{6,0,0}(u,v,w) = u^6, \mathcal{D}^6_{5,1,0}(u,v,w) = u^5v, \mathcal{D}^6_{5,0,1}(u,v,w) = u^5w,$
$\mathcal{D}^6_{4,2,0}(u,v,w) = u^4v, \mathcal{D}^6_{4,1,1}(u,v,w) = 6u^4vw, \mathcal{D}^6_{4,0,2}(u,v,w) = u^4w,$
$\mathcal{D}^6_{3,3,0}(u,v,w) = uv(2u+u^2+v(2+v)), \mathcal{D}^6_{3,2,1}(u,v,w) = \frac{1}{3}uv(u+v+12uvw+18u^2vw),$
$\mathcal{D}^6_{3,1,2}(u,v,w) = \frac{1}{3}uw(u+w+12uvw+18u^2vw),$
$\mathcal{D}^6_{3,0,3}(u,v,w) = uw(2u+u^2+w(2+w)), \mathcal{D}^6_{2,4,0}(u,v,w) = uv^4,$
$\mathcal{D}^6_{2,3,1}(u,v,w) = \frac{1}{3}uv(u+v+12uvw+18uv^2w),$
$\mathcal{D}^6_{2,2,2}(u,v,w) = \frac{1}{3}(vw(v+w)+u^2(v+w+12v^2w+12vw^2)+u(w^2+v^2(1+12w^2))),$
$\mathcal{D}^6_{2,1,3}(u,v,w) = \frac{1}{3}uw(u+w+12uvw+18uvw^2), \mathcal{D}^6_{2,0,4}(u,v,w) = uw^4,$
$\mathcal{D}^6_{1,5,0}(u,v,w) = uv^5, \mathcal{D}^6_{1,4,1}(u,v,w) = 6uv^4w,$
$\mathcal{D}^6_{1,3,2}(u,v,w) = \frac{1}{3}vw(v+w+12uvw+18uv^2w),$
$\mathcal{D}^6_{1,2,3}(u,v,w) = \frac{1}{3}vw(v+w+12uvw+18uvw^2),$
$\mathcal{D}^6_{1,1,4}(u,v,w) = 6uvw^4, \mathcal{D}^6_{1,0,5}(u,v,w) = uw^5, \mathcal{D}^6_{0,6,0}(u,v,w) = v^6,$
$\mathcal{D}^6_{0,5,1}(u,v,w) = v^5w, \mathcal{D}^6_{0,4,2}(u,v,w) = v^4w, \mathcal{D}^6_{0,3,3}(u,v,w) = vw(2v+v^2+w(2+w)),$
$\mathcal{D}^6_{0,2,4}(u,v,w) = vw^4, \mathcal{D}^6_{0,1,5}(u,v,w) = vw^5, \mathcal{D}^6_{0,0,6}(u,v,w) = w^6.$

For degree n=7,

$\mathcal{D}^7_{7,0,0}(u,v,w) = u^7, \mathcal{D}^7_{6,1,0}(u,v,w) = u^6 v, \mathcal{D}^7_{6,0,1}(u,v,w) = u^6 w,$
$\mathcal{D}^7_{5,2,0}(u,v,w) = u^5 v, \mathcal{D}^7_{5,1,2}(u,v,w) = 6u^5 vw, \mathcal{D}^7_{5,0,2}(u,v,w) = u^5 w,$
$\mathcal{D}^7_{4,3,0}(u,v,w) = \frac{1}{3}uv(2u + u^2 + 3u^3 + v(2+v)),$
$\mathcal{D}^7_{4,2,1}(u,v,w) = 6u^4 v^2 w, \mathcal{D}^7_{4,1,2}(u,v,w) = 6u^4 vw^2,$
$\mathcal{D}^7_{4,0,3}(u,v,w) = \frac{1}{3}uw(2u + u^2 + 3u^3 + w(2+w)),$
$\mathcal{D}^7_{3,4,0}(u,v,w) = \frac{1}{3}uv(2u + u^2 + 3v^3 + v(2+v)),$
$\mathcal{D}^7_{3,3,1}(u,v,w) = \frac{1}{3}uv(v(4+v) + u^2(1+18vw) + 2u(2+3vw(4+3v))),$
$\mathcal{D}^7_{3,2,2}(u,v,w) = \frac{1}{3}u(vw(v+w) + u^2(v+w+12vw(v+w)) + u(w^2 + v^2(1+12w^2))),$
$\mathcal{D}^7_{3,1,3}(u,v,w) = \frac{1}{3}uw(w(4+w) + u^2(1+18vw) + 2u(2+3vw(4+3w))),$
$\mathcal{D}^7_{3,0,4}(u,v,w) = \frac{1}{3}uw(2u + u^2 + 3w^3 + w(2+w)),$
$\mathcal{D}^7_{2,5,0}(u,v,w) = uv^5, \mathcal{D}^7_{2,4,1}(u,v,w) = 6u^2 v^4 w,$
$\mathcal{D}^7_{2,3,2}(u,v,w) = \frac{1}{3}v(vw(v+w) + u^2(v+w+12vw(v+w)) + u(w^2 + v^2(1+12w^2))),$
$\mathcal{D}^7_{2,2,3}(u,v,w) = \frac{1}{3}w(vw(v+w) + u^2(v+w+12vw(v+w)) + u(w^2 + v^2(1+12w^2))),$
$\mathcal{D}^7_{2,1,4}(u,v,w) = 6u^2 vw^4, \mathcal{D}^7_{2,0,5}(u,v,w) = uw^5, \mathcal{D}^7_{1,6,0}(u,v,w) = uv^6,$
$\mathcal{D}^7_{1,5,1}(u,v,w) = 6uv^5 w, \mathcal{D}^7_{1,4,2}(u,v,w) = 6uv^4 w^2,$
$\mathcal{D}^7_{1,3,3}(u,v,w) = \frac{1}{3}vw(w(4+w) + v^2(1+18uw) + 2v(2+3uw(4+3w))),$
$\mathcal{D}^7_{1,2,4}(u,v,w) = 6uv^2 w^4, \mathcal{D}^7_{1,1,5}(u,v,w) = 6uvw^5, \mathcal{D}^7_{1,0,6}(u,v,w) = uw^6,$
$\mathcal{D}^7_{0,7,0}(u,v,w) = v^7, \mathcal{D}^7_{0,6,1}(u,v,w) = v^6 w, \mathcal{D}^7_{0,5,2}(u,v,w) = v^5 w,$
$\mathcal{D}^7_{0,4,3}(u,v,w) = \frac{1}{3}vw(2v + v^2 + 3v^3 + w(2+w)),$
$\mathcal{D}^7_{0,3,4}(u,v,w) = \frac{1}{3}vw(2v + v^2 + 3w^3 + w(2+w)),$
$\mathcal{D}^7_{0,2,5}(u,v,w) = vw^5, \mathcal{D}^7_{0,1,6}(u,v,w) = vw^6, \mathcal{D}^7_{0,0,7}(u,v,w) = w^7.$

3.2.3 Recursive Formula

Definition 3.4

Recursive formula can be obtained from the interpolation as shown in figure 3.17. The following recursive algorithm used for evaluating a point on this triangular DP surface, denoted by $\mathcal{D}^n(u,v,w) = \mathbf{d}^n_{0,0,0}$, can be recursively defined as follows:

$$
\mathbf{d}^r_{i,j,k} = \begin{cases}
u \cdot \mathbf{d}^{r-1}_{i+1,j,k} + v \cdot \mathbf{d}^{r-1}_{i,j+1,k} + w \cdot \mathbf{d}^{r-1}_{i,j,k+1} & , \quad i \geq j \text{ or } j = k \text{ or } j \geq k \text{ or } k = i \text{ or } \\
 & \quad\; k \geq i \text{ or } i = j, \\
\frac{1}{3}(\mathbf{d}^{r-1}_{i+1,j,k} + \mathbf{d}^{r-1}_{i,j+1,k} + \mathbf{d}^{r-1}_{i,j,k+1}) & , \quad i = j \text{ or } j > k \text{ or } j = k \text{ or } k > i \text{ or } \\
 & \quad\; k = i \text{ or } i > j, \\
u \cdot \mathbf{d}^{r-1}_{i+1,j,k} & , \quad k > i > j \text{ or } k < i < j, \\
v \cdot \mathbf{d}^{r-1}_{i,j+1,k} & , \quad i > j > k \text{ or } i < j < k, \\
w \cdot \mathbf{d}^{r-1}_{i,j,k+1} & , \quad j > k > i \text{ or } j < k < i.
\end{cases} \tag{3.19}
$$

where $\mathbf{d}^0_{i,j,k} = \mathbf{d}_{i,j,k}$, $r = 1, ..., n$ and $i, j, k = 0, ..., n - r$. Hence, a point on this surface can be computed from $\mathbf{d}^n_{0,0,0}$.

From above recursive algorithm, we can calculate the number of 3-point interpolations. In Figure 3.17 (a) 1 interpolation, (b) 3+1 = 4 interpolations, (c) 4+6 = 10 interpolations, (d) 10+4 = 14 interpolations, (e) 14+9 = 23 interpolations, (f) 23+9 = 32 interpolations and (g) 32+10 = 42 interpolations.

We can find the number of interpolations for our second triangular DP surfaces by counting the points needed to interpolate from the schema or we can figure it out by a recursive algorithm as well. The equations for estimating the number of interpolations can be formulated as follows.

$$
\mathcal{A}_n = \begin{cases}
\frac{1}{12}(10n + 9n^2) & , \quad \text{when } n \bmod 6 = 0, \\
\frac{1}{12}(-7 + 10n + 9n^2) & , \quad n \bmod 6 = 1, \\
\frac{1}{12}(-8 + 10n + 9n^2) & , \quad n \bmod 6 = 2, \\
\frac{1}{12}(9 + 10n + 9n^2) & , \quad n \bmod 6 = 3, \\
\frac{1}{12}(-16 + 10n + 9n^2) & , \quad n \bmod 6 = 4, \\
\frac{1}{12}(1 + 10n + 9n^2) & , \quad n \bmod 6 = 5.
\end{cases} \tag{3.20}
$$

From equation (3.20), each equation has the maximum degree of 2. It is obvious that the evaluation of a point on this proposed triangular DP surface can be computed in the quadratic time, $O(n^2)$. The relationships between the number of interpolations for each surface and the degree from 1 to 13 are shown in Table 3.2.

Table 3.2 The number of interpolations for evaluating a point on our second proposed triangular DP surface compared to the existing triangular surfaces

Degree (n)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Bézier	1	4	10	20	35	56	84	120	165	220	286	364	455
Said-Ball	1	4	7	13	23	36	54	77	105	140	181	230	287
Hu's Wang-Ball	1	4	8	14	22	32	45	60	78	100	125	154	187
Our Second DP	1	4	10	14	23	32	42	54	69	82	100	118	137

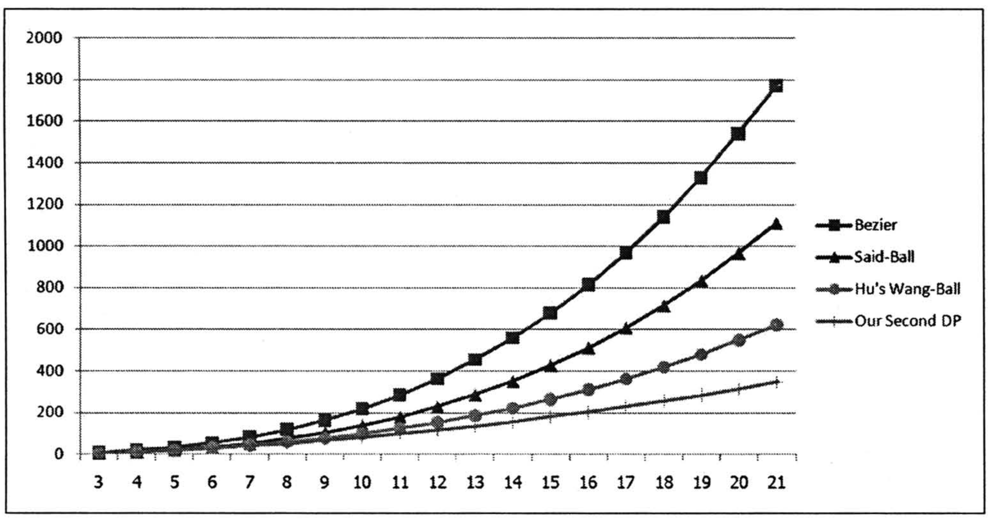

Figure 3.19 The Number of Interpolations for Evaluating a Point on Our Second Proposed Triangular DP Surface Compared to The Existing Triangular Surfaces

Our second triangular DP surface takes a smaller number of computations than the previous algorithms such as triangular Bézier surface, triangular Said-Ball surface as well as triangular Hu's Wang-Ball surface.

3.2.4 The Relationships between Bézier and DP Triangular Surfaces

In the previous section, it was shown that the new triangular DP surfaces requires about quarter of the computational time for triangular Bézier surfaces. Any triangular DP surface can be viewed as a triangular Bézier surface. This technique can help us to model surfaces by using our algorithms. Our method takes less computational time than directly using Bézier algorithms.

Proposition 3.1

Both triangular Bézier surface and triangular DP surface can be defined by their control points and their blending functions. The basis functions, denoted by $\mathcal{B}^n_{i,j,k}(u,v,w)$ and $\mathcal{D}^n_{i,j,k}(u,v,w)$, are two different polynomials of degree n, where $i+j+k=n$.

$$\sum_{i+j+k=n} \mathcal{D}^n_{i,j,k}(u,v,w)\cdot \mathbf{d}_{i,j,k} = \sum_{i+j+k=n} \mathcal{B}^n_{i,j,k}(u,v,w)\cdot \mathbf{b}_{i,j,k} \tag{3.21}$$

Conversion Matrix = $M_B \cdot M_D^{-1}$

Proof: This functions can be written in terms of matrices as follows:

$$G_B \cdot M_B \cdot T_B = G_D \cdot M_D \cdot T_D, \tag{3.22}$$

where G is represented by a matrix of control points, M is represented by a coefficient matrix and T is represented by a monomial matrix.

Since T_B and T_D are the same, we can drop these two terms from the equation for simplicity. M_B and M_D are coefficient matrices of two basis functions.

Thus, M_D^{-1} can be always derived.

$$G_B \cdot M_B = G_D \cdot M_D \tag{3.23}$$

$$G_B \cdot M_B \cdot M_D^{-1} = G_D \cdot M_D \cdot M_D^{-1} \tag{3.24}$$

$$G_D = G_B \cdot M_B \cdot M_D^{-1} \tag{3.25}$$

Given Bézier control points, denoted by $\mathbf{b}_{i,j,k}$ and DP control points, denoted by $\mathbf{d}_{i,j,k}$, the relationships of these two triangular surfaces for degree 4 can be shown as follows:

Example 3.3

The conversion from Bézier into DP triangular surface of degree 4 can be listed as follows:

$\mathbf{d}_{4,0,0} = \mathbf{b}_{4,0,0},$

$\mathbf{d}_{3,1,0} = \frac{3}{4}\mathbf{b}_{2,2,0} + \frac{1}{4}\mathbf{b}_{3,1,0},$

$\mathbf{d}_{3,0,1} = \frac{3}{4}\mathbf{b}_{2,0,2} + \frac{1}{4}\mathbf{b}_{3,0,1},$

$\mathbf{d}_{2,2,0} = \mathbf{b}_{2,2,0},$

$\mathbf{d}_{2,1,1} = \frac{1}{4}\mathbf{b}_{2,0,2} + \frac{1}{2}\mathbf{b}_{2,1,1} + \frac{1}{4}\mathbf{b}_{2,2,0},$

$\mathbf{d}_{2,0,2} = \mathbf{b}_{2,0,2},$

$\mathbf{d}_{1,3,0} = \frac{3}{4}\mathbf{b}_{2,2,0} + \frac{1}{4}\mathbf{b}_{1,3,0},$

$\mathbf{d}_{1,2,1} = \frac{1}{4}\mathbf{b}_{2,2,0} + \frac{1}{2}\mathbf{b}_{1,2,1} + \frac{1}{4}\mathbf{b}_{0,2,2},$

$\mathbf{d}_{1,1,2} = \frac{1}{4}\mathbf{b}_{2,0,2} + \frac{1}{2}\mathbf{b}_{1,1,2} + \frac{1}{4}\mathbf{b}_{0,2,2},$

$\mathbf{d}_{1,0,3} = \frac{3}{4}\mathbf{b}_{2,0,2} + \frac{1}{4}\mathbf{b}_{1,0,3},$

$\mathbf{d}_{0,4,0} = \mathbf{b}_{0,4,0},$

$\mathbf{d}_{0,3,1} = \frac{3}{4}\mathbf{b}_{0,2,2} + \frac{1}{4}\mathbf{b}_{0,3,1},$
$\mathbf{d}_{0,2,2} = \mathbf{b}_{0,2,2},$
$\mathbf{d}_{0,1,3} = \frac{3}{4}\mathbf{b}_{0,2,2} + \frac{1}{4}\mathbf{b}_{0,1,3},$
$\mathbf{d}_{0,0,4} = \mathbf{b}_{0,0,4}.$

Proposition 3.2

Bézier control points of a triangular surface can be obtained from the DP control points by

$$M_D \cdot M_B^{-1}. \tag{3.26}$$

Proof: The proof is similar to the proof in Proposition 3.1.

Therefore, the conversion from DP into Bézier triangular surface of degree 4
$\mathbf{b}_{4,0,0} = \mathbf{d}_{4,0,0},$
$\mathbf{b}_{3,1,0} = 4\mathbf{d}_{3,1,0} - 3\mathbf{d}_{2,2,0},$
$\mathbf{b}_{3,0,1} = 4\mathbf{d}_{3,0,1} - 3\mathbf{d}_{2,0,2},$
$\mathbf{b}_{2,2,0} = \mathbf{d}_{2,2,0},$
$\mathbf{b}_{2,1,1} = 2\mathbf{d}_{2,1,1} - \frac{1}{2}\mathbf{d}_{2,0,2} - \frac{1}{2}\mathbf{d}_{2,2,0},$
$\mathbf{b}_{2,0,2} = \mathbf{d}_{2,0,2},$
$\mathbf{b}_{1,3,0} = 4\mathbf{d}_{1,3,0} - 3\mathbf{d}_{2,2,0},$
$\mathbf{b}_{1,2,1} = 2\mathbf{d}_{1,2,1} - \frac{1}{2}\mathbf{d}_{2,2,0} - \frac{1}{2}\mathbf{d}_{0,2,2},$
$\mathbf{b}_{1,1,2} = 2\mathbf{d}_{1,1,2} - \frac{1}{2}\mathbf{d}_{2,0,2} - \frac{1}{2}\mathbf{d}_{0,2,2},$
$\mathbf{b}_{1,0,3} = 4\mathbf{d}_{1,0,3} - 3\mathbf{d}_{2,0,2},$
$\mathbf{b}_{0,4,0} = \mathbf{d}_{0,4,0},$
$\mathbf{b}_{0,3,1} = 4\mathbf{d}_{0,3,1} - 3\mathbf{d}_{0,2,2},$
$\mathbf{b}_{0,2,2} = \mathbf{d}_{0,2,2},$
$\mathbf{b}_{0,1,3} = 4\mathbf{d}_{0,1,3} - 3\mathbf{d}_{0,2,2},$
$\mathbf{b}_{0,0,4} = \mathbf{d}_{0,0,4}.$
These formulae can be applied to the case of degree 5 which is provided in Appendix A.

3.2.5 Degree Elevation

Degree elevation is a technique that can increase the flexibility of the polygon with identical surfaces by adding another vertex to surfaces when it does not possess sufficient flexibility to model the desired shape.

Proposition 3.3

The n^{th}-degree triangular DP surface, denoted by $\mathcal{D}(u,v,w)$ can be raised into the same triangular DP surface of degree $n+1$, i.e.,

$$\sum_{i+j+k=n} \mathcal{D}^n_{i,j,k}(u,v,w)\mathbf{d}_{i,j,k} = \sum_{i+j+k=n+1} \mathcal{D}^{n+1}_{i,j,k}(u,v,w)\mathbf{d}^{(1)}_{i,j,k} \tag{3.27}$$

where $\{\mathbf{d}^{(1)}_{i,j,k}\}(i+j+k=n+1)$ is a new control point after degree elevation.

Degree Elevation Matrix = $M^n_D \cdot [M^{n,n+1}_D]^{-1}$

Proof: This functions can be written in terms of matrices as follows.

$$G^n_D \cdot M^n_D \cdot T^n_D = G^{n+1}_D \cdot M^{n+1}_D \cdot T^{n+1}_D \tag{3.28}$$

Since T^n_D and T^{n+1}_D are the same, we can drop these two terms from the equation for simplicity.

$$G^n_D \cdot M^n_D = G^{n+1}_D \cdot M^{n+1}_D \tag{3.29}$$

$$G^n_D \cdot M^n_D \cdot [M^{n,n+1}_D]^{-1} = G^{n+1}_D \cdot M^{n+1}_D \cdot [M^{n,n+1}_D]^{-1} \tag{3.30}$$

where $[M^{n,n+1}_D]^{-1}$ is extended monomial matrix [1].

Thus,

$$G^{n+1}_D = G^n_D \cdot M^n_D \cdot [M^{n,n+1}_D]^{-1} \tag{3.31}$$

A given set of control points for triangular DP surface of degree n can be raised to degree $n+1$ by using the conversion, then raising the control points of triangular Bézier surface to degree $n+1$ and convert to triangular DP surface.

Example 3.4

For of degree 4, the relationships for the degree elevation of a cubic DP surface into the same surface of degree 4 are

$\mathbf{d}^{(1)}_{4,0,0} = \mathbf{d}_{3,0,0},$

$\mathbf{d}^{(1)}_{3,1,0} = \mathbf{d}_{3,0,0} + \frac{3}{2}\mathbf{d}_{2,1,0} - \frac{3}{2}\mathbf{d}_{1,2,0},$

$\mathbf{d}^{(1)}_{3,0,1} = \mathbf{d}_{3,0,0} + \frac{3}{2}\mathbf{d}_{2,0,1} - \frac{3}{2}\mathbf{d}_{1,0,2},$

$\mathbf{d}^{(1)}_{2,2,0} = \frac{1}{2}\mathbf{d}_{2,1,0} + \frac{1}{2}\mathbf{d}_{1,2,0},$

$\mathbf{d}^{(1)}_{2,1,1} = \mathbf{d}_{1,1,1} + \frac{1}{4}\mathbf{d}_{2,1,0} + \frac{1}{4}\mathbf{d}_{2,0,1} - \frac{1}{4}\mathbf{d}_{1,2,0} - \frac{1}{4}\mathbf{d}_{1,0,2},$

$\mathbf{d}^{(1)}_{2,0,2} = \frac{1}{2}\mathbf{d}_{2,0,1} + \frac{1}{2}\mathbf{d}_{1,0,2},$

$\mathbf{d}^{(1)}_{1,3,0} = \mathbf{d}_{0,3,0} + \frac{3}{2}\mathbf{d}_{1,2,0} - \frac{3}{2}\mathbf{d}_{2,1,0},$

$\mathbf{d}^{(1)}_{1,2,1} = \mathbf{d}_{1,1,1} + \frac{1}{4}\mathbf{d}_{1,2,0} + \frac{1}{4}\mathbf{d}_{0,2,1} - \frac{1}{4}\mathbf{d}_{2,1,0} - \frac{1}{4}\mathbf{d}_{0,1,2},$

$\mathbf{d}^{(1)}_{1,1,2} = \mathbf{d}_{1,1,1} + \frac{1}{4}\mathbf{d}_{1,0,2} + \frac{1}{4}\mathbf{d}_{0,1,2} - \frac{1}{4}\mathbf{d}_{2,0,1} - \frac{1}{4}\mathbf{d}_{0,2,1},$

$\mathbf{d}^{(1)}_{1,0,3} = \mathbf{d}_{0,0,3} + \frac{3}{2}\mathbf{d}_{1,0,2} - \frac{3}{2}\mathbf{d}_{2,0,1},$

$\mathbf{d}^{(1)}_{0,4,0} = \mathbf{d}_{0,3,0},$

$$
\begin{aligned}
\mathbf{d}^{(1)}_{0,3,1} &= \mathbf{d}_{0,3,0} + \tfrac{3}{2}\mathbf{d}_{0,2,1} - \tfrac{3}{2}\mathbf{d}_{0,1,2},\\
\mathbf{d}^{(1)}_{0,2,2} &= \tfrac{1}{2}\mathbf{d}_{0,2,1} + \tfrac{1}{2}\mathbf{d}_{0,1,2},\\
\mathbf{d}^{(1)}_{0,1,3} &= \mathbf{d}_{0,0,3} + \tfrac{3}{2}\mathbf{d}_{0,1,2} - \tfrac{3}{2}\mathbf{d}_{0,2,1},\\
\mathbf{d}^{(1)}_{0,0,4} &= \mathbf{d}_{0,0,3}.
\end{aligned}
$$