CHAPTER 2 LITERATURE REVIEW

In this chapter, essential theories related to the study are reviewed in detail in order to provide the history and concepts for univariate and bivariate bases in curve and surface modeling, respectively. The first technique is the most popular and well-known curve which is Bézier/Bernstein curve. Then, the definitions, degree elevation and degree reduction for Said-Ball curve, Wang-Ball curve and DP curve are described respectively. Finally, Bézier triangular surface, Said-Ball triangular surface, Hu's Wang-Ball triangular surface and DP triangular surface are surveyed in detail for the constructions of triangular surfaces.

2.1 Curve Modeling

2.1.1 Bézier Curve

Bézier curve provides a simple model for constructing a parametric curve by Bernstein polynomials. It was first independently developed by Paul de Casteljau in 1959 and P. Bézier in 1962. Parametric equations of a Bézier curve can be expressed by sum of products of the Bernstein polynomials and its corresponding control points. Both Bernstein polynomials and the coordinates of the control points directly influence the shape of Bézier curve. The Bézier curve of degree n with n+1 control points, denoted by $\{\mathbf{b}_i\}_{i=0}^{n}$, can be defined as follows:

$$\mathcal{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i \cdot \mathcal{B}_i^n(t), \quad 0 \leq t \leq 1, \tag{2.1}$$

where $\mathcal{B}_i^n(t)$ is a Bernstein polynomial of degree n indexed at i, denoted by

$$\mathcal{B}_i^n(t) = \binom{n}{i} \cdot t^i \cdot (1-t)^{n-i}, \tag{2.2}$$

and binomial coefficients,

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}. \tag{2.3}$$

2.1.1.1 Degree Elevation of Bézier Curve

The degree elevation of a Bézier curve is used for increasing the number of curve's degrees by adding control points. Since the control points are used for determining the shape of the curve, a single degree elevation will be used to increase the flexibility of adding another vertices into the curve control polygon without changing the shape of the curve. The new Bézier control points of degree n+1, denoted by $\{\mathbf{b}_i^{(1)}\}_{i=0}^{n+1}$, from a given Bézier curve of degree n can be defined in terms of its original control points, $\{\mathbf{b}_i\}_{i=0}^{n}$, [13]

$$\mathbf{b}_i^{(1)} = \frac{i}{n+1} \cdot \mathbf{b}_{i-1} + (1 - \frac{i}{n+1}) \cdot \mathbf{b}_i, \tag{2.4}$$

where $i = 0, 1, 2, .., n+1$.

2.1.1.2 Degree Reduction of Bézier Curve

Degree reduction is the opposite operation to degree elevation. Degree reduction is used for reducing the number of control points while the shape of the curve remains the same. The new set of Bézier control points of degree n-1, denoted by $\{\mathbf{b}_i^{(-1)}\}_{i=0}^{n-1}$, from an n^{th} Bézier curve can be given in terms of its control points, $\{\mathbf{b}_i\}_{i=0}^{n}$, [13]

$$\mathbf{b}_i^{(-1)} = \frac{n-i}{n} \cdot \mathbf{b}_i + \frac{i}{n} \cdot \mathbf{b}_i\,, \tag{2.5}$$

where

$$\mathbf{b}_i = \tfrac{n}{n-i} \cdot \mathbf{b}_i - \tfrac{i}{n-i} \cdot \mathbf{b}_{i-1}, \quad i = 0, 1, 2, ..., n-1,$$

$$\mathbf{b}_i = \tfrac{n}{i} \cdot \mathbf{b}_i - \tfrac{n-i}{i} \cdot \mathbf{b}_{i-1}, \quad i = 0, 1, 2, ..., n.$$

2.1.2 Said-Ball Curves

In 1974, Ball [2][3][4] defined a set of basis functions for cubic curves. In 1989, Said [22] generalized the Ball model to higher degrees and Hu et.al. (1996) [16] developed the recursive algorithms for Said-Ball curves.

The Said-Ball curve of degree n with n+1 control points, denoted by $\{\mathbf{V}_i\}_{i=0}^{n}$, can be given by

$$S(t) = \sum_{i=0}^{n} \mathbf{V}_i S_i^n(t), \tag{2.6}$$

where $S_i^n(t)$ are the Said-Ball basis functions [16] defined as follows:

$$S_i^n(t) = \begin{cases} \binom{\lfloor - \rfloor + i}{i} t^i (1-t)^{\lfloor - \rfloor + 1} & , \quad \text{for } 0 \le i \le \lceil \frac{n}{2} \rceil - 1, \\ \binom{n}{-} t^{-} (1-t)^{-} & , \quad \text{for } i = \frac{n}{2}, \\ \binom{\lfloor - \rfloor + n - i}{n-i} t^{\lfloor - \rfloor + 1} (1-t)^{n-i} & , \quad \text{for } \lfloor \frac{n}{2} \rfloor + 1 \le i \le n. \end{cases} \tag{2.7}$$

2.1.2.1 Degree Elevation of Said-Ball Curves

Degree elevation and degree reduction for Said-Ball curves were provided by Hu et al. [16] in 1996. The degree elevation of Said-Ball curve is useful for raising the number of degrees in order to adjust the shape of the curve. The Said-Ball curve of degree n with n+1 control points, denoted by $\{\mathbf{V}_i\}_{i=0}^{n}$, can be expressed in terms of a Said-Ball curve of degree n+1 with n+2 control points, given by $\{\mathbf{V}_i^{(1)}\}_{i=0}^{n+1}$, by using the following relationships of the Said-Ball degree elevation.

if n is odd,

$$\mathbf{V}_i^{(1)} = \begin{cases} \mathbf{V}_i & , \quad \text{for } 0 \leq i \leq \frac{n}{2}, \\ \mathbf{V}_{-} & , \quad \text{for } i = \frac{n+2}{2}, \\ \mathbf{V}_{i-1} & , \quad \text{for } \frac{n+4}{2} \leq i \leq n+1, \end{cases} \tag{2.8}$$

if n is even,

$$\mathbf{V}_i^{(1)} = \frac{i}{\frac{n+1}{2}+i}\mathbf{V}_{i-1}^{(1)} + \frac{\frac{n+1}{2}}{\frac{n+1}{2}+i}\mathbf{V}_i, \tag{2.9}$$

$$\mathbf{V}_{n+1-i}^{(1)} = \frac{i}{\frac{n+1}{2}+i}\mathbf{V}_{n+2-i}^{(1)} + \frac{\frac{n+1}{2}}{\frac{n+1}{2}+i}\mathbf{V}_{n-i}, \tag{2.10}$$

for $i = 0, 1, .., \frac{n-1}{2}$, and

$$\mathbf{V}_{\_\_}^{(1)} = \mathbf{V}_{\_\_}^{(1)} + \mathbf{V}_{\_\_}^{(1)}, \tag{2.11}$$

2.1.2.2 Degree Reduction of Said-Ball Curves

Typically, the degree reduction of Said-Ball curve will only produce an approximation to the original curve. However, it can produce the exact solution for the Said-Ball degree reduction if and only if

$$\mathbf{V}_{--1} + \mathbf{V}_{-+1} - 2\mathbf{V}_{-} = 0, \tag{2.12}$$

when n is even or

$$\mathbf{V}_{\_\_} - \mathbf{V}_{\_\_} = 0, \tag{2.13}$$

when n is odd.

The Said-Ball curve of degree n with n+1 control points, denoted by $\{\mathbf{V}_i\}_{i=0}^{n}$, can be approximately reduced into the Said-Ball curve of degree n-1 with n control points, expressed by $\{\mathbf{V}_i^{(-1)}\}_{i=0}^{n-1}$, as follows:

$$\mathbf{V}_{--1} = \mathbf{V}_{--1} + \frac{\mathbf{V}_{--1} + \mathbf{V}_{-+1} - 2\mathbf{V}_{-}}{2}, \tag{2.14}$$

$$\mathbf{V}_{-+1} = \mathbf{V}_{-+1} + \frac{\mathbf{V}_{--1} + \mathbf{V}_{-+1} - 2\mathbf{V}_{-}}{2}, \tag{2.15}$$

$$\mathbf{V}_{-} = \mathbf{V}_{-} + \frac{\mathbf{V}_{--1} + \mathbf{V}_{-+1} - 2\mathbf{V}_{-}}{2}. \tag{2.16}$$

if n is even,

$$\mathbf{V}_i^{(-1)} = \frac{n+2i}{n}\mathbf{V}_i + \frac{2i}{n}V_{i-1},$$

$$\mathbf{V}_{n-1-i}^{(-1)} = \frac{n+2i}{n}\mathbf{V}_{n-i} + \frac{2i}{n}V_{n+1-i}, \tag{2.17}$$

for $i = 0, 1, 2, .., \frac{n}{2} - 1$.
if n is odd,

$$\mathbf{V}_i^{(-1)} = \begin{cases} \mathbf{V}_i & , \quad \text{for } 0 \le i \le \frac{n-3}{2}, \\ \frac{\mathbf{V}_{\_\_} + V_{\_\_}}{2} & , \quad \text{for } i = \frac{n-1}{2}, \\ \mathbf{V}_{i+1} & , \quad \text{for } \frac{n+1}{2} \le i \le n-1, \end{cases} \tag{2.18}$$

for $i = 0, 1, 2, .., n$-1.

2.1.3 Wang-Ball Curves

Wang-Ball curves were implemented by Wang [25] in 1987 but publicized later in 1996 by Hu et al.[16] with the degree elevation and the degree reduction. After that Dejdumrong et al. [7] extended the formulation to the rational Wang-Ball curves in 2001.
Wang-Ball curves with n+1 control points, denoted by $\{\mathbf{p}_i\}_{i=0}^{n}$ can be formulated by

$$W(t) = \sum_{i=0}^{n} \mathbf{p}_i A_i^n(t), \quad 0 \le t \le 1, \tag{2.19}$$

where $\{\mathbf{A}_i^n(t)\}_{i=0}^{n}$ are the Wang-Ball polynomials rewritten from Hu et.al. (1996) [16], which can be defined as follows:

$$A_i^n(t) = \begin{cases} (1-t)^{2+i}(2t)^i & , \quad \text{for } 0 \le i \le \frac{n-3}{2}, \\ (1-t)^{\_\_}(2t)^{\_\_} & , \quad \text{for } i = \frac{n-1}{2}, \\ (2(1-t))^{\_\_}t^{\_\_} & , \quad \text{for } i = \frac{n+1}{2}, \\ (2t(1-t))^{n-i}t^{n-i+2} & , \quad \text{for } \frac{n+3}{2} \le i \le n, \end{cases} \tag{2.20}$$

when n is odd or

$$A_i^n(t) = \begin{cases} (1-t)^{2+i}(2t)^i & , \quad \text{for } 0 \le i \le \frac{n}{2} - 1, \\ (2t(1-t))^{-} & , \quad \text{for } i = \frac{n}{2}, \\ (2(1-t))^{n-i}t^{n-i+2} & , \quad \text{for } \frac{n}{2} + 1 \le i \le n, \end{cases} \tag{2.21}$$

when n is even.

2.1.3.1 Degree Elevation of Wang-Ball Curves

The new control points of Wang-Ball curve with degree n+1, denoted by $\{\mathbf{p}_i^{(1)}\}_{i=0}^{n+1}$, from the Wang-Ball control points with degree n, given by $\{\mathbf{p}_i\}_{i=0}^{n}$, can be obtained by the following:

$$\mathbf{p}_i^{(1)} = \begin{cases} \mathbf{p}_i & , \quad \text{for } 0 \le i \le \frac{n}{2}, \\ \mathbf{p}_{-} & , \quad \text{for } i = \frac{n}{2} + 1, \\ \mathbf{p}_{i-1} & , \quad \text{for } \frac{n}{2} + 2 \le i \le n+1, \end{cases} \tag{2.22}$$

when n is even or

$$\mathbf{p}_i^{(1)} = \begin{cases} \mathbf{p}_i & , \quad \text{for } 0 \leq i \leq \frac{n-1}{2}, \\ \frac{\mathbf{p}_{\_\_} + \mathbf{p}_{\_\_}}{2} & , \quad \text{for } i = \frac{n+1}{2}, \\ \mathbf{p}_{i-1} & , \quad \text{for } \frac{n+3}{2} \leq i \leq n+1, \end{cases} \tag{2.23}$$

when n is odd.

2.1.3.2 Degree Reduction of Wang-Ball Curves

The new control points of Wang-Ball curve with degree n-1, denoted by $\{\mathbf{p}_i^{(-1)}\}_{i=0}^{n-1}$, from the Wang-Ball control points with degree n, given by $\{\mathbf{p}_i\}_{i=0}^{n}$, can be obtained by the following:

$$\mathbf{p}_i^{(-1)} = \begin{cases} \mathbf{p}_i & , \quad \text{for } 0 \leq i \leq \frac{n}{2} - 2, \\ \mathbf{p}_{-\,-1} + \frac{(\mathbf{p}_{-} \;\; + \mathbf{p}_{-} \;\; -2\mathbf{p}_{-})}{2} & , \quad \text{for } i = \frac{n}{2} - 1, \\ \mathbf{p}_{-+1} + \frac{(\mathbf{p}_{-} \;\; + \mathbf{p}_{-} \;\; -2\mathbf{p}_{-})}{2} & , \quad \text{for } i = \frac{n}{2}, \\ \mathbf{p}_{i+1} & , \quad \text{for } \frac{n}{2} + 1 \leq i \leq n-1, \end{cases} \tag{2.24}$$

when n is even or

$$\mathbf{p}_i^{(-1)} = \begin{cases} \mathbf{p}_i & , \quad \text{for } 0 \leq i \leq \frac{n-3}{2}, \\ \frac{\mathbf{p}_{\_\_} + \mathbf{p}_{\_\_}}{2} & , \quad \text{for } i = \frac{n-1}{2}, \\ \mathbf{p}_{i+1} & , \quad \text{for } \frac{n+1}{2} \leq i \leq n-1, \end{cases} \tag{2.25}$$

when n is odd.

2.1.4 DP Curve

DP curves were developed by Delgado and Peña in 2003. They introduced a new curve representation which has the properties of Normalized Totally Positive Property (NTP) basis functions, corner cutting algorithm, linear computational complexity and a smaller number of coefficient calculations. An n^{th}-degree DP curve with n+1 control points, given by $\{\mathbf{d}_i\}_{i=0}^{n}$, can be expressed by

$$\mathcal{D}(t) = \sum_{i=0}^{n} \mathbf{d}_i \mathcal{D}_i^n(t), \quad 0 \leq t \leq 1, \tag{2.26}$$

where $\mathcal{D}_i^n(t)$ are the DP blending functions as given by [11]

$$\mathcal{D}_i^n(t) = \begin{cases} (1-t)^n & , \quad \text{for } i = 0, \\ t(1-t)^{n-i} & , \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ \mathcal{K}_1^n(t) + \mathcal{K}_2^n(t) & , \quad \text{for } i = \left\lfloor \frac{n}{2} \right\rfloor, \\ \mathcal{K}_1^n(t) + \mathcal{K}_3^n(t) & , \quad \text{for } i = \left\lceil \frac{n}{2} \right\rceil, \\ \mathcal{D}_{n-i}^n(1-t) & , \quad \text{for } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n, \end{cases} \tag{2.27}$$

where

$$\mathcal{K}_1^n(t) = \frac{1}{2}^{\lceil - \rceil - \lfloor - \rfloor}(1 - t^{\lfloor - \rfloor + 1} - (1-t)^{\lfloor - \rfloor + 1}),$$

$$\mathcal{K}_2^n(t) = (\lceil \tfrac{n}{2} \rceil - \lfloor \tfrac{n}{2} \rfloor) t (1-t)^{\lfloor - \rfloor + 1},$$

$$\mathcal{K}_3^n(t) = (\lceil \tfrac{n}{2} \rceil - \lfloor \tfrac{n}{2} \rfloor) t^{\lfloor - \rfloor + 1}(1-t).$$

2.1.4.1 Degree Elevation of DP Curve

Itsariyawanich [18] proposed the degree elevation for DP curve in 2007. The new control points of DP curve with degree $n+k$ can be denoted by $\{\mathbf{d}_i^{(k)}\}_{i=0}^{n+k}$, the DP control points with degree n can be defined by

$$\left[\mathbf{d}_0^{(k)}\ \mathbf{d}_1^{(k)}\ \cdots\ \mathbf{d}_{n+k}^{(k)}\right] = [\mathbf{d}_0\ \mathbf{d}_1\ \cdots\ \mathbf{d}_n] \cdot \mathbf{c}_n^\top \cdot \left[\mathcal{E}_n^{n+k}\right]^\top \cdot \left[\mathcal{C}_{n+k}^{-1}\right]^\top, \tag{2.28}$$

where $\mathcal{E}_n^{n+k}$ is a matrix of Bézier curve's degree elevation from degree n to degree $n+k$,

$$\mathcal{E}_n^{n+k} = \begin{bmatrix} \mathbf{e}_{0,0} & \mathbf{e}_{0,1} & \cdots & \mathbf{e}_{0,n} \\ \mathbf{e}_{1,0} & \mathbf{e}_{1,1} & \cdots & \mathbf{e}_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{e}_{n,0} & \mathbf{e}_{n,1} & \cdots & \mathbf{e}_{n,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{e}_{n+k,0} & \mathbf{e}_{n+k,1} & \cdots & \mathbf{e}_{n+k,n} \end{bmatrix}, \tag{2.29}$$

$$\mathbf{e}_{i,j} = \begin{cases} \frac{(\ )(\ )}{(\ )} & , \text{ for } 0 \le i \le n+k,\ i-k \le j \le i \text{ and } j \ge 0, \\ 0 & , \text{ otherwise,} \end{cases} \tag{2.30}$$

and $\mathcal{C}_n$ is a conversion matrix from DP into Bézier curve of degree n [18],

$$\mathcal{C}_n = \begin{bmatrix} \mathbf{c}_{0,0} & \mathbf{c}_{0,1} & \cdots & \mathbf{c}_{0,n} \\ \mathbf{c}_{1,0} & \mathbf{c}_{1,1} & \cdots & \mathbf{c}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}_{n,0} & \mathbf{c}_{n,1} & \cdots & \mathbf{c}_{n,n} \end{bmatrix}, \tag{2.31}$$

where

$$\mathbf{c}_{i,j} = \begin{cases} 1 & , \text{ for } i=j=0 \text{ or } i=j=n, \\ \frac{j(\ \ )}{n(\ \ )} & , \text{ for } 1 \le i \le \lfloor \frac{n}{2} \rfloor - 1 \text{ and } 1 \le j \le i, \\ \frac{(n-j)(\ \ )}{(n-i)(\ \ )} & , \text{ for } \lceil \frac{n}{2} \rceil + 1 \le i \le n-1 \text{ and } i \le j \le n-1. \end{cases} \tag{2.32}$$

If n is even,

$$
\mathbf{c}_{i,j} = \begin{cases} 1 - \frac{(\lfloor - \rfloor\ ) - (\lfloor - \rfloor\ )}{(\lfloor - \rfloor\ )} & , \quad \text{for } i = \lfloor \frac{n}{2} \rfloor \text{ and } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor, \\ 1 - \frac{(\lfloor - \rfloor\ ) - (\lfloor - \rfloor\ )}{(\lfloor - \rfloor\ )} & , \quad \text{for } i = \lfloor \frac{n}{2} \rfloor \text{ and } \lceil \frac{n}{2} \rceil \leq j \leq n-1, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.33}
$$

If n is odd,

$$
\mathbf{c}_{i,j} = \begin{cases} \frac{1}{2}\left[1 - \frac{(\lfloor - \rfloor\ ) - (\lfloor - \rfloor\ )}{(\lfloor - \rfloor\ )}\right] + \frac{2j(\lfloor - \rfloor\ )}{n-1(\lfloor - \rfloor\ )} & , \quad \text{for } i = \lfloor \frac{n}{2} \rfloor \text{ and } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor, \\ \frac{1}{2}\left[1 - \frac{(\lfloor - \rfloor\ ) - (\lfloor - \rfloor\ )}{(\lfloor - \rfloor\ )}\right] + \frac{n-j(\lfloor - \rfloor\ )}{n(\lfloor - \rfloor\ )} & , \quad \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } \lfloor \frac{n}{2} \rfloor \leq j \leq \lfloor \frac{n}{2} \rfloor, \\ \frac{1}{2}\left[1 - \frac{(\lfloor - \rfloor\ ) - (\lfloor - \rfloor\ )}{(\lfloor - \rfloor\ )}\right] + \frac{j(\lfloor - \rfloor\ )}{n(\lfloor - \rfloor\ )} & , \quad \text{for } i = \lfloor \frac{n}{2} \rfloor \text{ and } \lceil \frac{n}{2} \rceil \leq j \leq n-1, \\ \frac{1}{2}\left[1 - \frac{(\lfloor - \rfloor\ ) - (\lfloor - \rfloor\ )}{(\lfloor - \rfloor\ )}\right] + \frac{2(n-j)(\lfloor - \rfloor\ )}{n-1(\lfloor - \rfloor\ )} & , \quad \text{for } i = \lceil \frac{n}{2} \rceil \text{ and } \lceil \frac{n}{2} \rceil \leq j \leq n-1, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.34}
$$

2.1.4.2 Degree Reduction of DP Curve

Itsariyawanich [18] also proposed the exact degree reduction for DP curve. The new control points of DP curve with degree $n-k$ can be denoted by $\{\mathbf{d}_i^{(-k)}\}_{i=0}^{n-k}$, the DP control points with degree n can be defined by

$$
\left[\mathbf{d}_0^{(-k)}\ \mathbf{d}_1^{(-k)}\ \cdots\ \mathbf{d}_{n-k}^{(-k)}\right] = [\mathbf{d}_0\ \mathbf{d}_1\ \cdots\ \mathbf{d}_n] \cdot \mathcal{D}_n^{\top} \cdot \left[\mathcal{R}_n^{n-k}\right]^{\top} \cdot \left[\mathcal{D}_{n-k}^{-1}\right]^{\top}, \tag{2.35}
$$

where $\mathcal{R}_n^{n-k}$ is the degree reduction matrix from degree n to degree $n-k$ for Bézier curve,

$$
\mathcal{R}_n^{n-k} = \begin{bmatrix} \mathbf{r}_{0,0} & \mathbf{r}_{0,1} & \cdots & \mathbf{r}_{0,n} \\ \mathbf{r}_{1,0} & \mathbf{r}_{1,1} & \cdots & \mathbf{r}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{r}_{n-k,0} & \mathbf{r}_{n-k,1} & \cdots & \mathbf{r}_{n-k,n} \end{bmatrix}, \tag{2.36}
$$

and $\mathcal{R}_n^{n-k}$ can be obtained by inverting a square matrix $\mathcal{F}_n^n$ and then erasing k last rows of the inversion matrix. The matrix $\mathcal{F}_n^n$ can be defined by

$$
\mathcal{F}_n^n = \begin{bmatrix} \mathbf{f}_{0,0} & \mathbf{f}_{0,1} & \cdots & \mathbf{f}_{0,n} \\ \mathbf{f}_{1,0} & \mathbf{f}_{1,1} & \cdots & \mathbf{f}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{f}_{n,0} & \mathbf{f}_{n,1} & \cdots & \mathbf{f}_{n,n} \end{bmatrix}, \tag{2.37}
$$

where

$$\mathbf{f}_{i,j} = \begin{cases} \frac{(\ )(\ )}{(\ )} & , \quad \text{for } 0 \le i \le n-k,\ i-k \le j \le i \text{ and } j \ge 0, \\ 1 & , \quad \text{for } i = j \text{ and } n-k+1 \le i \le n, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.38}$$

$\mathcal{C}_n$ is a conversion matrix from DP to Bézier curves of degree n as defined in Equations 2.31, 2.32, 2.33 and 2.34.

2.2 Triangular Surfaces

2.2.1 Bézier Triangular Surfaces

The Bézier triangular surface was developed in 1959 by de Casteljau at Citröen. Bézier triangular surfaces have been specifically used in several CAD/CAM and CAGD applications, especially in subtle geometric and shape design. However, they have cubic computational complexity, $O(n^3)$.

2.2.1.1 Basis Function

A Bézier triangular patch is defined by

$$\mathbf{b}(u,v,w) = \sum_{i+j+k=n} \mathbf{b}_{i,j,k} \frac{n!}{i!j!k!} u^i v^j w^k = \sum_{i+j+k=n} \mathbf{b}_{i,j,k} \mathcal{B}^n_{i,j,k}(u,v,w), \tag{2.39}$$

where $u, v, w \ge 0, u + v + w = 1$ and the Bernstein polynomials of degree n are obtained by

$$\mathcal{B}^n_{i,j,k}(u,v,w) = \frac{n!}{i!j!k!} u^i v^j w^k. \tag{2.40}$$

2.2.2 Said-Ball Triangular Surfaces

The Said-Ball triangular surface was introduced in 1991 by Goodman and Said [15]. This surface still requires cubic time complexity, $O(n^3)$. That is, the complexity is the same as that of the Bézier triangular surface.

2.2.2.1 Basis Function

A Said-Ball triangular surface can be defined by basis $\mathcal{S}^n_{i,j,k}(u,v,w)$ where (u,v,w) is the barycentric coordinates and control points, denoted by $\mathbf{ba}_{i,j,k}$ for degree n.
For $m \ge 1$ and $0 \le r \le m$,

$$\mathbf{ba}(u,v,w) = \sum_{i=1}^{4} \mathcal{S}^i_{2m+1,r}(u,v,w), \tag{2.41}$$

where,

$$S^1_{2m+1,r}(u,v,w) = u^{m+1} \sum_{i+j+k=2m+1-r, i\geq m+1} S^r_{i,j,k} S^m_{j,k}(v,w), \tag{2.42}$$

$$S^2_{2m+1,r}(u,v,w) = v^{m+1} \sum_{i+j+k=2m+1-r, j\geq m+1} S^r_{i,j,k} S^m_{k,i}(w,u), \tag{2.43}$$

$$S^3_{2m+1,r}(u,v,w) = w^{m+1} \sum_{i+j+k=2m+1-r, k\geq m+1} S^r_{i,j,k} S^m_{i,j}(u,v), \tag{2.44}$$

$$S^4_{2m+1,r}(u,v,w) = u^{m+1} \sum_{i+j+k=2m+1-r, i,j,k\leq m} S^r_{i,j,k} S^{2m+1-r}_{i,j,k}(u,v,w), \tag{2.45}$$

in which
$V^0_{i,j,k} = V_{i,j,k}$
For $r \geq 1$ and $i + j + k = 2m + 1 - r$

The first condition $(i \geq m+1 \text{ or } j \geq m+1 \text{ or } k \geq m+1)$:
$V^r_{i,j,k} = V^{r-1}_{i+1,j,k}$ if $i \geq m+1$
$V^r_{i,j,k} = V^{r-1}_{i,j+1,k}$ if $j \geq m+1$
$V^r_{i,j,k} = V^{r-1}_{i,j,k+1}$ if $k \geq m+1$

The second condition $(i,j,k \leq m)$: $V^r_{i,j,k} = uV^{r-1}_{i+1,j,k} + vV^{r-1}_{i,j+1,k} + wV^{r-1}_{i,j,k+1}$ if $i,j,k \leq m$.

When $r = m$, becomes:

$$\mathbf{ba}(u,v,w) = \sum_{i+j+k=m+1} \mathbf{ba}^m_{i,j,k} S^{m+1}_{i,j,k}(u,v,w), \tag{2.46}$$

where $u, v, w \geq 0, u + v + w = 1$ and the basis functions are Bernstein bivariate polynomials of degree $m + 1$, denoted by

$$S^{m+1}_{i,j,k}(u,v,w) = \frac{(m+1)!}{i!j!k!} u^i v^j w^k \tag{2.47}$$

2.2.3 Hu's Wang-Ball Triangular Surfaces

Hu's Wang-Ball Triangular surface was presented in 1998 by Hu et. al. Nevertheless, the computational time of this surface cannot be reduced to quadratic complexity, $O(n^2)$. This surface still requires cubic computation time, $O(n^3)$.

2.2.3.1 Basis Function

A Hu's Wang-Ball triangular surface can be defined by use of Hu's Wang-Ball basis denoted by, $\mathcal{W}^n_{i,j,k}(u,v,w)$, and control points $\mathbf{ba}_{i,j,k}$, i.e.

$$\mathbf{ba}(u,v,w) = \sum_{i+j+k=n} \mathbf{ba}^n_{i,j,k} \mathcal{W}^n_{i,j,k}(u,v,w), \tag{2.48}$$

where $u,v,w \geq 0, u+v+w=1$ and Hu's Wang-Ball basis functions are recursively given as follows:

Let $\mathcal{W}^1_{1,0,0} = u, \mathcal{W}^1_{0,1,0} = v, \mathcal{W}^1_{0,0,1} = w$.

For $i+j+k = n+1$

if $i,j,k \leq [\frac{n+1}{2}]$

$$\mathcal{W}^{n+1}_{i,j,k}(u,v,w) = u\mathcal{W}^n_{i-1,j,k}(u,v,w) + v\mathcal{W}^n_{i,j-1,k}(u,v,w) + w\mathcal{W}^n_{i,j,k-1}(u,v,w) \tag{2.49}$$

otherwise

$$\mathcal{W}^{n+1}_{i,j,k}(u,v,w) = \begin{cases} u\mathcal{W}^n_{i-1,j,k}(u,v,w) & , \quad i = [\frac{n+1}{2}]+1, \\ v\mathcal{W}^n_{i,j-1,k}(u,v,w) & , \quad j = [\frac{n+1}{2}]+1, \\ w\mathcal{W}^n_{i,j,k-1}(u,v,w) & , \quad k = [\frac{n+1}{2}]+1, \\ \mathcal{W}^n_{i-1,j,k}(u,v,w) & , \quad i \geq [\frac{n+1}{2}]+2, \\ \mathcal{W}^n_{i,j-1,k}(u,v,w) & , \quad j \geq [\frac{n+1}{2}]+2, \\ \mathcal{W}^n_{i,j,k-1}(u,v,w) & , \quad k \geq [\frac{n+1}{2}]+2. \end{cases} \tag{2.50}$$

2.2.4 DP Triangular Surfaces (2008)

In 2008, Chen provided the first model of a DP Triangular surface [6]. Chen's model, however, does not satisfy the convexity condition for geometric modeling and does not reduce computational complexity to $O(n^2)$.

2.2.4.1 Basis Function

For degree 1,

$\mathcal{D}^1_{1,0,0}(u,v,w) = u, \mathcal{D}^1_{0,1,0}(u,v,w) = v, \mathcal{D}^1_{0,0,1}(u,v,w) = w.$

For degree 2,

$\mathcal{D}^2_{2,0,0}(u,v,w) = u^2, \mathcal{D}^2_{1,1,0}(u,v,w) = 2uv, \mathcal{D}^2_{1,0,1}(u,v,w) = 2vw,$

$\mathcal{D}^2_{0,2,0}(u,v,w) = v^2, \mathcal{D}^2_{0,1,1}(u,v,w) = 2vw, \mathcal{D}^2_{0,0,2}(u,v,w) = w^2.$

For degree 3,
$\mathcal{D}^3_{3,0,0}(u,v,w) = u^3, \mathcal{D}^3_{2,1,0}(u,v,w) = u^2v, \mathcal{D}^3_{2,0,1}(u,v,w) = u^2w,$
$\mathcal{D}^3_{1,2,0}(u,v,w) = uv^2, \mathcal{D}^3_{1,1,1}(u,v,w) = 1-(u^2+v^2+w^2), \mathcal{D}^3_{1,0,2}(u,v,w) = uw^2,$
$\mathcal{D}^3_{0,3,0}(u,v,w) = v^3, \mathcal{D}^3_{0,2,1}(u,v,w) = v^2w, \mathcal{D}^3_{0,1,2}(u,v,w) = vw^2,$
$\mathcal{D}^3_{0,0,3}(u,v,w) = w^3.$

For $n \geq 4$, suppose $i+j+k=n$. Then, the triangular DP basis of degree n are recursively given as follows:

(1) When n is even, if $i=n$ or $j=n$ or $k=n$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \Delta^n_u & , \quad i=n \text{ and } j=0 \text{ and } k=0, \\ \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \Delta^n_v & , \quad j=n \text{ and } i=0 \text{ and } k=0, \\ \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) + \Delta^n_w & , \quad k=n \text{ and } i=0 \text{ and } j=0, \end{cases} \tag{2.51}$$

where

$$\Delta^n_\xi = (\frac{1}{3})^{n-4}[\frac{1}{3}(\xi^n - \xi^-) + \frac{2}{3}(\xi^- - \xi^{n-1})], \xi = u \text{ or } v \text{ or } w, \tag{2.52}$$

for each $i,j,k<n$, if $i=0$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \frac{1}{3}v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad j>k, \\ \frac{1}{3}(\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w)) & , \quad j=k, \\ \frac{1}{3}w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad j<k, \end{cases} \tag{2.53}$$

if $j=0$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \frac{1}{3}u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad i>k, \\ \frac{1}{3}(\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w)) & , \quad i=k, \\ \frac{1}{3}w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) & , \quad i<k, \end{cases} \tag{2.54}$$

if $k=0$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \frac{1}{3}u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) & , \quad i>j, \\ \frac{1}{3}(\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w)) & , \quad i=j, \\ \frac{1}{3}v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) & , \quad i<j, \end{cases} \tag{2.55}$$

if $0<i,j,k<n$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \frac{1}{3}(\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w)). \tag{2.56}$$

(2)When n is odd, if $i = n$ or $j = n$ or $k = n$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \Phi^n_u &, \quad i = n \text{ and } j = 0 \text{ and } k = 0, \\ \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \Phi^n_v &, \quad j = n \text{ and } i = 0 \text{ and } k = 0, \\ \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) + \Phi^n_w &, \quad k = n \text{ and } i = 0 \text{ and } j = 0, \end{cases} \tag{2.57}$$

where

$$\Phi^n_\xi = (\frac{1}{3})^{n-4}[\frac{1}{3}(\xi^n - \xi^{\text{—}}) + \frac{2}{3}(\xi^{\text{—}} - \xi^{n-1})], \xi = u \text{ or } v \text{ or } w, \tag{2.58}$$

for each $i, j, k < n$, if $i = 0$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \frac{1}{3}v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) &, \quad j > \frac{n+1}{2}, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) &, \quad j = \frac{n+1}{2}, \\ \frac{1}{3}w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) &, \quad k > \frac{n+1}{2}, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) &, \quad k = \frac{n+1}{2}, \end{cases} \tag{2.59}$$

if $j = 0$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \frac{1}{3}u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) &, \quad i > \frac{n+1}{2}, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) &, \quad i = \frac{n+1}{2}, \\ \frac{1}{3}w\mathcal{D}^{n-1}_{i,j,k-1}(u,v,w) &, \quad k > \frac{n+1}{2}, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) &, \quad k = \frac{n+1}{2}, \end{cases} \tag{2.60}$$

if $k = 0$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \begin{cases} \frac{1}{3}u\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) &, \quad i > \frac{n+1}{2}, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) &, \quad i = \frac{n+1}{2}, \\ \frac{1}{3}v\mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) &, \quad j > \frac{n+1}{2}, \\ \frac{1}{3}\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) &, \quad j = \frac{n+1}{2}, \end{cases} \tag{2.61}$$

if $0 < i, j, k < n$, then

$$\mathcal{D}^n_{i,j,k}(u,v,w) = \frac{1}{3}(\mathcal{D}^{n-1}_{i-1,j,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j-1,k}(u,v,w) + \mathcal{D}^{n-1}_{i,j,k-1}(u,v,w)). \tag{2.62}$$

From the basis function of triangular DP surface (2008), the polynomials of this model are defined by coefficient values that do not relate to the polynomials of DP curve.

For example in degree 4,
$\mathcal{D}^4_{4,0,0}(u,v,w) = \frac{1}{3}u^4, \mathcal{D}^4_{3,1,0}(u,v,w) = \frac{1}{3}u^3v, \mathcal{D}^4_{3,0,1}(u,v,w) = \frac{1}{3}u^3w,$
$\mathcal{D}^4_{2,2,0}(u,v,w) = \frac{1}{3}(u^2v + uv^2), \mathcal{D}^4_{2,1,1}(u,v,w) = \frac{1}{3} - \frac{1}{3}(u^2 + v^2 + w^2),$
$\mathcal{D}^4_{2,0,2}(u,v,w) = \frac{1}{3}(u^2w + uw^2), \mathcal{D}^4_{1,3,0}(u,v,w) = \frac{1}{3}uv^3,$
$\mathcal{D}^4_{1,2,1}(u,v,w) = \frac{1}{3} - \frac{1}{3}(u^2 + v^2 + w^2), \mathcal{D}^4_{1,1,2}(u,v,w) = \frac{1}{3} - \frac{1}{3}(u^2 + v^2 + w^2),$
$\mathcal{D}^4_{1,0,3}(u,v,w) = \frac{1}{3}uw^3, \mathcal{D}^4_{0,4,0}(u,v,w) = \frac{1}{3}v^4, \mathcal{D}^4_{0,3,1}(u,v,w) = \frac{1}{3}v^3w,$
$\mathcal{D}^4_{0,2,2}(u,v,w) = \frac{1}{3}(v^2w + vw^2), \mathcal{D}^4_{0,1,3}(u,v,w) = \frac{1}{3}vw^3, \mathcal{D}^4_{0,0,4}(u,v,w) = \frac{1}{3}w^4$

For example in degree 5,
$\mathcal{D}^5_{5,0,0}(u,v,w) = \frac{1}{9}(u^5+u^4+2u^3+5u^2), \mathcal{D}^5_{4,1,0}(u,v,w) = \frac{1}{9}u^4v, \mathcal{D}^5_{4,0,1}(u,v,w) = \frac{1}{9}u^4w,$
$\mathcal{D}^5_{3,2,0}(u,v,w) = \frac{1}{9}u^3v, \mathcal{D}^5_{3,1,1}(u,v,w) = \frac{1}{9} - \frac{1}{9}(u^4 + v^2 + w^2),$
$\mathcal{D}^5_{3,0,2}(u,v,w) = \frac{1}{9}u^3w, \mathcal{D}^5_{2,3,0}(u,v,w) = \frac{1}{9}uv^3,$
$\mathcal{D}^5_{2,2,1}(u,v,w) = \frac{2}{9} - \frac{1}{9}(2u^3 + u^2w + 2v^3 + v^2w + 2w^2),$
$\mathcal{D}^5_{2,1,2}(u,v,w) = \frac{2}{9} - \frac{1}{9}(2u^3 + u^2v + 2w^3 + vw^2 + 2v^2),$
$\mathcal{D}^5_{2,0,3}(u,v,w) = \frac{1}{9}uw^3, \mathcal{D}^5_{1,4,0}(u,v,w) = \frac{1}{9}uv^4, \mathcal{D}^5_{1,3,1}(u,v,w) = \frac{1}{9} - \frac{1}{9}(u^2 + v^4 + w^2),$
$\mathcal{D}^5_{1,2,2}(u,v,w) = \frac{2}{9} - \frac{1}{9}(2v^3+uv^2+2w^3+uw^2+2u^2), \mathcal{D}^5_{1,1,3}(u,v,w) = \frac{1}{9} - \frac{1}{9}(u^2+v^2+w^4),$
$\mathcal{D}^5_{1,0,4}(u,v,w) = \frac{1}{9}uw^4, \mathcal{D}^5_{0,5,0}(u,v,w) = \frac{1}{9}(v^5 + v^4 + 2v^3 + 5v^2),$
$\mathcal{D}^5_{0,4,1}(u,v,w) = \frac{1}{9}v^4w, \mathcal{D}^5_{0,3,2}(u,v,w) = \frac{1}{9}v^3w, \mathcal{D}^5_{0,2,3}(u,v,w) = \frac{1}{9}vw^3,$
$\mathcal{D}^5_{0,1,4}(u,v,w) = \frac{1}{9}vw^4, \mathcal{D}^5_{0,0,5}(u,v,w) = \frac{1}{9}(w^5 + w^4 + 2w^3 + 5w^2)$

Thus, this model does not directly apply the concepts of DP curves.

2.2.4.2 Convex Hull Property

The convex hull property is an important property that can guarantee the surface to be inside the convex hull of its control net. However, Chen's triangular DP surface (2008) does not satisfy the affine condition, i.e., $\sum_{i+j+k=n} \mathcal{D}^n_{i,j,k}(u,v,w) = 1$, and thus does not have the convex hull property.

For example in degree 3,
$\mathcal{D}^3_{3,0,0}(u,v,w) = u^3, \mathcal{D}^3_{2,1,0}(u,v,w) = u^2v, \mathcal{D}^3_{2,0,1}(u,v,w) = u^2w,$
$\mathcal{D}^3_{1,2,0}(u,v,w) = uv^2, \mathcal{D}^3_{1,1,1}(u,v,w) = 1 - (u^2 + v^2 + w^2), \mathcal{D}^3_{1,0,2}(u,v,w) = uw^2,$
$\mathcal{D}^3_{0,3,0}(u,v,w) = v^3, \mathcal{D}^3_{0,2,1}(u,v,w) = v^2w, \mathcal{D}^3_{0,1,2}(u,v,w) = vw^2, \mathcal{D}^3_{0,0,3}(u,v,w) = w^3$

$$\sum_{i+j+k=3} \mathcal{D}^3_{i,j,k}(u,v,w) \neq 1 \tag{2.63}$$

In cases of high degrees, the summation of polynomials is not equal to 1.

$$\sum_{i+j+k=n} \mathcal{D}^n_{i,j,k}(u,v,w) \neq 1 \text{ for } n \geq 3 \qquad (2.64)$$

Thus, this triangular DP surface (2008) lacks the convexity property, which is one of the most important characteristics for geometric modeling.

2.2.4.3 Computational Complexity

We can find the number for interpolations of Chen's triangular DP surface (2008) by counting the points needed to interpolate from the recursive algorithms [6]. The relationships between the number of interpolations for each surface and the degree from 1 to 13 can be shown in Table 2.1

Table 2.1 The number of interpolations for evaluating a point on Chen's triangular DP surface (2008) compared to the existing triangular surfaces

Degree (n)	1	2	3	4	5	6	7	8	9	10	11	12	13
Bézier	1	4	10	20	35	56	84	120	165	220	286	364	455
Said-Ball	1	4	7	13	23	36	54	77	105	140	181	230	287
Hu's Wang-Ball	1	4	8	14	22	32	45	60	78	100	125	154	187
DP (2008)	1	4	7	14	24	40	61	90	126	172	227	294	372

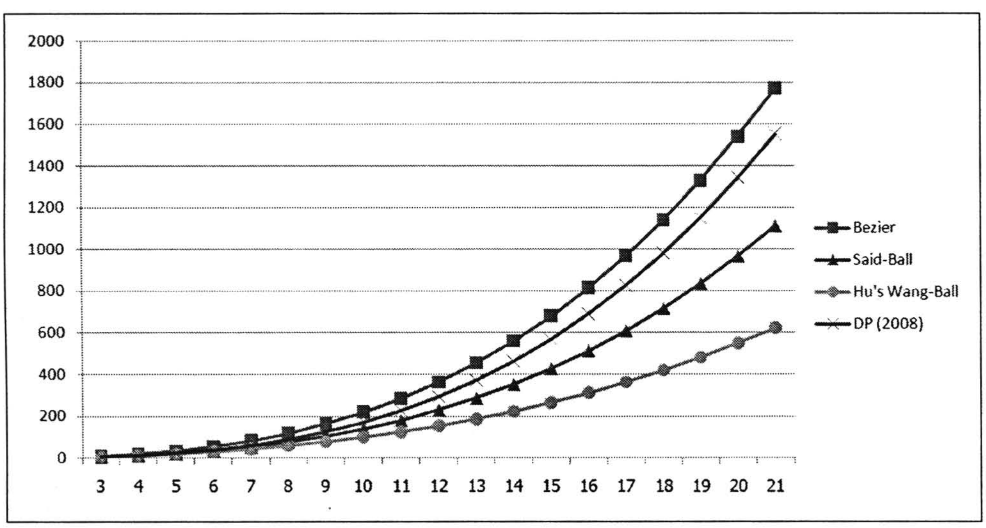

Figure 2.1 The Number of Interpolations for Evaluating a Triangular DP Surface (Chen, 2008) Compared to The Previous Triangular Surfaces

Thus, the computation time of Chen's triangular DP surface (2008) is nearly equal to the computation time of triangular Bézier surface which is $O(n^3)$. Compared to triangular Said-Ball surface and triangular Hu's Wang-Ball surface, Chen's model takes longer time for evaluating a triangular DP surface, as shown in Figure 2.1.

2.2.4.4 Degree Elevation

The degree elevation of the triangular DP surface (2008) has been provided only for the case of degree $n = 2$. It is not a generalized function for any degree.

The quadratic triangular DP surface $\mathcal{D}(u, v, w)$ can be raised into a triangular DP surface of degree 3, i.e.,

$$\sum_{i+j+k=2} \mathcal{D}^2_{i,j,k}(u, v, w)\mathbf{d}_{i,j,k} = \sum_{i+j+k=3} \mathcal{D}^3_{i,j,k}(u, v, w)\mathbf{d}^{(1)}_{i,j,k} \tag{2.65}$$

where $\{\mathbf{d}^{(1)}_{i,j,k}\}(i + j + k = 3)$ is the new control point after degree elevation and can be expressed as follows:

$\mathbf{d}^{(1)}_{3,0,0}(u, v, w) = \mathbf{d}_{2,0,0}$,
$\mathbf{d}^{(1)}_{2,1,0}(u, v, w) = \mathbf{d}_{2,0,0} + 2\mathbf{d}_{1,1,0} - \frac{2}{3}(\mathbf{d}_{1,1,0} + \mathbf{d}_{1,0,1} + \mathbf{d}_{0,1,1})$,
$\mathbf{d}^{(1)}_{2,0,1}(u, v, w) = \mathbf{d}_{2,0,0} + 2\mathbf{d}_{1,0,1} - \frac{2}{3}(\mathbf{d}_{1,1,0} + \mathbf{d}_{1,0,1} + \mathbf{d}_{0,1,1})$,
$\mathbf{d}^{(1)}_{1,2,0}(u, v, w) = \mathbf{d}_{0,2,0} + 2\mathbf{d}_{1,1,0} - \frac{2}{3}(\mathbf{d}_{1,1,0} + \mathbf{d}_{1,0,1} + \mathbf{d}_{0,1,1})$,
$\mathbf{d}^{(1)}_{1,1,1}(u, v, w) = \frac{1}{3}(\mathbf{d}_{1,1,0} + \mathbf{d}_{1,0,1} + \mathbf{d}_{0,1,1})$,
$\mathbf{d}^{(1)}_{1,0,2}(u, v, w) = \mathbf{d}_{0,0,2} + 2\mathbf{d}_{1,0,1} - \frac{2}{3}(\mathbf{d}_{1,1,0} + \mathbf{d}_{1,0,1} + \mathbf{d}_{0,1,1})$,
$\mathbf{d}^{(1)}_{0,3,0}(u, v, w) = \mathbf{d}_{0,2,0}$,
$\mathbf{d}^{(1)}_{0,2,1}(u, v, w) = \mathbf{d}_{0,2,0} + 2\mathbf{d}_{0,1,1} - \frac{2}{3}(\mathbf{d}_{1,1,0} + \mathbf{d}_{1,0,1} + \mathbf{d}_{0,1,1})$,
$\mathbf{d}^{(1)}_{0,1,2}(u, v, w) = \mathbf{d}_{0,0,2} + 2\mathbf{d}_{0,1,1} - \frac{2}{3}(\mathbf{d}_{1,1,0} + \mathbf{d}_{1,0,1} + \mathbf{d}_{0,1,1})$,
$\mathbf{d}^{(1)}_{0,0,3}(u, v, w) = \mathbf{d}_{0,0,2}$

Unfortunately, triangular DP surface (2008) does not provide degree reduction and the conversions which are important properties in triangular surface modeling.