บทที่ 3

Solutions of the field equation

Solution to the field equation (2.17),

$$-\frac{T'}{2\Gamma}(\Gamma-1)^2+\frac{\Box^2\phi}{\Gamma}-\frac{g^{\mu\nu}}{\Gamma^2}\partial_\mu\phi\partial_\nu\Gamma=V'+\frac{\beta\rho}{M_p}(1-3\omega)e^{(1-3\omega)\beta\phi/M_p}$$

is major ingredient of the project since this encodes both dynamical behavior (temporal dependent) and spatial cosmography of the universe.

3.1 Cosmological solution

Let us assume that the metric is just that of the flat FRW form, and that the scalar field is homogeneous, i.e. the field equation is only-time dependent. The equation of motion therefore reduces to the following

$$\ddot{\phi}+3H\Gamma^2\dot{\phi}-\frac{T_\phi}{2}\left(2\Gamma^3+1-3\Gamma^2\right)+V_{\rm eff}\Gamma^3=0 \tag{3.1}$$

or in another form,

$$\ddot{\phi}+\Gamma^2 3H\dot{\phi}+\Gamma^3\left(V'-T'\right)+T'-\frac{3}{2}\frac{T'}{T}\dot{\phi}^2+\Gamma^3\frac{\beta\rho}{M_{\rm P}}e^{\beta\phi/M_{\rm P}}=0 \tag{3.2}$$

Let us consider the regime where $\dot{\phi}^2 \ll T(\phi)$ which is the non-relativistic regime. Keeping the fourth order terms, one finds that the equation of motion admits the following expansion

$$\ddot{\phi} + 3H\dot{\phi}\left(1 - \frac{\dot{\phi}^2}{T}\right) - \frac{3T_\phi \dot{\phi}^2}{8T^2} + V_{\rm eff}\left(1 - \frac{3\dot{\phi}^2}{2T} + \frac{3\dot{\phi}^4}{8T^2}\right) \simeq 0 \tag{3.3}$$

3.2 Radial solution

Assuming isotropy of the universe but not homogeneity, the non-zero component of the FRW metric for the field equation is radial, i.e.

$$\begin{aligned} &\phi_{rr}\left[1 - \frac{\phi_r^2}{T\,(1+\phi_r^2/T)}\right] \\ &+ \frac{2}{r}\phi_r\left[1 + \frac{\phi_r^3}{T(1+{\phi_r}^2/T)}\frac{r}{4}\left(\frac{T'}{T}\right)\right] = T'\left(1+\frac{\phi_r^2}{2T}\right) - T'\left(\sqrt{1+\frac{\phi_r^2}{T}}\right) \\ &\qquad + V'\Gamma + \Gamma\frac{\beta}{M_{\rm P}}\rho(r)e^{\beta(\phi)/M_{\rm P}} \end{aligned} \tag{3.4}$$

Limit I: $\phi_r^2/T \ll 1$

Expanding $\sqrt{1+\phi_r^2/T}$ and keeping the lowest order term,

$$\phi_{rr}\left(1 - \frac{\phi_r^2}{T}\right) + \frac{2}{r}\phi_r\left[1 + \frac{\phi_r^3}{T}\frac{r}{4}\left(\frac{T'}{T}\right)\right] \approx V'\Gamma + \Gamma\frac{\beta}{M_{\rm P}}\rho(r)e^{\beta\phi/M_{\rm P}} \tag{3.5}$$

or

$$\phi_{rr} + \frac{2}{r}\phi_r\left[1 + \frac{\phi_r^3}{T}\frac{r}{4}\left(\frac{T'}{T}\right)\right] \approx \Gamma\left[V' + \frac{\beta}{M_{\rm P}}\rho(r)e^{\beta\phi/M_{\rm P}}\right] \tag{3.6}$$

Limit II: $\phi_r^2/T \gg 1$

Under the limit $\phi_r^2/T \gg 1$, approximations

$$\Gamma \equiv \sqrt{1+\frac{\phi_r^2}{T}} \simeq \frac{\phi_r}{\sqrt{T}} \tag{3.7}$$

and

$$\frac{\phi_r^2}{T} = \Gamma^2 - 1 \simeq \Gamma^2 \tag{3.8}$$

are made. Field equation then takes the form,

$$\frac{2}{r} = \frac{1}{\sqrt{T}}\left(V' - T'\right) \tag{3.9}$$

3.3 Spherically symmetric with locally flat solution

Let us assume that $g_{\mu\nu} = \eta_{\mu\nu}$, where we work in polar coordinates on the world-volume. The resulting expression for the equation of motion becomes

$$\phi_{rr} + \frac{2}{r}\Gamma^2\phi_r + \frac{T_\phi}{2}(2\Gamma^3 + 1 - 3\Gamma^2) - \Gamma^3 V_{\text{eff}} = 0 \tag{3.10}$$

Let us first consider the case of constant T and the pressureless fluid. We split the solution into those that are outside the sphere, and those that are inside the sphere. Outside the sphere we may approximate the potential terms by $m_\infty^2(\phi - \phi_\infty)$ because the scalar field is driven towards ϕ_∞. The resulting equation of motion therefore takes the form

$$\phi_{rr} + \frac{2}{r}\Gamma^2\,\phi_r^2 \simeq \Gamma^3 m_\infty^2(\phi - \phi_\infty) \tag{3.11}$$

which we must study in the two asymptotic limits.

We first expand for small velocities, introducing a perturbation ϵ which satisfies the condition that $\Gamma = 1$ as $\epsilon \to 0$. The scalar field therefore admits the following solution

$$\begin{aligned}
\phi(r) \simeq\; & \phi_\infty \\
& + \frac{A}{r}\, e^{-m_\infty (r-R)}(1+\epsilon) \\
& - \frac{3\epsilon A^2}{4}\frac{m_\infty^2 \phi_\infty}{r^2 T}\, e^{-2m_\infty (r-R)} \\
& - \frac{3\epsilon A}{8}\frac{m_\infty^3 \phi_\infty}{rT}\left[e^{m_\infty (r+2R)}\mathrm{Ei}(-3m_\infty r) + e^{-m_\infty (r-2R)}\mathrm{Ei}(-m_\infty r)\right] \\
& + \ldots
\end{aligned} \tag{3.12}$$

where we have introduced the exponential integral function.

In the relativistic case we see that $\Gamma \gg 1$ implies that $\phi_r^2 \gg T$. We can find a solution to this equation under the assumption that $\phi_r \gg 1$ with $\phi_{rr} \sim 0$. The resulting expression for the scalar field becomes

$$\phi(r) \simeq \phi_\infty + \frac{2\sqrt{T}}{r m_\infty^2} \tag{3.13}$$

In the limit, $\phi_r \ll T(\phi)$, we find that $\Gamma \sim 1 + \ldots$ where the corrections are a derivative expansion, resulting in the approximation

$$\phi_{rr} + \frac{2\phi_r}{r}\left(1 + \frac{\phi_r^2}{T}\right) + \frac{T_\phi}{2}\frac{3\phi_r^4}{8T^2} - V_{\mathrm{eff}}\left(1 + \frac{3\phi_r^2}{2T} + \frac{3\phi_r^4}{8T^2}\right) \simeq 0 \tag{3.14}$$