CHAPTER II

PRELIMINARIES

In this chapter, we review some basic knowledge in probability theory, graph theory and a model of random graph which will be used in our work. The proofs are omitted but can be found in [6], [13] and [16].

2.1 Probability Space and Random Variables

Definition 2.1.1. A measure space $(\Omega, \mathcal{F}, P)$ is called **probability space** if $P(\Omega) = 1$.

For probability space $(\Omega, \mathcal{F}, P)$, the measure P is called a **probability measure**. The set Ω will be referred as a **sample space** and its elements are called **points** or **elementary events**. The elements of $\mathcal{F}$ are called **events**. For any event A, the value $P(A)$ is called the **probability of A**.

Definition 2.1.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if for every Borel set B in $\mathbb{R}$, $X^{-1}(B)$ belongs to $\mathcal{F}$.

We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega \mid X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Definition 2.1.3. Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

A random variable X with the distribution function F is said to be a **discrete random variable** if the image of X is countable and it is called a

continuous random variable if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$, and f is called **probability function** or **density function** of X.

Now we will give some examples of random variables.

A random variable X, taking on one of the values $0, 1, 2, \ldots$ is said to be a **Poisson** random variable with parameter $\lambda, \lambda > 0$, written as $X \sim Poi_\lambda$, if

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots.$$

We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} exp(-\frac{1}{2\sigma^2}(x-\mu)^2).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

2.2 Independence

Definition 2.2.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_\alpha$ be a sub σ-algebra of $\mathcal{F}$ for every $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha \mid \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J = \{j_1, j_2, \ldots, j_k\}$ of Λ,

$$P(\bigcap_{m=1}^{k} A_m) = \prod_{m=1}^{k} P(A_m)$$

where $A_m \in \mathcal{F}_{j_m}$ for $m = 1, 2, \ldots, k$.

Definition 2.2.2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\varepsilon_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\varepsilon_\alpha \mid \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\varepsilon_\alpha) \mid \alpha \in \Lambda\}$ is independent where $\sigma(\varepsilon)$ is the smallest σ-algebra with $\varepsilon_\alpha \subseteq \sigma(\varepsilon_\alpha)$.

Definition 2.2.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X_α is a random variable on $(\Omega, \mathcal{F}, P)$ for every $\alpha \in \Lambda$. We say that the set of random variables

$\{X_\alpha \mid \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha) \mid \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B$ is a Borel subset of $\mathbb{R}\}$.

Theorem 2.2.4. *Random variables $X_1, X_2, \ldots, X_n$ are* **independent** *if for any Borel sets $B_1, B_2, \ldots, B_n$, we have*

$$P\left(\bigcap_{i=1}^{n}\{X_i \in B_i\}\right) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2.5. *If $X_{ij}; i = 1, 2, \ldots, n, j = 1, 2, \ldots, m_i$ are independent and $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ are measurable, then $\{f_i(X_{i1}, X_{i2}, \ldots, X_{im_i}),\ i = 1, 2, \ldots, n\}$ is independent.*

2.3 Expectation, Variance and Conditional Expectation

Definition 2.3.1. Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathbb{E}(X) = \displaystyle\int_\Omega X dP$. If $\mathbb{E}(|X|) < \infty$, then $\mathbb{E}(X)$ is called **expected value** of X.

Proposition 2.3.2.

1. *If X is a discrete random variable, then $\mathbb{E}(X) = \displaystyle\sum_{x \in ImX} xP(X = x)$.*

2. *If X is a continuous random variable with probability function f, then*

$$\mathbb{E}(X) = \int_{\mathbb{R}} xf(x)dx.$$

Proposition 2.3.3. *Let X and Y be random variables such that $\mathbb{E}(|X|) < \infty$ and $\mathbb{E}(|Y|) < \infty$ and $a, b \in \mathbb{R}$. Then we have the followings :*

1. *$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$.*

2. *If $X \leq Y$, then $\mathbb{E}(X) \leq \mathbb{E}(Y)$.*

3. *$|\mathbb{E}(X)| \leq \mathbb{E}(|X|)$.*

Let X be a random variable which $\mathbb{E}(|X|^k) < \infty$. Then $\mathbb{E}(|X|^k)$ is called the k-**th moment** of X about the origin and $\mathbb{E}[(X-\mathbb{E}(X))^k]$ or $\mathbb{E}[X-\mathbb{E}(X)]^k$ the k-**th moment** about the mean.

We call the second moment of X about the mean, the **variance** of X, denoted by $Var(X)$. Then

$$Var(X) = \mathbb{E}[X - \mathbb{E}(X)]^2.$$

We note that

1. $Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X)$.

2. If $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$ and $Var(X) = \sigma^2$.

3. If $X \sim Poi_\lambda$, then $\mathbb{E}(X) = \lambda$ and $Var(X) = \lambda$.

Proposition 2.3.4. *If $X_1, X_2, \ldots, X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, \ldots, n$, then*

1. $\mathbb{E}(X_1 X_2 \cdots X_n) = \mathbb{E}(X_1)\mathbb{E}(X_2)\cdots\mathbb{E}(X_n)$,

2. $Var(a_1X_1 + a_2X_2 + \cdots + a_nX_n) = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \cdots + a_n^2 Var(X_n)$ *for any real numbers $a_1, a_2, \ldots, a_n$.*

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}|X| < \infty$ and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(E) = P(E)$$

and a sign-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP.$$

Then, by Radon-Nikodym theorem we have $\mathcal{Q}_X \ll P_{\mathcal{D}}$ and there exists a unique measurable function $\mathbb{E}(X|\mathcal{D})$ on $(\Omega, \mathcal{F}, P_{\mathcal{D}})$ such that

$$\int_E \mathbb{E}(X|\mathcal{D}) dP_{\mathcal{D}} = \mathcal{Q}_X(E) = \int_E X dP \quad \text{for any } \; E \in \mathcal{D}.$$

We call $\mathbb{E}(X|\mathcal{D})$ the **conditional expectation** of X with respect to $\mathcal{D}$.

Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}(|X|) < \infty$, we will denote $\mathbb{E}(X|\sigma(Y))$ by $\mathbb{E}(X|Y)$.

Theorem 2.3.5. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}(|X|) < \infty$, then the following hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.*

1. *If X is a random variable on $(\Omega, \mathcal{D}, P_{\mathcal{D}})$, then $\mathbb{E}(X|\mathcal{D}) = X$ a.s.$[P_{\mathcal{D}}]$.*

2. *$\mathbb{E}(X|\mathcal{F}) = X$ a.s.$[P]$.*

3. *If $\sigma(X)$ and $\mathcal{D}$ are independent, then $\mathbb{E}(X|\mathcal{D}) = \mathbb{E}(X)$ a.s.$[P_{\mathcal{D}}]$.*

2.4 Graph Theory

Definition 2.4.1. A **graph** G is an ordered pair $G(V(G), E(G))$ consists of a **vertex set** $V(G)$ and an **edge set** $E(G)$. If $\{u, v\}$ is an edge e, for some vertices u and v in G then u and v are said to be **adjacent** or edge e is said to be **incident** u and v.

Definition 2.4.2. A graph H is a **subgraph** of a graph G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq V(G)$.

Definition 2.4.3. An **isomorphism** from G to H is a bijection $f : V(G) \to V(H)$ such that $\{u, v\}$ is an edge in G if and only if $\{f(u), f(v)\}$ is an edge in H. G is isomorphic to H, denoted by $G \cong H$, if there is an **isomorphism** from G to H.

Definition 2.4.4. A **walk** in a graph is an alternating sequence, $v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n$. The walk which begins with v_0 and ends with v_n is referred to as a $v_0 v_n$-walk, and a walk in which no vertex is repeated is called a **path**.

Definition 2.4.5. A graph G is called **connected** if for any two given vertices $\{v_i, v_j\}$, there is a path from v_i to v_j, and G is disconnected if G is not connected.

Definition 2.4.6. Let G be a graph. A subgraph H' of graph H is called a **copy** of G in H if H' is isomorphic to G.

Definition 2.4.7. Let G be a connected graph. A subgraph H' of a graph H is called an **isolated copy** of G in H if H' is a connected component of H which are isomorphic to G .

2.5 Models of Random Graphs

The notion of a random graph originated in a paper of Erdős ([7], [8], [9]), which is considered by some as the first conscious application of the probabilistic method. It was used there to prove the existence of a graph with a specific Ramsey property.

The model introduced by Erdős is very natural and can be described as choosing a graph at random, with equal probabilities, from the set of all $2^{\binom{n}{2}}$ graphs whose vertex set is $\{1, 2, \dots, n\}$. In other words, it can be described as the probability space $(\Omega, \mathcal{F}, P)$, where Ω is the set of all graphs with vertex set $\{1, 2, \dots, n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $\omega \in \Omega$

$$P(\omega) = 2^{-\binom{n}{2}}.$$

Generally speaking, a random graph is a graph constructed by a random procedure. In accordance with standard definitions in probability theory, this is formalized by representing the "random procedure" by a probability space $(\Omega, \mathcal{F}, P)$ and the "construction" by a function from the probability space into a suitable family of graphs. The *distribution* of a random graph is the induced probability distribution of the family of graphs: for many purposes this is the only relevant feature of the construction and we usually do not distinguish between different random graphs with the same distribution. Indeed, it is often convenient to define a random graph by specifying its distribution.

The word "model" is used rather loosely in theory of random graphs. It may refer to a specific class of random graphs, defined as above, or perhaps to a specific distribution. Nowadays, among several models of random graphs, there are two basic ones, the binomial model and the uniform model, both originating in the simple model introduced by Erdős (1947).

Given a real number p, $0 \le p \le 1$, the *binomial random graph*, denoted by $\mathbb{G}(n, p)$, is defined by taking Ω as the set of all graphs on vertex set $\{1, 2, \dots, n\}$ and setting

$$P(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|},$$

where $|E(G)|$ stands for the number of edges of a graph G. For $p = \dfrac{1}{2}$ this is the model of 1947. However, most of the random graph literature is devoted to cases in which $p = p(n)$ as $n \to \infty$.

Given an integer $M, 0 \leq M \leq \binom{n}{2}$, the *uniform random graph*, denoted by $\mathbb{G}(n, M)$, is defined by taking Ω as the family of all graphs on the vertex set $\{1, 2, \ldots, n\}$ with exactly M edges, and the uniform probability on Ω,

$$P(G) = \binom{\binom{n}{2}}{M}^{-1}, \qquad G \in \Omega.$$

In this work, we are interested in one of two models which is the *binomial random graph*, is shorthand for a random graph.