CHAPTER III

MAIN RESULTS

In this chapter, we give bound for approximating the probability of number of isolated copies of a fixed connected graph in a random graph by using the Stein-Chen method.

Let $\mathbb{G}(n,p)$ be a random graph on n labeled vertices $\{1,2,\ldots,n\}$ where possible edge $\{i,j\}$ is presented randomly and independent with the probability p such that $0<p<1$, G be a fixed connected graph and

$$D=\{i=:\{i_1,i_2,\ldots,i_{v_G}\}\mid 1\le i_1<\ldots<i_{v_G}\le n\}$$

be the set of all possible combinations of v_G vertices, where $v_G:=|V(G)|$.

For each $i\in D$, we define the indicator random variable X_i, as follows:

$$X_i=\begin{cases}1 & \text{if there is an isolated copy of } G \text{ in } \mathbb{G}(n,p) \text{ that spans the vertices}\\ & i=\{i_1,\ldots,i_{v_G}\},\\ 0 & \text{otherwise,}\end{cases}$$

and set

$$W=\sum_{i\in D}X_i.$$

Then W is the number of isolated copy of G in $\mathbb{G}(n,p)$.

In 1981, Karoński and Ruciński [12] used the method of moments to prove that the distribution of W can be approximated by Poisson distribution with parameter

$$\lambda:=\mathbb{E}(W)=\binom{n}{v_G}P(X_i=1)=\binom{n}{v_G}\frac{v_G!}{Aut(G)}p^{e_G}q^{\binom{v_G}{2}-e_G+v_G(n-v_G)} \qquad (3.0.1)$$

where $q=1-p$, $e_G:=|E(G)|$ and $\dfrac{v_G!}{Aut(G)}$ is a number of possible copies of G which spans the vertex i, $Aut(G)$ stands for the number of automorphisms of G.

In 1992, Barbour, Holst and Janson[4] studied about the number of isolated copies of fixed connected graph in a random graph, and they gave the uniform bound as the following.

Theorem 3.0.1. *Let W be the number of isolated copies of G in $\mathbb{G}(n,p)$. Then*

$$d_{TV}(\mathcal{L}(W), Poi_\lambda) \leq (1-e^{-\lambda})\left(\frac{VarW}{\lambda} - 1 + 2\pi_i|G_i|\right)$$

where $d_{TV}(\mathcal{L}(W), Poi_\lambda) := \sup\{|P(W \in A) - Poi_\lambda(A)| : A \subseteq \mathbb{N}\}$, *$Poi_\lambda$ is a Poisson distribution with parameter λ,* $\pi_i = p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-v_G)}$ *and* $G_i = \{j \in D \mid j \cap i \neq \emptyset\}$.

In this chapter, we give non-uniform bound of this approximation by using Stein-Chen method. In section 3.2, we introduce Stein-Chen and coupling methods which are used in our work. The proofs of Theorem 3.1.1 and Corollary 3.1.2 are given in section 3.3 and 3.4, respectively.

3.1 Main Theorem

Theorem 3.1.1. *Let G be a fixed connected graph consisting of v_G vertices and W be the number of isolated copies of G in random graph $\mathbb{G}(n,p)$. For $A \subseteq \mathbb{N}$ and $n > 2v_G$, we have*

$$|P(W \in A) - Poi_\lambda(A)| \leq C_{\lambda,A} \frac{(np)^{v_G}}{q^{2v_G^2} e^{np}} \left[\frac{1+np}{np}\right] \tag{3.1.1}$$

where $C_{\lambda,A} = \frac{v_G^2}{Aut(G)} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}$,

$$\Delta(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\ 2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A, \end{cases}$$

when $C_w = \{0, 1, \ldots, w\}$.

Corollary 3.1.2. *Let W be the number of isolated copies of a fixed connected graph G consisting of v_G vertices in random graph $\mathbb{G}(n,p)$ and $p = \dfrac{1}{n^\gamma}$ for some $\gamma \in \mathbb{R}^+ \backslash \{1\}$. Then, for $A \subseteq \mathbb{N}$,*

1. *if* $\gamma > 1$ *then*

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C(\lambda, A, v_G)}{n^{(\gamma-1)v_G}}, \qquad (3.1.2)$$

where $C(\lambda, A, v_G) = \dfrac{2v_G^2}{Aut(G)q^{2v_G^2}} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}.$

2. *if* $0 < \gamma < 1$ *then*

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C(\lambda, A, v_G)}{n^{(1-\gamma)v_G}}. \qquad (3.1.3)$$

where $C(\lambda, A, v_G) = \dfrac{2v_G^2(2v_G)!}{Aut(G)q^{2v_G^2}} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}.$

3.2 Stein-Chen and Coupling Method

In 1972, Stein [18] gave a new technique to find a bound in the normal approximation to a distribution of a sum of dependent random variables. His technique was relied instead on the elementary differential equation. In 1975, Chen [5] applied Stein's idea to the Poisson case. The central idea of the Stein-Chen method is the difference equation

$$I_A(j) - \mathcal{P}_\lambda(I_A) = \lambda g_{\lambda,A}(j+1) - j g_{\lambda,A}(j) \qquad (3.2.1)$$

where $\lambda > 0$ and $\mathcal{P}_\lambda(I_A) = e^{-\lambda}\sum_{l=0}^{\infty} I_A(l)\frac{\lambda^l}{l!}$ and I_A is bounded real-valued function on $\mathbb{N} \cup \{0\}$. For $A \subseteq \mathbb{N} \cup \{0\}$, let $I_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$I_A(w) = \begin{cases} 1 & ; w \in A, \\ 0 & ; w \notin A. \end{cases}$$

We always call equation (3.2.1) Stein's equation for Poisson distribution function and it is well-known that the solution $g_{\lambda,A}(w)$ of is of (3.2.1) the form,

$$g_{\lambda,A}(w) = \begin{cases} (w-1)!\lambda^{-w}e^{\lambda}[\mathcal{P}_\lambda(I_{A\cap C_{w-1}}) - \mathcal{P}_\lambda(I_A)\mathcal{P}_\lambda(I_{C_{w-1}})] & ; w \geq 1, \\ 0 & ; w = 0 \end{cases}$$

where

$$C_{w-1} = \{0, 1, \ldots, w-1\}.$$

By substituting j and λ in (3.2.1) by any integer-valued random variable W and $\lambda = \mathbb{E}(W)$, we have

$$P(W \in A) - Poi_\lambda(A) = \mathbb{E}(\lambda g_{\lambda,A}(W+1)) - \mathbb{E}(W g_{\lambda,A}(W)). \qquad (3.2.2)$$

Let X_i be a random indicator with the probability $P(X_i = 1) = 1 - P(X_i = 0) = p_i$, where i ranges over some finite index set Γ, and $W = \sum\limits_{i\in\Gamma} X_i$ and $\lambda = \mathbb{E}(W) = \sum\limits_{i\in\Gamma} p_i$.

In 1992, Barbour, Holst and Janson [4] used Stein-Chen method and constructed coupling random variable W_i to find the bound in Poisson approximation. He assumed that for each i we can construct a random variable W_i, on the same probability space as W, such that the distribution of W_i equals to the conditional distribution of $W - X_i$ given $X_i = 1$, we refer to this as coupling method. Hence, for each $i \in \Gamma$,

$$\begin{aligned}
\mathbb{E}(X_i g_{\lambda,A}(W)) &= \mathbb{E}(\mathbb{E}(X_i g_{\lambda,A}(W)|X_i)) \\
&= \mathbb{E}(X_i g_{\lambda,A}(W)|X_i = 0)P(X_i = 0) \\
&\quad + \mathbb{E}(X_i g_{\lambda,A}(W)|X_i = 1)P(X_i = 1) \\
&= \mathbb{E}(g_{\lambda,A}(W)|X_i = 1)P(X_i = 1) \\
&= p_i \mathbb{E}(g_{\lambda,A}(W_i + 1)). \qquad (3.2.3)
\end{aligned}$$

Then by (3.2.2) and (3.2.3), we have

$$\begin{aligned}
&|P(W \in A) - Poi_\lambda(A)| \\
&= |\mathbb{E}(\lambda g_{\lambda,A}(W+1)) - \mathbb{E}(W g_{\lambda,A}(W))| \\
&= |\lambda \mathbb{E}(g_{\lambda,A}(W+1)) - \mathbb{E}(W g_{\lambda,A}(W))| \\
&= |\sum_{i\in\Gamma} p_i \mathbb{E}(g_{\lambda,A}(W+1)) - \sum_{i\in\Gamma} \mathbb{E}(X_i g_{\lambda,A}(W))| \\
&= |\sum_{i\in\Gamma} p_i \mathbb{E}(g_{\lambda,A}(W+1)) - \sum_{i\in\Gamma} p_i \mathbb{E}(g_{\lambda,A}(W_i+1))| \\
&\le \sum_{i\in\Gamma} p_i \mathbb{E}(|g_{\lambda,A}(W+1) - g_{\lambda,A}(W_i+1)|) \\
&\le \sum_{i\in\Gamma} p_i \mathbb{E}(|\sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)][(W+1) - (W_i+1)]|) \\
&\le \sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)] \sum_{i\in\Gamma} p_i \mathbb{E}(|W - W_i|).
\end{aligned}$$

From the estimates above, we arrive at our fundamental result.

Theorem 3.2.1. *If W and W_i are defined as above, then*

$$|P(W \in A) - Poi_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i\in\Gamma} p_i \mathbb{E}(|W - W_i|) \tag{3.2.4}$$

where $\| g_{\lambda,A} \| := \sup_w [g_{\lambda,A}(w+1) - g_{\lambda,A}(w)]$.

In order to justify the Poisson approximation we therefore have to

1. bound $\| g_{\lambda,A} \|$ and
2. find couplings (W, W_i) such that $\mathbb{E}(|W - W_i|)$ is small.

Many authors would like to determine a bound of $\| g_{\lambda,A} \|$. For $A \subseteq \mathbb{N} \cup \{0\}$, Chen ([5], 1975) proved that

$$\| g_{\lambda,A} \| \leq \min\{1, \lambda^{-1}\}$$

and Janson ([11], 1994) showed that

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}).$$

In case of non-uniform bound, Neammanee ([14], 2003) showed that

$$\| g_{\lambda,\{w_0\}} \| \leq \min\left\{\frac{1}{w_0}, \lambda^{-1}\right\}$$

and Teerapabolarn and Neammanee ([19], 2005) gave bound of $\| g_{\lambda,A} \|$ where $A = \{0, 1, \ldots, w_0\}$ in the terms of

$$\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^\lambda}{w_0 + 1}\right\}.$$

In general case for any subset A of $\{0, 1, \ldots, n\}$, Santiwipanont and Teerapabolarn ([17], 2006) gave a bound in the form of

$$\| g_{\lambda,A} \| \leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \tag{3.2.5}$$

where

$$\Delta(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\ 2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases}$$

The difficult part in apply Theorem 3.2.1 is to construct W_i which makes $\mathbb{E}|W-W_i|$ small. This has not the solution in general. For the case of $X_1, \ldots, X_n$ are independent, we let $W_i = W - X_i$. Then $\mathbb{E}|W-W_i| = p_i$ and from (3.2.4), we have $|P(W \in A) - Poi_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i^2$. The problem of the construction of W_i is difficult in the case of dependent indicator summand.

In next section, we will use Theorem 3.2.1 to prove our main result by constructing the random variable W_i.

3.3 Proof of Theorem 3.1.1

Let $A \subseteq \mathbb{N}$. By (3.2.4), it suffices to bound $\mathbb{E}|W - W_i|$ for $i \in D$ where the distribution of W_i equals the conditional distribution of $W - X_i$ given $X_i = 1$

Let W_i be the number of isolated copies of G in a random graph $\mathbb{G}(n,p) - i$, $\mathbb{G}(n,p) - i$ obtained from $\mathbb{G}(n,p)$ by dropping the vertices in i and all edges incident to these vertices. Then for $w_0 \in \{0, 1, \ldots, \lfloor \frac{n-v_G}{v_G} \rfloor\}$, we have

$$P(W_i = w_0) = \binom{\binom{n-v_G}{v_G}}{w_0} \left[\frac{v_G!}{AutG} p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-2v_G)} \right]^{w_0} \tag{3.3.1}$$

and

$$\begin{aligned} & P(W - X_i = w_0 \mid X_i = 1) \\ = & \frac{P(W - X_i = w_0, X_i = 1)}{P(X_i = 1)} \\ = & \frac{P(W = w_0 + 1, X_i = 1)}{P(X_i = 1)} \\ = & \frac{\binom{\binom{n-v_G}{v_G}}{w_0} \left[\frac{v_G!}{AutG} p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-2v_G)} \right]^{w_0} \frac{v_G!}{AutG} p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-v_G)}}{\frac{v_G!}{AutG} p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-v_G)}} \\ = & \binom{\binom{n-v_G}{v_G}}{w_0} \left[\frac{v_G!}{AutG} p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-2v_G)} \right]^{w_0} . \end{aligned} \tag{3.3.2}$$

From (3.3.1) and (3.3.2), W_i has the distribution of $W - X_i$ conditional on $X_i = 1$.

For $i, j \in D$ such that $i \neq j$, we define the indicator random variable $X_j^{(i)}$ and E_{ij} by

$$X_j^{(i)} = \begin{cases} 1 & \text{if there is isolated copy of } G \text{ in } \mathbb{G}(n,p) - i \text{ that spans the vertices} \\ & j = (j_1, \ldots, j_{vG}), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$E_{ij} = \begin{cases} 1 & \text{if there exists adjacent between } i_k \in i \text{ and } j_l \in j, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that in case $X_i = 1$, that is we have an isolated copy of G in $\mathbb{G}(n,p)$ that span vertices $i = \{i_1, \ldots, i_{vG}\}$. Thus the number of isolated copies of G in a random graph $\mathbb{G}(n,p) - i$ equals the number of isolated copies of G in a random graph $\mathbb{G}(n,p)$ minus 1, i.e.,

$$W_i = W - 1. \tag{3.3.3}$$

In case $X_i = 0$. For $j \in D$ such that $j \cap i \neq \emptyset$ and $j \neq i$ we have $X_j^{(i)} = 0$, then

$$W_i = W + \sum_{\substack{j \in D \\ j \cap i = \emptyset}} E_{ij} X_j^{(i)} \tag{3.3.4}$$

that is the number of isolated copies in $\mathbb{G}(n,p) - i$ equals the sum of the number of isolated copies of G in $\mathbb{G}(n,p)$ and the number of isolated copies of G in $\mathbb{G}(n,p) - i$ which are connected to i.

We know that

$$|W - W_i| = (W - W_i)^+ + (W - W_i)^-$$

where $(W - W_i)^+ = \max\{W - W_i, 0\}$ and $(W - W_i)^- = -\min\{W - W_i, 0\}$. Since $-\min\{W - W_i, 0\} = \max\{W_i - W, 0\} = (W_i - W)^+$, we have

$$\mathbb{E}|W - W_i| = \mathbb{E}(W - W_i)^+ + \mathbb{E}(W_i - W)^+.$$

Form (3.3.3) and (3.3.4), we have

$$\begin{aligned}(W-W_i)^+ &\le X_i\\(W_i-W)^+ &\le \sum_{\substack{j\in D\\ j\cap i=\emptyset}} E_{ij}X_j^{(i)}.\end{aligned}$$

We note that,

$$\begin{aligned}\mathbb{E}(X_i) &= \frac{\mathbb{E}(W)}{\binom{n}{v_G}}\\ &\le \frac{v_G^2}{n}\mathbb{E}(W)\\ &= \frac{v_G^2}{n}\binom{n}{v_G}\frac{v_G!}{Aut(G)}p^{e_G}q^{\binom{v_G}{2}-e_G+v_G(n-v_G)}\\ &\le \frac{v_G^2}{n}\frac{n^{v_G}}{v_G!}\frac{v_G!}{Aut(G)}p^{e_G}q^{nv_G-v_G^2}\\ &\le \frac{v_G^2(np)^{v_G-1}q^{nv_G-v_G^2}}{Aut(G)} \qquad (3.3.5)\end{aligned}$$

and

$$\begin{aligned}\sum_{j\in D, j\cap i=\emptyset}\mathbb{E}[E_{ij}X_j^{(i)}] &= \sum_{j\in D, j\cap i=\emptyset} P(E_{ij}=1, X_j^{(i)}=1)\\ &= \binom{n-v_G}{v_G}(1-q^{v_G^2})\frac{v_G!}{Aut(G)}p^{e_G}q^{\binom{v_G}{2}-e_G+v_G(n-2v_G)}\\ &\le \frac{n^{v_G}}{v_G!}pv_G^2\frac{v_G!}{Aut(G)}p^{e_G}q^{nv_G-2v_G^2}\\ &\le \frac{v_G^2(np)^{v_G}q^{nv_G-2v_G^2}}{Aut(G)} \qquad (3.3.6)\end{aligned}$$

From (3.3.5), (3.3.6) and use the fact that $1-p \leq \frac{1}{e^p}$, we have

$$\begin{aligned}
\mathbb{E}(|W-W_i|) &= \mathbb{E}(W-W_i)^+ + \mathbb{E}(W_i-W)^+ \\
&\leq \mathbb{E}(X_i) + \sum_{j\in D, j\cap i=\emptyset} \mathbb{E}(E_{ij}X_j^{(i)}) \\
&\leq \frac{v_G^2(np)^{v_G-1}q^{nv_G-v_G^2}}{Aut(G)} + \frac{v_G^2(np)^{v_G}q^{nv_G-2v_G^2}}{Aut(G)} \\
&= \frac{v_G^2(np)^{v_G-1}q^{nv_G-2v_G^2}}{Aut(G)}[q^{v_G^2}+np] \\
&\leq \frac{v_G^2(np)^{v_G-1}q^{nv_G-2v_G^2}}{Aut(G)}[1+np] \\
&\leq \frac{v_G^2(np)^{v_G-1}}{Aut(G)q^{2v_G^2}e^{np}}(1+np) \\
&= \frac{v_G^2(np)^{v_G}}{Aut(G)q^{2v_G^2}e^{np}}\left[\frac{1+np}{np}\right] \qquad (3.3.7)
\end{aligned}$$

Hence, by (3.2.4), (3.2.5) and (3.3.7), we have

$$|P(W\in A) - Poi_\lambda(A)| \leq C_{\lambda,A}\frac{(np)^{v_G}}{q^{2v_G^2}e^{np}}\left[\frac{1+np}{np}\right]$$

where $C_{\lambda,A} = \frac{v_G^2}{Aut(G)}\min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

3.4 Proof of Corollary 3.1.2

From (3.3.7), we have

$$\mathbb{E}(|W-W_i|) \leq \frac{v_G^2(np)^{v_G}}{Aut(G)q^{2v_G^2}e^{np}}\left[\frac{1+np}{np}\right].$$

Suppose that $p = \frac{1}{n^\gamma}$ for any $\gamma \in \mathbb{R}^+\backslash\{1\}$.

1. If $\gamma > 1$ then we observe that

$$\begin{aligned}
\mathbb{E}(|W-W_i|) &\leq \frac{v_G^2(np)^{v_G}}{Aut(G)q^{2v_G^2}e^{np}}\left[\frac{1+np}{np}\right] \\
&\leq \frac{v_G^2(np)^{v_G}}{Aut(G)q^{2v_G^2}e^{np}}\left[\frac{2}{np}\right] \\
&= \frac{2v_G^2(np)^{v_G-1}}{Aut(G)q^{2v_G^2}e^{np}} \\
&\leq \frac{2v_G^2(np)^{v_G-1}}{Aut(G)q^{2v_G^2}} \\
&= \frac{2v_G^2}{Aut(G)q^{2v_G^2}}\frac{1}{n^{(\gamma-1)(v_G-1)}} \qquad (3.4.1)
\end{aligned}$$

By (3.2.4), (3.2.5) and (3.4.1), we have

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C(\lambda, A, v_G)}{n^{(\gamma-1)(v_G-1)}},$$

where $C(\lambda, A, v_G) = \dfrac{2v_G^2}{Aut(G)q^{2v_G^2}} \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

2. If $0 < \gamma < 1$ then we observe that

$$\begin{aligned}
\mathbb{E}(|W - W_i|) &\leq \frac{v_G^2 (np)^{v_G}}{Aut(G)q^{2v_G^2} e^{np}} \left[\frac{1+np}{np}\right] \\
&\leq \frac{2v_G^2 (np)^{v_G}}{Aut(G)q^{2v_G^2} e^{np}} \\
&\leq \frac{2v_G^2 (2v_G)!(np)^{v_G}}{Aut(G)q^{2v_G^2} (np)^{2v_G}} \\
&= \frac{2v_G^2 (2v_G)!}{Aut(G)q^{2v_G^2}} \frac{1}{(np)^{(1-\gamma)v_G}}
\end{aligned} \tag{3.4.2}$$

By (3.2.4), (3.2.5) and (3.4.2), we have

$$|P(W \in A) - Poi_\lambda(A)| \leq \frac{C(\lambda, A, v_G)}{n^{(1-\gamma)v_G}}$$

where $C(\lambda, A, v_G) = \dfrac{2v_G^2 (2v_G)!}{Aut(G)q^{2v_G^2}} \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.