CHAPTER I

INTRODUCTION

A random graph is a graph generated by some random procedure. The theory of random graph originated in a series of paper published in the period 1959-1968 by two outstanding Hungarian mathematicians, Pual Erdős and Alfred Rényi ([7], [8], [9]). A random graph developed into an independent and fast-growing branch of discrete mathematics and has produced a prodigious number of results, many of which are highly ingenious, describing some of the statistical properties of graph such as the distribution of component sizes of a giant component and typical vertex-vertex distances.

In 1972, Stein [18] introduced a new powerful technique for obtaining the rate of convergence to the standard normal distribution. His approach was subsequently extended to cover a convergence to Poisson distribution by Chen [5]. Both methods were illustrated, in the context of random graph theory, in Barbour [1]. The method for proving Poisson convergence has since been widely taken up, but results for random graphs subsequently obtained by the method for normal convergence seem to be limited to example in Barbour and Eagleson ([2], [3]).

Stein-Chen method has been developed for approximating the distribution of a sum of random indicator. In contrast to many asymptotic methods, this approximation carries with it explicit error bounds. Let X_α be a random indicator with the probability $P(X_\alpha = 1) = 1 - P(X_\alpha = 0) = p_\alpha$, where α ranges over some finite index set Γ, and let $W = \sum_{\alpha\in\Gamma} X_\alpha$ and $\lambda = \sum_{\alpha\in\Gamma} p_\alpha$. If $\Gamma = \{1, \ldots, n\}$ and X_α's are independent, then W has the Poisson binomial distribution, and in case where p_α's are identical to p, W has the binomial distribution with parameter n and p. It is well known that the Poisson distribution is a good model for counting the number of occurrences of rare, or exceptional, events in an experiment with many trials. That is, if the probabilities p_α's are small, then the distribution of W is approximately Poisson with parameter

$\lambda = \mathbb{E}(W) = \sum\limits_{\alpha\in\Gamma} p_\alpha$. Many authors used the Stein-Chen method to investigate bounds for approximating the distribution of W.

In this thesis, we will approximate the distribution functions of the number of isolated copies of a fixed connected graph in a random graph by Poisson distribution functions.

Let $\mathbb{G}(n,p)$ be a random graph on n labeled vertices $\{1,2,\ldots,n\}$ where possible edge $\{i,j\}$ is present randomly and independently with the probability p, $0 < p < n$.

In 1981, Karoński and Ruciński [12] used the method of moments to prove that the distribution of W, the number of isolated copies of a fixed connected graph in $\mathbb{G}(n,p)$, can be approximated by Poisson distribution with parameter

$$\lambda := \mathbb{E}(W) = \binom{n}{v_G} P(X_i = 1) = \binom{n}{v_G} \frac{v_G!}{Aut(G)} p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-v_G)}$$

where $q = 1-p$, G is a fixed connected graph, e_G is the number of edges in G and v_G is the number of vertices in G, and $Aut(G)$ stands for the number of automorphisms of G. In 1992, Babour, Holst and Janson [4] gave the uniform bound is given by the following theorem.

Theorem 1.1.1. *Let W be the number of isolated copies of G in $\mathbb{G}(n,p)$. Then*

$$d_{TV}(\mathcal{L}(W), Poi_\lambda) \le (1 - e^{-\lambda}) \left(\frac{VarW}{\lambda} - 1 + 2\pi_i |G_i| \right)$$

where $d_{TV}(\mathcal{L}(W), Poi_\lambda) := \sup\{|P(W \in A) - Poi_\lambda(A)| : A \subseteq \mathbb{N}\}$, Poi_λ is a Poisson distribution with parameter λ, $\pi_i = p^{e_G} q^{\binom{v_G}{2} - e_G + v_G(n-v_G)}$, $G_i = \{j \in D \mid j \cap i \neq \emptyset\}$ and $D = \{i =: \{i_1, i_2, \ldots, i_{v_G}\} \mid 1 \le i_1 < \ldots < i_{v_G} \le n\}$.

In this work, we a give a non-uniform bound of this approximation of the number of isolated copies of a fixed connected graph in $\mathbb{G}(n,p)$ by using Stein-Chen method. The followings are our main results.

Theorem 1.1.2. *Let G be a fixed connected graph consisting of v_G vertices and W be the number of isolated copies of G in random graph $\mathbb{G}(n,p)$. For $A \subseteq \mathbb{N}$ and $n > 2v_G$, we have*

$$|P(W \in A) - Poi_\lambda(A)| \le C_{\lambda,A} \frac{(np)^{v_G}}{q^{2v_G^2} e^{np}} \left[\frac{1 + np}{np} \right]$$

where $C_{\lambda,A} = \frac{v_G^2}{Aut(G)} \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$,

$$\triangle(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \le M_A, \\ 2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A, \end{cases}$$

and

$$M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A, \end{cases}$$

when $C_w = \{0, 1, \ldots, w\}$.

Corollary 1.1.3. *Let* W *be the number of isolated copies of a fixed connected graph* G *consisting of* v_G *vertices in random graph* $\mathbb{G}(n,p)$ *and* $p = \frac{1}{n^\gamma}$ *for some* $\gamma \in \mathbb{R}^+ \backslash \{1\}$. *Then, for* $A \subseteq \mathbb{N}$,

1. *if* $\gamma > 1$ *then*

$$|P(W \in A) - Poi_\lambda(A)| \le \frac{C(\lambda, A, v_G)}{n^{(\gamma-1)v_G}},$$

where $C(\lambda, A, v_G) = \dfrac{2v_G^2}{Aut(G)q^{2v_G^2}} \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

2. *if* $0 < \gamma < 1$ *then*

$$|P(W \in A) - Poi_\lambda(A)| \le \frac{C(\lambda, A, v_G)}{n^{(1-\gamma)v_G}}.$$

where $C(\lambda, A, v_G) = \dfrac{2v_G^2(2v_G)!}{Aut(G)q^{2v_G^2}} \min\left\{1, \lambda, \frac{\triangle(\lambda)}{M_A+1}\right\}$.

This thesis is organized as follows. Preliminaries are given in Chapter II. In Chapter III, we give a bound of Poisson approximation of the number of isolated copies of a fixed connected graph in a random graph. In Chapter IV, we conclude about the results.