Maejo Int. J. Sci. Technol. **2015**, *9*(01), 82-92

Maejo International
Journal of Science and Technology

ISSN 1905-7873
Available online at www.mijst.mju.ac.th

Full Paper

Weak efficiency of non-smooth multiobjective programming via an η-approximation method

Rekha Gupta * and Manjari Srivastava

Department of Mathematics, University of Delhi, Delhi 110007, India

* Corresponding author, e-mail: rekhagupta1983@yahoo.com

Received: 29 January 2014/ Accepted: 18 March 2015 / Published: 19 March 2015

Abstract: A non-smooth vector optimisation problem (VOP) over cones is considered. An η-approximated vector optimisation problem is constructed by modifying the objective and constraints of the VOP at a feasible point. An equivalence between their (weak) efficient solutions is given by assuming the functions involved in the VOP to be a generalised type I. Further, the definitions of Lagrange function L_η and saddle point are introduced for the η-approximated problem and saddle-point results are deduced. At the end, sufficient conditions for the existence of (weak) efficient solutions of the VOP and the saddle point of L_η are derived. Examples are given to support the results.

Keywords: non-smooth vector optimisation, η-approximation method, generalised type I functions, Lagrange function, saddle point

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INTRODUCTION

Multiobjective programming has become an important area of investigation in recent times. This is because of its practical usage in the fields of economics, decision theory, optimal control, game theory and many more. Most of the optimisation problems are actually multiobjective programming problems where the objectives are conflicting. As a result, there is no single solution which optimises all objectives simultaneously. The concept of (weak) efficiency has played a useful role in the analysis of solutions of this type of optimisation problems. Many authors have studied necessary and sufficient optimal conditions of Fritz-John and Karush-Kuhn-Tucker type of (weak) efficient solutions of a multiobjective programming problem ([1-8] and references therein).

Recently, considerable attention has been given for devising new methods of solving a mathematical programming problem with the help of some associated optimisation problems which are in general easier to solve. One such method makes use of a modified objective function introduced by Antczak [9] to solve a differentiable multiobjective programming problem involving functions which are invex. Antczak [10] introduced an η-approximation method of solving a differentiable multiobjective programming problem. This method is an extension of an approach with a modified objective function in the sense that here the η-approximation problem is obtained by a modification of both the objective and constraint functions in the original multiobjective programming problem at an arbitrary but fixed feasible point. Antczak [11] then extended the η-approximation method to the non-smooth case by constructing a family of η-approximated vector optimisation problems (VOPs) in terms of Clarke's generalised gradients of the objective and constraint functions [12]. Further Antczak [13] introduced a vector valued Lagrange function for the same η-approximation problem as considered previously [11] and obtained saddle-point results under invexity assumptions. Recently Suneja et al. [14] studied a modified objective function method for non-smooth VOP over cones and established the equivalence between the original problem and its modified objective function problem. Scalar valued Lagrange function was also introduced and saddle-point results were obtained under cone invex and cone pseudo-invex assumptions on the functions involved in the original problem.

Motivated by the above research work, the present paper develops an η-approximation method for solving a non-smooth VOP over cones. An η-approximated VOP over cones constructed in terms of Clarke's generalised gradients of the objective and constraint functions is associated with the original problem, and the equivalence between (weak) efficient solutions of both problems is established assuming the functions involved in the original problem is a generalised type I. Further, the definitions of Lagrange function and saddle point are given for the η-approximated problem and saddle-point results are deduced under generalised type I assumptions on the functions involved. Finally, existing results for (weak) efficient solutions of the considered problem and saddle points of the Lagrange function for the η-approximated problem under some Karush-Kuhn-Tucker type conditions are obtained. Examples are given to illustrate the results.

PRELIMINARIES AND DEFINITIONS

Let $K \subseteq \mathbb{R}^p$ be a closed convex cone with $\operatorname{int} K \neq \phi$, where $\operatorname{int} K$ denotes the interior of K. The positive dual cone K^* and the strict positive dual cone K^{s^*} of K are defined as follows:

$$K^* = \{y \in \mathbb{R}^p : x^T y \geq 0, \ \forall\, x \in K\}$$

and

$$K^{s^*} = \{y \in \mathbb{R}^p : x^T y > 0, \ \forall\, x \in K \setminus \{0\}\}.$$

Let X be a non-empty open subset of $\mathbb{R}^n$.

Definition 1. A real valued function $\phi : X \to \mathbb{R}$ is said to be locally Lipschitz at a point $\bar{x} \in X$ if there exists a real number $l > 0$ such that

$$|\,\phi(x) - \phi(y)\,| \leq l \,\|\, x - y \,\|$$

for all x, y in a neighbourhood of $\bar{x}$. A function ϕ is said to be locally Lipschitz on X if it is locally Lipschitz at each point of X.

Definition 2 [12]. Let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function; then $\phi^o(\bar{x};v)$ denotes the Clarke's generalised directional derivative of ϕ at $\bar{x} \in X$ in the direction v and is defined as

$$\phi^o(\bar{x};v) = \limsup_{\substack{y \to \bar{x} \\ t \downarrow 0}} \frac{\phi(y + tv) - \phi(y)}{t}.$$

The Clarke's generalised subdifferential of ϕ at $\bar{x} \in X$ is denoted by $\partial\phi(\bar{x})$ and is defined as

$$\partial\phi(\bar{x}) = \{\xi \in \mathbb{R}^n : \phi^o(\bar{x};v) \geq \langle \xi, v \rangle, \ \forall\, v \in \mathbb{R}^n\}.$$

Let $f=(f_1,f_2,\ldots,f_p):X\to\mathbb{R}^p$ be a vector valued function. Then f is said to be locally Lipschitz on X if each f_i is locally Lipschitz on X.

The Clarke's generalised directional derivative of a locally Lipschitz function $f:X\to\mathbb{R}^p$ at $\bar{x}\in X$ in the direction v is given by

$$f^o(\bar{x};v)=(f_1^o(\bar{x};v),f_2^o(\bar{x};v),\ldots,f_p^o(\bar{x};v)).$$

The Clarke's generalised subdifferential of $f:X\to\mathbb{R}^p$ at $\bar{x}\in X$ is the set

$$\partial f(\bar{x})=\partial f_1(\bar{x})\times\partial f_2(\bar{x})\times\ldots\times\partial f_p(\bar{x}),$$

where $\partial f_i(\bar{x})$ is the Clarke's generalised subdifferential of f_i at $\bar{x}$.

Every element $A=(A_1,A_2,\ldots,A_p)\in\partial f(\bar{x})$ is a continuous linear operator from $\mathbb{R}^n$ to $\mathbb{R}^p$. Let *u* be a vector in Euclidean space ¡ ^{n}, then define

$$Au=(\langle A_1,u\rangle,\langle A_2,u\rangle,\ldots,\langle A_p,u\rangle)^T.$$

Definition 3 [14]**.** A locally Lipschitz function $f:X\to\mathbb{R}^p$ is said to be K-generalised invex with respect to η at $\bar{x}\in X$ on X, if there exists $\eta:X\times X\to\mathbb{R}^n$ such that for every $x\in X$ and $\xi\in\partial f(\bar{x})$,

$$f(x)-f(\bar{x})-\xi\eta(x,\bar{x})\in K.$$

Consider the following non-smooth VOP:

$$K\text{-Minimise } f(x)$$
$$\text{subject to } -g(x)\in Q,$$

where $f:X\to\mathbb{R}^p$, $g:X\to\mathbb{R}^m$ are locally Lipschitz functions on X, and K and Q are closed convex pointed cones with non-empty interiors in $\mathbb{R}^p$ and $\mathbb{R}^m$ respectively. Let $D=\{x\in X:-g(x)\in Q\}$ denote the set of all feasible solutions of VOP.

Definition 4. A point $\bar{x}\in D$ is said to be a weak efficient solution of VOP if there exists no $x\in D$ such that

$$f(x)-f(\bar{x})\in-\operatorname{int}K.$$

Definition 5. A point $\bar{x}\in D$ is said to be an efficient solution of VOP if there exists no $x\in D$ such that

$$f(x)-f(\bar{x})\in-K\setminus\{0\}.$$

Now on the lines of Suneja et al. [6], we have the following definition:

Definition 6. (f,g) is said to be $(K\times Q)$ generalised type I with respect to η at a point $\bar{x}\in X$ if there exists $\eta:D\times X\to\mathbb{R}^n$ such that for every $x\in D$, $\xi\in\partial f(\bar{x})$ and $\zeta\in\partial g(\bar{x})$,

$$f(x)-f(\bar{x})-\xi\eta(x,\bar{x})\in K$$

and

$$-g(\bar{x})-\zeta\eta(x,\bar{x})\in Q.$$

Taking $X=\mathbb{R}^n$, Suneja et al. [15] proved the following result giving Karush-Kuhn-Tucker necessary optimality conditions for VOP.

Lemma 1. *Let f be K-generalised invex and g be Q-generalised invex with respect to the same $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ at $\bar{x}\in D$ on $\mathbb{R}^n$. Suppose that the generalised Slater constraint qualification is satisfied; that is, there exists $x^*\in D$ such that $g(x^*)\in-\operatorname{int}Q$. If $\bar{x}$ is a weak efficient solution of VOP, then there exist $\bar{\lambda}\in K^*\setminus\{0\}$ and $\bar{\mu}\in Q^*$ such that*

$$0\in\partial f(\bar{x})\bar{\lambda}+\partial g(\bar{x})\bar{\mu},\tag{1}$$

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$$\bar{\mu}^T g(\bar{x}) = 0. \quad (2)$$

EQUIVALENT VOP AND OPTIMALITY CONDITIONS

Let $\bar{x} \in D$ and $\bar{\xi}, \bar{\zeta}$ be Clarke's generalised gradients of the objective function f and constraint function g in VOP at $\bar{x}$ respectively. We consider an η-approximated VOP given by

$$\text{VOP}_\eta(\bar{x}, \bar{\xi}, \bar{\zeta}) \qquad K\text{-Minimise } \bar{\xi}\eta(x, \bar{x})$$
$$\text{subject to } -(g(\bar{x}) + \bar{\zeta}\eta(x, \bar{x})) \in Q,$$

where f, g, X are as defined in VOP, and $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_p)$, $\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_m)$, $\bar{\xi}_i, i = 1, \ldots, p$, $\bar{\zeta}_j, j = 1, \ldots, m$ are Clarke's generalised gradients of $f_i, i = 1, \ldots, p$ and $g_j, j = 1, \ldots, m$ respectively at $\bar{x}$; that is, $\bar{\xi}_i \in \partial f_i(\bar{x}), i = 1, \ldots, p$, $\bar{\zeta}_j \in \partial g_j(\bar{x}), j = 1, \ldots, m$, and $\eta : X \times X \to \mathbb{R}^n$ is a vector valued function. Let $D(\bar{x}, \bar{\zeta}) = \{x \in X : -g(\bar{x}) - \bar{\zeta}\eta(x, \bar{x}) \in Q\}$ denote a feasible set of $\text{VOP}_\eta(\bar{x}, \bar{\xi}, \bar{\zeta})$.

Theorem 1. *Let $X = \mathbb{R}^n$, f be K-generalised invex and g be Q-generalised invex with respect to the same $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ at $\bar{x} \in D$ on $\mathbb{R}^n$ with $\eta(\bar{x}, \bar{x}) = 0$. Also assume that the generalised Slater constraint qualification is satisfied. If $\bar{x}$ is a weak efficient solution of VOP, then $\bar{x}$ is also a weak efficient solution of $VOP_\eta(\bar{x}, \bar{\xi}, \bar{\zeta})$, where $\bar{\xi} \in \partial f(\bar{x})$ and $\bar{\zeta} \in \partial g(\bar{x})$ are Clarke's generalised gradients of f and g at $\bar{x}$ respectively, satisfying the Karush-Kuhn-Tucker conditions* (1) *and* (2) at $\bar{x}$ *with Lagrange multipliers $\bar{\lambda}$ and $\bar{\mu}$.*

Proof. Suppose that $\bar{x}$ is not a weak efficient solution of $\text{VOP}_\eta(\bar{x}, \bar{\xi}, \bar{\zeta})$. Then there exists some $\hat{x} \in D(\bar{x}, \bar{\zeta})$ such that

$$\bar{\xi}\eta(\hat{x}, \bar{x}) - \bar{\xi}\eta(\bar{x}, \bar{x}) \in -\text{int}\, K.$$

Since $\eta(\bar{x}, \bar{x}) = 0$, we get

$$\bar{\xi}\eta(\hat{x}, \bar{x}) \in -\text{int}\, K.$$

As $\bar{\lambda} \in K^* \setminus \{0\}$, we obtain

$$\bar{\lambda}^T \bar{\xi}\eta(\hat{x}, \bar{x}) < 0. \quad (3)$$

Further $\hat{x} \in D(\bar{x}, \bar{\zeta})$ implies

$$-g(\bar{x}) - \bar{\zeta}\eta(\hat{x}, \bar{x}) \in Q.$$

As $\bar{\mu} \in Q^*$, we get

$$\bar{\mu}^T(g(\bar{x}) + \bar{\zeta}\eta(\hat{x}, \bar{x})) \leq 0.$$

Using (2), the above inequality gives

$$\bar{\mu}^T \bar{\zeta}\eta(\hat{x}, \bar{x}) \leq 0. \quad (4)$$

Adding (3) and (4), we get

$$(\bar{\lambda}^T \bar{\xi}^T + \bar{\mu}^T \bar{\zeta}^T)\eta(\hat{x}, \bar{x}) < 0.$$

This contradicts (1). Hence $\bar{x}$ is a weak efficient solution of $\text{VOP}_\eta(\bar{x}, \bar{\xi}, \bar{\zeta})$.

Theorem 2. *Let (f, g) be $(K \times Q)$ generalised type I at $\bar{x}$ with respect to η with $\eta(\bar{x}, \bar{x}) = 0$. If $\bar{x}$ is a weak efficient solution of $VOP_\eta(\bar{x}, \bar{\xi}, \bar{\zeta})$, then $\bar{x}$ is a weak efficient solution of VOP.*

Proof. Suppose, on the contrary, that $\bar{x}$ is not a weak efficient solution of VOP. Then there exists some $\hat{x} \in D$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int}\, K$$

$$\Rightarrow \quad f(\bar{x}) - f(\hat{x}) \in \text{int}\, K. \quad (5)$$

Since (f, g) is $(K \times Q)$ generalised type I at $\bar{x}$; therefore, we get

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$$f(\hat{x}) - f(\bar{x}) - \bar{\xi}\eta(\hat{x},\bar{x}) \in K \tag{6}$$

and
$$-g(\bar{x}) - \bar{\zeta}\eta(\hat{x},\bar{x}) \in Q, \tag{7}$$

as $\bar{\xi} \in \partial f(\bar{x})$ and $\bar{\zeta} \in \partial g(\bar{x})$.

Now (7) shows that $\hat{x} \in D(\bar{x},\bar{\zeta})$. From (5) and (6), we get

$$-\bar{\xi}\eta(\hat{x},\bar{x}) \in \operatorname{int} K$$

$$\Rightarrow \quad \bar{\xi}\eta(\hat{x},\bar{x}) \in -\operatorname{int} K.$$

As $\eta(\bar{x},\bar{x}) = 0$, we obtain

$$\bar{\xi}\eta(\hat{x},\bar{x}) - \bar{\xi}\eta(\bar{x},\bar{x}) \in -\operatorname{int} K,$$

which contradicts the fact that $\bar{x}$ is a weak efficient solution of $\text{VOP}_\eta(\bar{x},\bar{\xi},\bar{\zeta})$. Hence the desired result follows.

Example 1. Consider the VOP:

$$K\text{-Minimise } f(x)$$
$$\text{subject to } -g(x) \in Q,$$

where $f: X \to \mathbb{R}^2$ $g: X \to \mathbb{R}^2$, $X =]-1,1[$, $f(x) = (f_1(x), f_2(x))$, $g(x) = (g_1(x), g_2(x))$, $K = \{(x,y):\ y \le x, y \le 0\}$ and $Q = \{(x,y):\ y \le -x, x \le 0\}$.

Define $\eta : D \times X \to \mathbb{R}$ as $\eta(x,\bar{x}) = x^2 - \bar{x}^2$. Let

$$f_1(x) = \begin{cases} x^2, & x \ge 0 \\ -x^2 - x, & x < 0 \end{cases}, \qquad f_2(x) = \begin{cases} -x^3, & x \ge 0 \\ x, & x < 0 \end{cases}$$

and

$$g_1(x) = \begin{cases} x^4 + x^2, & x \ge 0 \\ 5x, & x < 0 \end{cases}, \qquad g_2(x) = \begin{cases} x^2, & x \ge 0 \\ 2x, & x < 0. \end{cases}$$

Here $-g(x) \in Q \Rightarrow 0 \le x < 1$. Hence feasible set $D = \{x \in \mathbb{R} : 0 \le x < 1\}$.

Let $\bar{x} = 0 \in D$. Then $\partial f_1(0) = [-1,0]$, $\partial f_2(0) = [0,1]$, $\partial g_1(0) = [0,5]$ and $\partial g_2(0) = [0,2]$. Now (f,g) is $(K \times Q)$ generalised type I at $\bar{x} = 0$ because for every $x \in D$, $\xi \in \partial f(\bar{x})$ and $\zeta \in \partial g(\bar{x})$, we have

$$f(x) - f(0) - \xi\eta(x,0) \in K$$

and

$$-g(0) - \zeta\eta(x,0) \in Q.$$

Also, $\eta(\bar{x},\bar{x}) = 0$.

Now we construct a modified $\text{VOP}_\eta(\bar{x},\bar{\xi},\bar{\zeta})$ for $\bar{\xi} = (\bar{\xi}_1,\bar{\xi}_2) = (-\frac{1}{2},0)$ and $\bar{\zeta} = (\bar{\zeta}_1,\bar{\zeta}_2) = (\frac{3}{2},1)$ as follows:

$$K\text{-Minimise } \bar{\xi}\eta(x,\bar{x}) = (-\tfrac{1}{2}x^2, 0)^T$$
$$\text{subject to } -(\tfrac{3}{2}x^2, x^2)^T \in Q.$$

Here $(-\frac{3}{2}x^2, -x^2)^T \in Q$ for every $x \in X$. Therefore, feasible set of modified problem is X; that is, $D(\bar{x},\bar{\zeta}) = X$. Since

$$\bar{\xi}\eta(x,\bar{x}) - \bar{\xi}\eta(\bar{x},\bar{x}) = \bar{\xi}\eta(x,\bar{x}) = (-\tfrac{1}{2}x^2, 0)^T \notin -\operatorname{int} K$$

for any $x \in D(\bar{x},\bar{\zeta})$, $\bar{x} = 0$ is therefore a weak efficient solution of $\text{VOP}_\eta(\bar{x},\bar{\xi},\bar{\zeta})$. Hence by Theorem 2, $\bar{x} = 0$ is a weak efficient solution of VOP.

Theorem 3. *Let (f,g) be $(K \times Q)$ generalised type I at $\bar{x}$ with respect to η, with $\eta(\bar{x},\bar{x}) = 0$. If $\bar{x}$ is an efficient solution of $VOP_\eta(\bar{x},\bar{\xi},\bar{\zeta})$, then $\bar{x}$ is an efficient solution of VOP.*

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Proof. The proof follows on the lines of Theorem 2.

SADDLE-POINT CRITERIA

In this section we use the η-approximation method to obtain saddle-point criteria for a class of non-smooth VOPs. First, we define the Lagrange function L_η for $\text{VOP}_\eta(\bar{x},\bar{\xi},\bar{\zeta})$ associated with the original problem VOP as follows:

$$L_\eta(x,\lambda,\mu,\bar{\xi},\bar{\zeta}) = \lambda^T\bar{\xi}\eta(x,\bar{x}) + \mu^T(g(\bar{x}) + \bar{\zeta}\eta(x,\bar{x})),$$

for all $x \in D, \lambda \in K^*, \mu \in Q^*$.

Now for the above Lagrange function, we give the definition of saddle point as follows.

Definition 7. A point $(\bar{x},\bar{\lambda},\bar{\mu}) \in D \times K^* \setminus \{0\} \times Q^*$ is said to be a saddle point of the Lagrange function L_η if for any $x \in D, \lambda \in K^*, \mu \in Q^*$,

$$L_\eta(\bar{x},\lambda,\mu,\bar{\xi},\bar{\zeta}) \le L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) \le L_\eta(x,\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}).$$

Theorem 4. *Let (f,g) be $(K \times Q)$ generalised type I with respect to η at $\bar{x} \in D$ and $\eta(\bar{x},\bar{x}) = 0$. If $(\bar{x},\bar{\lambda},\bar{\mu})$ is a saddle point of the Lagrange function L_η of $VOP_\eta(\bar{x},\bar{\xi},\bar{\zeta})$, then $\bar{x}$ is a weak efficient solution of VOP.*

Proof. Let $(\bar{x},\bar{\lambda},\bar{\mu})$ be a saddle point of L_η. Then

$$L_\eta(\bar{x},\lambda,\mu,\bar{\xi},\bar{\zeta}) \le L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}), \;\; \forall\; \lambda \in K^*,\; \mu \in Q^* \tag{8}$$

and

$$L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) \le L_\eta(x,\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}), \;\; \forall\; x \in D. \tag{9}$$

From (8) and $\eta(\bar{x},\bar{x}) = 0$, we get that

$$\mu^T g(\bar{x}) \le \bar{\mu}^T g(\bar{x}), \;\; \forall\; \mu \in Q^*.$$

Therefore, in particular for $\mu = 0$, we get $\bar{\mu}^T g(\bar{x}) \ge 0.$ Also, as $\bar{\mu} \in Q^*$ and $\bar{x} \in D$, we have $\bar{\mu}^T g(\bar{x}) \le 0.$ From these two inequalities, it follows that

$$\bar{\mu}^T g(\bar{x}) = 0. \tag{10}$$

Now let us suppose that $\bar{x}$ is not a weak efficient solution of VOP. Then there exists some $\hat{x} \in D$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int}\, K$$

$$\Rightarrow \qquad f(\bar{x}) - f(\hat{x}) \in \text{int}\, K. \tag{11}$$

Since (f,g) is $(K \times Q)$ generalised type I at $\bar{x}$, we get

$$f(\hat{x}) - f(\bar{x}) - \bar{\xi}\eta(\hat{x},\bar{x}) \in K \tag{12}$$

and

$$-g(\bar{x}) - \bar{\zeta}\eta(\hat{x},\bar{x}) \in Q, \tag{13}$$

as $\bar{\xi} \in \partial f(\bar{x})$ and $\bar{\zeta} \in \partial g(\bar{x})$.

From (11) and (12), we get

$$-\bar{\xi}\eta(\hat{x},\bar{x}) \in \text{int}\, K.$$

As $\bar{\lambda} \in K^* \setminus \{0\}$, we obtain

$$\bar{\lambda}^T\bar{\xi}\eta(\hat{x},\bar{x}) < 0. \tag{14}$$

Now from (13) and $\bar{\mu} \in Q^*$, we get

$$\bar{\mu}^T(g(\bar{x}) + \bar{\zeta}\eta(\hat{x},\bar{x})) \le 0. \tag{15}$$

Using (10), (14), (15) and $\eta(\bar{x},\bar{x})) = 0$, we get

$$\begin{aligned} L_\eta(\hat{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) &= \bar{\lambda}^T\bar{\xi}\eta(\hat{x},\bar{x}) + \bar{\mu}^T(g(\bar{x}) + \bar{\zeta}\eta(\hat{x},\bar{x})) \\ &< 0 \\ &= \bar{\lambda}^T\bar{\xi}\eta(\bar{x},\bar{x}) + \bar{\mu}^T(g(\bar{x}) + \bar{\zeta}\eta(\bar{x},\bar{x})) \\ &= L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}), \end{aligned}$$

which contradicts (9). Hence $\bar{x}$ is a weak efficient solution of VOP.

Theorem 5. *Let (f,g) be $(K\times Q)$ generalised type I with respect to η at $\bar{x}\in D$ and $\eta(\bar{x},\bar{x})=0$. If $(\bar{x},\bar{\lambda},\bar{\mu})$ is a saddle point of L_η with $\bar{\lambda}\in K^{s^*}$, then $\bar{x}$ is an efficient solution of VOP.*

Proof. The proof follows on the lines of Theorem 4.

Example 2. Consider the VOP:

$$\begin{aligned} &K\text{-Minimise } f(x) \\ &\text{subject to } -g(x)\in Q, \end{aligned}$$

where $f: X\to\mathbb{R}^2$, $g: X\to\mathbb{R}^2$, $X =]-2,2[$, $f(x)=(f_1(x),f_2(x))$, $g(x)=(g_1(x),g_2(x))$, $K=\{(x,y): x\ge y, x\ge 0\}$ and $Q=\{(x,y): y\le -x, y\le 0\}$.

Define $\eta: D\times X\to\mathbb{R}$ as $\eta(x,\bar{x})=(x-\bar{x})^3$; then $\eta(\bar{x},\bar{x})=0$. Let

$$f_1(x)=\begin{cases} x^3, & x\ge 0 \\ -x^2-x, & x<0 \end{cases}, \quad f_2(x)=\begin{cases} -\frac{x}{6}, & x\ge 0 \\ 0, & x<0 \end{cases}$$

and

$$g_1(x)=\begin{cases} x-x^2, & x\ge 0 \\ -x^3, & x<0 \end{cases}, \quad g_2(x)=\begin{cases} \frac{7}{2}x, & x\ge 0 \\ -x^2, & x<0. \end{cases}$$

Here $-g(x)\in Q \Rightarrow 0\le x<2$. Hence feasible set $D=\{x\in\mathbb{R}: 0\le x<2\}$.
Let $\bar{x}=0\in D$. Then $\partial f_1(0)=[-1,0]$, $\partial f_2(0)=[-\frac{1}{6},0]$, $\partial g_1(0)=[0,1]$ and $\partial g_2(0)=[0,\frac{7}{2}]$.

Now (f,g) is $(K\times Q)$ generalised type I at $\bar{x}=0$ because for every $x\in D$, $\xi\in\partial f(\bar{x})$ and $\zeta\in\partial g(\bar{x})$, we have

$$f(x)-f(0)-\xi\eta(x,0)\in K$$

and

$$-g(0)-\zeta\eta(x,0)\in Q.$$

We construct the Lagrange function L_η for the $\text{VOP}_\eta(\bar{x},\bar{\xi},\bar{\zeta})$ for $\bar{\xi}=(\bar{\xi}_1,\bar{\xi}_2)=(-\frac{1}{10},-\frac{1}{6})$ and $\bar{\zeta}=(\bar{\zeta}_1,\bar{\zeta}_2)=(\frac{1}{2},\frac{1}{12})$, which is given as

$$\begin{aligned} L_\eta(x,\lambda,\mu,\bar{\xi},\bar{\zeta}) &= \lambda^T\bar{\xi}\eta(x,\bar{x}) + \mu^T(g(\bar{x}) + \bar{\zeta}\eta(x,\bar{x})) \\ &= (\lambda_1,\lambda_2)\,(-\tfrac{1}{10}x^3,-\tfrac{1}{6}x^3)^T + (\mu_1,\mu_2)\,(\tfrac{1}{2}x^3,\tfrac{1}{12}x^3)^T. \end{aligned}$$

It is easy to see that for $\bar{\lambda}^T=(\bar{\lambda}_1,\bar{\lambda}_2)=(\frac{1}{8},-\frac{1}{8})\in K^*\setminus\{0\}$ and $\bar{\mu}^T=(\bar{\mu}_1,\bar{\mu}_2)=(0,-\frac{1}{12})\in Q^*$, $(\bar{x},\bar{\lambda},\bar{\mu})$ is a saddle point of L_η. Hence by Theorem 4, $\bar{x}$ is a weak efficient solution of VOP.

Remark 1. If in Example 2, we take $\bar{\lambda}^T=(\bar{\lambda}_1,\bar{\lambda}_2)=(\frac{1}{8},-\frac{1}{9})\in K^{s^*}$, $\bar{\xi}=(\bar{\xi}_1,\bar{\xi}_2)=(-\frac{1}{12},-\frac{1}{6})$ and everything else as in Example 2, then by Theorem 5, it can be proved that $\bar{x}$ is an efficient solution of VOP.

Theorem 6. *Let $X=\mathbb{R}^n$, f be K-generalised invex and g be Q-generalised invex with respect to the same $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ at $\bar{x}\in D$ on $\mathbb{R}^n$ with $\eta(\bar{x},\bar{x})=0$. Also, assume that the generalised Slater constraint qualification is satisfied. If $\bar{x}$ is a weak efficient solution of* VOP, *then $(\bar{x},\bar{\lambda},\bar{\mu})$ is a saddle point of the Lagrange function L_η of* $\text{VOP}_\eta(\bar{x},\bar{\xi},\bar{\zeta})$, *where $\bar{\xi}\in\partial f(\bar{x})$ and*

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$\bar{\zeta} \in \partial g(\bar{x})$ *are Clarke's generalised gradients of* f *and* g *at* $\bar{x}$ *respectively, satisfying the Karush-Kuhn-Tucker conditions* (1) *and* (2) *at* $\bar{x}$ *with Lagrange multipliers* $\bar{\lambda}$ *and* $\bar{\mu}$.

Proof. Since $\bar{x}$ is a weak efficient solution of VOP, therefore from Lemma 1 we have

$$\bar{\xi}\ \bar{\lambda} + \bar{\zeta}\ \bar{\mu} = 0 \tag{16}$$

and

$$\bar{\mu}^T g(\bar{x}) = 0. \tag{17}$$

Now for any $x \in D$, we have by using (16), (17) and $\eta(\bar{x},\bar{x}) = 0$ that

$$\begin{aligned} L_\eta(x,\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) &= \bar{\lambda}^T \bar{\xi}\eta(x,\bar{x}) + \bar{\mu}^T (g(\bar{x}) + \bar{\zeta}\eta(x,\bar{x})) \\ &= (\bar{\lambda}^T \bar{\xi}^T + \bar{\mu}^T \bar{\zeta}^T)\eta(x,\bar{x}) + \bar{\mu}^T g(\bar{x}) \\ &= 0 \\ &= \bar{\lambda}^T \bar{\xi}\eta(\bar{x},\bar{x}) + \bar{\mu}^T (g(\bar{x}) + \bar{\zeta}\eta(\bar{x},\bar{x})) \end{aligned}$$

$$\Rightarrow \quad L_\eta(x,\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) = L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}),\ \forall\ x \in D. \tag{18}$$

Again, as $\bar{x} \in D$, we have for all $\mu \in Q^*$,

$$0 = \bar{\mu}^T g(\bar{x}) \geq \mu^T g(\bar{x}). \tag{19}$$

Now using $\eta(\bar{x},\bar{x}) = 0$ and (19), we have, for any $\lambda \in K^*$ and $\mu \in Q^*$,

$$\begin{aligned} L_\eta(\bar{x},\lambda,\mu,\bar{\xi},\bar{\zeta}) &= \lambda^T \bar{\xi}\eta(\bar{x},\bar{x}) + \mu^T (g(\bar{x}) + \bar{\zeta}\eta(\bar{x},\bar{x})) \\ &\leq \bar{\lambda}^T \bar{\xi}\eta(\bar{x},\bar{x}) + \bar{\mu}^T (g(\bar{x}) + \bar{\zeta}\eta(\bar{x},\bar{x})) \\ &= L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) . \end{aligned} \tag{20}$$

Combining (18) and (20), we get

$$L_\eta(\bar{x},\lambda,\mu,\bar{\xi},\bar{\zeta}) \leq L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) = L_\eta(x,\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}),$$

for all $x \in D$, $\lambda \in K^*$ and $\mu \in Q^*$. Hence $(\bar{x},\bar{\lambda},\bar{\mu})$ is a saddle point of L_η.

EXISTENCE OF WEAK EFFICIENT SOLUTIONS OF VOP AND SADDLE POINT OF L_η

Theorem 7. *Let* $\bar{x} \in D$, *for which there exist* $\bar{\lambda} \in K^* \setminus \{0\}$ *and* $\bar{\mu} \in Q^*$ *such that* (1) *and* (2) *hold and* (f,g) *is* $(K \times Q)$ *generalised type I at* $\bar{x}$ *with respect to* η. *Then* $\bar{x}$ *is a weak efficient solution of* VOP.

Proof. Suppose that $\bar{x}$ is not a weak efficient solution of VOP. Then there exists some $\hat{x} \in D$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\operatorname{int} K$$

$$\Rightarrow \quad f(\bar{x}) - f(\hat{x}) \in \operatorname{int} K. \tag{21}$$

Since (f,g) is $(K \times Q)$ generalised type I at $\bar{x}$, we have for all $\xi \in \partial f(\bar{x})$ and $\zeta \in \partial g(\bar{x})$ that

$$f(\hat{x}) - f(\bar{x}) - \xi\eta(\hat{x},\bar{x}) \in K \tag{22}$$

and

$$-g(\bar{x}) - \zeta\eta(\hat{x},\bar{x}) \in Q. \tag{23}$$

From (21) and (22), we get

$$-\xi\eta(\hat{x},\bar{x}) \in \operatorname{int} K,\ \forall\ \xi \in \partial f(\bar{x}).$$

As $\bar{\lambda} \in K^* \setminus \{0\}$, we obtain

$$\bar{\lambda}^T \xi\eta(\hat{x},\bar{x}) < 0,\ \forall\ \xi \in \partial f(\bar{x}) . \tag{24}$$

Using (23) and $\bar{\mu} \in Q^*$, we get

$$\bar{\mu}^T (g(\bar{x}) + \zeta\eta(\hat{x},\bar{x})) \leq 0,\ \forall\ \zeta \in \partial g(\bar{x}).$$

As (2) holds, the above inequality becomes

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$$\bar{\mu}^T \zeta \eta(\hat{x},\bar{x}) \le 0, \ \forall \ \zeta \in \partial g(\bar{x}). \tag{25}$$

Adding (24) and (25), we get

$$(\bar{\lambda}^T \xi^T + \bar{\mu}^T \zeta^T) \ \eta(\hat{x},\bar{x}) < 0, \ \ \forall \ \xi \in \partial f(\bar{x}), \ \zeta \in \partial g(\bar{x}) \ ,$$

which contradicts (1). Hence $\bar{x}$ is a weak efficient solution of VOP.

Theorem 8. *Let $\bar{x} \in D$, for which there exist $\bar{\lambda} \in K^{s^*}$ and $\bar{\mu} \in Q^*$ such that* (1) *and* (2) *hold and* (f,g) *is* $(K \times Q)$ *generalised type I at $\bar{x}$ with respect to η. Then $\bar{x}$ is an efficient solution of* VOP.

Proof. The proof follows on the lines of Theorem 7.

Theorem 9. *Let $\bar{x} \in D$ at which Karush-Kuhn-Tucker conditions* (1) *and* (2) *are satisfied with Lagrange multipliers $\bar{\lambda} \in K^* \setminus \{0\}$* and *$\bar{\mu} \in Q^*$. Then $(\bar{x},\bar{\lambda},\bar{\mu})$ is a saddle point of the Lagrange function L_η of* $\mathrm{VOP}_\eta(\bar{x},\bar{\xi},\bar{\zeta})$, *where $\eta(\bar{x},\bar{x}) = 0$ and $\bar{\xi} \in \partial f(\bar{x})$, $\bar{\zeta} \in \partial g(\bar{x})$ are Clarke's generalised gradients of f and g at $\bar{x}$ respectively, satisfying conditions* (1) *and* (2) *at $\bar{x}$ with Lagrange multipliers $\bar{\lambda}$ and $\bar{\mu}$.*

Proof. If $(\bar{x},\bar{\lambda},\bar{\mu})$ is not a saddle point of L_η, then at least one of the following two statements holds:

(a) There exists $(\hat{\lambda},\hat{\mu}) \in K^* \times Q^*$ such that

$$L_\eta(\bar{x},\hat{\lambda},\hat{\mu},\bar{\xi},\bar{\zeta}) > L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) \ .$$

(b) There exists $\hat{x} \in D$ such that

$$L_\eta(\bar{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) > L_\eta(\hat{x},\bar{\lambda},\bar{\mu},\bar{\xi},\bar{\zeta}) \ .$$

If (a) holds, then using the definition of L_η and $\eta(\bar{x},\bar{x}) = 0$, we get

$$\hat{\mu}^T g(\bar{x}) > \bar{\mu}^T g(\bar{x}).$$

Since $\bar{x} \in D$ and $\hat{\mu} \in Q^*$, therefore from (2) we have

$$0 \ge \hat{\mu}^T g(\bar{x}) > \bar{\mu}^T g(\bar{x}) = 0,$$

which is a contradiction. If (b) holds, then from $\eta(\bar{x},\bar{x}) = 0$ we get

$$\bar{\mu}^T g(\bar{x}) > \bar{\lambda}^T \bar{\xi}(\hat{x},\bar{x}) + \bar{\mu}^T (g(\bar{x}) + \bar{\zeta}\ \eta(\hat{x},\bar{x}))$$

$$\Rightarrow \qquad (\bar{\lambda}^T \bar{\xi}^T + \bar{\mu}^T \bar{\zeta}^T)\, \eta(\hat{x},\bar{x}) < 0,$$

which contradicts (1). Hence $(\bar{x},\bar{\lambda},\bar{\mu})$ is a saddle point of L_η.

Example 3. Consider the VOP:

$$K\text{-Minimise } f(x)$$
$$\text{subject to } -g(x) \in Q,$$

where $f: X \to \mathbb{R}^2$, $g: X \to \mathbb{R}^2$, $X =]-2,2[$, $f(x) = (f_1(x), f_2(x))$, $g(x) = (g_1(x), g_2(x))$, $K = \{(x,y): y \le x, x \ge 0\}$ and $Q = \{(x,y): y \le -x, y \ge 0\}$.

Define $\eta: D \times X \to \mathbb{R}$ as $\eta(x,\bar{x}) = (x-\bar{x})^2$; then $\eta(\bar{x},\bar{x}) = 0$. Let

$$f_1(x) = \begin{cases} -x^2, & x < 1 \\ -x, & x \ge 1 \end{cases}, \qquad f_2(x) = \begin{cases} -x^3, & x < 1 \\ -x, & x \ge 1 \end{cases}$$

and

$$g_1(x) = \begin{cases} x, & x < 1 \\ 1, & x \ge 1 \end{cases}, \qquad g_2(x) = \begin{cases} 0, & x < 1 \\ -x+1, & x \ge 1. \end{cases}$$

Now $-g(x) \in Q \Rightarrow 0 \le x < 2$. Hence $D = \{x \in \mathbb{R}: 0 \le x < 2\}$.

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Let us take $\bar{x} = 1 \in D$; then $\partial f_1(1) = [-2,-1]$, $\partial f_2(1) = [-3,-1]$, $\partial g_1(1) = [0,1]$ and $\partial g_2(1) = [-1,0]$

Now it is easy to see that for $\bar{\lambda}^T = (\bar{\lambda}_1, \bar{\lambda}_2) = (1, -\frac{2}{3}) \in K^* \setminus \{0\}$, $\bar{\mu}^T = (\bar{\mu}_1, \bar{\mu}_2) = (0, \frac{1}{2}) \in Q^*$, $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2) = (-\frac{7}{4}, -3) \in \partial f(1)$ and $\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2) = (\frac{1}{3}, -\frac{1}{2}) \in \partial g(1)$, we have

$$\bar{\lambda}^T \bar{\xi}^T + \bar{\mu}^T \bar{\zeta}^T = 0$$

and
$$\bar{\mu}^T g(\bar{x}) = 0.$$

Therefore, conditions (1) and (2) are satisfied at $\bar{x} = 1$. We construct the Lagrange function L_η for the $\text{VOP}_\eta(\bar{x}, \bar{\xi}, \bar{\zeta})$ (where $\bar{\xi}, \bar{\zeta}$ are as given above):

$$L_\eta(x, \lambda, \mu, \bar{\xi}, \bar{\zeta}) = \left(-\frac{7}{4}\lambda_1 - 3\lambda_2\right)(x-1)^2 + \left(\frac{1}{3}\mu_1 - \frac{1}{2}\mu_2\right)(x-1)^2 + \mu_1 .$$

Then with simple calculations, it can be seen that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a saddle point of L_η, where $\bar{\lambda}, \bar{\mu}$ are the Lagrange multipliers given above, satisfying Karush-Kuhn-Tucker conditions (1) and (2).

ACKNOWLEDGEMENTS

The first author is thankful to the Council of Scientific and Industrial Research (CSIR), India for the financial support provided during the work.

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